Statistical model for quantum error correction without ancillas

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Abstract

One of the most promising physical properties for implementing quantum technology is light polarization. However, since light polarization is fragile, it is crucial to use quantum error correction (QEC) in order to make quantum information over optical networks feasible. This paper performs a statistical analysis of a redundant-free technique to correct errors on information sent through light polarization. We discuss the performance of the error-correction scheme in a very noisy channel. Finally, we propose an expression to measure the efficiency of the analyzed setup.

1 Introduction

Quantum communication and computing are new areas of information processing that make use of quantum mechanics in order to realize new ways of communication and computation without counterpart in the classical world, as quantum key distribution [2, 1, 7], quantum teleportation [15, 3], and quantum searching [9].

One of the most promising physical property in experimental realization of quantum technologies is light polarization. This happens because a polarization-encoded qubit is easy to generate, detect, and transform. On the other hand, it is well known that light polarization is fragile and changes in an unpredictable way during light propagation in common single-mode fibers. Thus, in order to make quantum technology based on light polarization feasible, quantum error correction (QEC) schemes must be employed, that is, the unpredicted light-polarization changes must be controlled. QEC can be achieved by using quantum codes [8, 12, 14]. Most quantum codes are based on introducing redundancy. Additional qubits are included and entangled with the qubit that carries the useful information. For a polarization-encoded qubit using single-photons, each qubit is represented by the polarization of an optical pulse containing only one photon. This means that a quantum code of n qubits will use n single-photon pulses, where $n-1$ are ancillas. The n photons must be entangled through an n-qubit quantum circuit. All of them are sent through a noisy channel and disentangled at the receiver by another quantum circuit. The ancillas are measured and, according to their values, a selected single-qubit operation (a polarization change) is applied in the signal qubit in order to recover the correct polarization.

Herein the concept of quantum time-bin state follows the definition presented in [4]. The quantum states $|S\rangle$ and $|L\rangle$ represent the pulses traveling through the short and the long paths, or similarly, early and late time slot. The quantum state $|S\rangle$ represents the state which takes the short path and the quantum state $|L\rangle$ represents the state which takes the long path. Any state of the two-dimensional Hilbert space spanned by the basic states $|S\rangle$ and $|L\rangle$ can be prepared ($\alpha|S\rangle + \beta|L\rangle$, where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$).

Our paper focuses on a scheme for quantum error correction without ancilla photons. The first scheme was proposed in [10] and other schemes were proposed later [6, 5]. The essential idea in these schemes is to separate the components of polarization-encoded qubit in time slots as in [4]. The slow variation of the channel transformation in the time does not change the information when the quantum state travels trough it. In this work, a statistical analysis is done considering probabilistic variations between the components of the time-bin quantum state. An expression to measure the efficiency of the quantum error correction setup is proposed to the statistical model of the discussed channel, showing that the efficiency increases when the channel length increases.

The paper is outlined as follows: In Section 2 the quantum error correction system without ancillas is presented. In Section 3 the correction presented in the Section 2 is analyzed when small statistical variations between the components of the time-bin state happen. We draw some conclusions in Section 4. Finally, some analytical equalities required for Section 3 are presented in Appendix.

2 Quantum error correction system without ancillas

We start by discussing the single-photon linear-optical scheme for quantum error proposed in [6]. The scheme is depicted in Figure 1.

The transmitter, Alice, has a single photon in an unknown polarization state $|\psi\rangle = \alpha|H\rangle + \beta|V\rangle$, where $|H\rangle$ ($|V\rangle$) represents the horizontal (vertical) state. After the unbalanced polarization interferometer the state is $\alpha|H, S\rangle + \beta|V, L\rangle$, since the horizontal component takes the short path, S, while the vertical com-

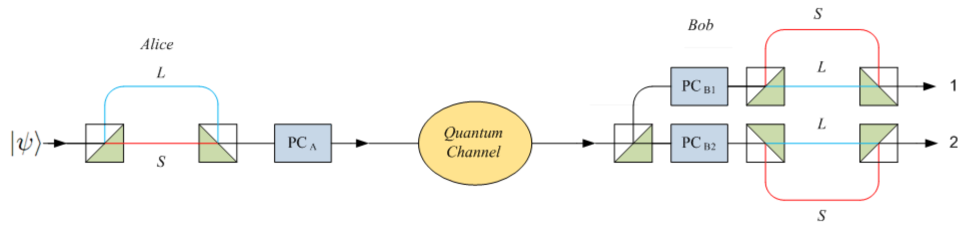

Figure 1: Optical scheme for single-photon quantum error correction: PBS (polarization beam splitter) and PC (Pockels cell).

ponent takes the long path, L. Alice turns on her Pockels cell only when the L-path component is present, effecting the transmission $|V, L\rangle \to |H, L\rangle$. Hence, the state that Alice sends to Bob is $\alpha|H, S\rangle + \beta|H, L\rangle$.

The quantum communication protocols work well for fully polarized light; however, when a fully polarized light propagates in a common optical fiber, the light experiments several random rotations in its polarization due to random birefringence in the fiber that can be produced, for example, by mechanical stress like bending, impurities during the fabrication process, and noncircularity of the core. This effect can be expressed by a unitary transformation $U(\phi, \chi)$, in such way that $U(\phi, \chi)|H\rangle = \cos\phi|H\rangle + e^{i\chi}\sin\phi|V\rangle$ and $U(\phi, \chi)|V\rangle = -\sin\phi|H\rangle + e^{i\chi}\cos\phi|V\rangle$ which describes a general qubit transformation (excluding a global phase which does not have physical significance in this context). Where ϕ denotes the angle rotation and χ is a shift phase between the two polarization components. So, we define the channel as a product of two matrices with one random variable each

$$U(\phi, \chi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\chi} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{1}$$

Since the time separation between the components of the time-bin qubit is taken to be much less than the time of fluctuation of the channel parameters. Both components, S and L, will see, in first hypothesis, the same stationary quantum channel modeled by the unitary transformation U. The general form of U denotes the transformation

$$\begin{aligned} U(\phi, \chi)\alpha|H, S\rangle + U(\phi, \chi)\beta|H, L\rangle \quad &= \quad \alpha(\cos\phi|H, S\rangle + e^{i\chi}\sin\phi|V, S\rangle) \\ &+ \quad \beta(\cos\phi|H, L\rangle + e^{i\chi}\sin\phi|V, L\rangle). \end{aligned} \tag{2}$$

As can be seen in Figure 1, the Pockels cell PC_{B1} is activated only when the S-path component is present; similarly PC_{B2} is activated only when the L-path component is present. At each mode, 1 (upper arm) and 2 (lower arm), there exists an unbalanced polarization interferometer. In these interferometers, the horizontal component propagates through the long path while the vertical component propagates through the short path. When the corrupted state 2 arrives at Bob's place, after passing through the first PBS and Pockels cells, it is transformed into

$$\alpha(\cos\phi|H,S\rangle^2 + e^{i\chi}\sin\phi|H,S\rangle^1) + \beta(\cos\phi|V,L\rangle^2 + e^{i\chi}\sin\phi|V,L\rangle^1). \quad (3)$$

At last, passing through the unbalanced polarization interferometers, the final state is

$$\cos\phi(\alpha|H\rangle^2 + \beta|V\rangle^2) + e^{i\chi}\sin\phi(\alpha|H\rangle^1 + \beta|V\rangle^1). \quad (4)$$

In the equations (3) and (4), the superscripts 1 and 2 denote the paths in the direction of the output modes 1 and 2, respectively. Each received qubit emerges randomly in either one of the two output modes (1 or 2) according to a distribution that depends on the channel parameter ϕ. In both setups Bob obtains the uncorrupted state in mode 1 with probability $\cos^2\phi$ and in mode 2 with probability $\sin^2\phi$. When the channel is approximately an ideal channel Bob receives the uncorrupted state most likely in mode 1, on the other hand, when ϕ is allowed to vary over its range according to a uniform distribution, the probability of obtaining the uncorrupted state in either mode tends to 1/2. However, using an optical delay and an electro-optic switch to form time multiplexing, one can have the state always at the same output (but for different times).

3 The correction when there is fast variation between the pulses

Here, we analyze the performance of the error-correction setup when the channel is very noisy, implying that the fiber birefringence has local fast variations and it can change during the time interval between the short and long pulses. In other words, we analyze the case when there is a variation in a point of the channel after the passage of the short pulse and before the passage of the long pulse. In this case, which parameter can be used to measure the probability of channel varies between the components $|H,S\rangle$ and $|H,L\rangle$? For this analysis, we choose the degree of polarization. Degree of polarization is a quantity used to describe the portion of an electromagnetic wave which is polarized, $1-\xi(z)$. Where $1-\xi(z)$ is the degree of polarization in function of the channel length z. A perfectly polarized wave has $1-\xi(z)=1$, whereas a totally unpolarized wave has $\xi(z)=1$. A wave that is partially polarized, and therefore can be represented by a superposition of a polarized and unpolarized component, has degree of polarization $1-\xi(z)$, a value somewhere in between 0 and 1. For a single polarized photon the same idea is applied to describe if a quantum state is totally mixed or totally pure. A pure quantum state has degree of purity equal to 1, whereas a totally mixed state has degree of purity equal to 0. So, a quantum state can be also represented as superposition of a pure quantum state and a totally mixed state.

$$\xi(z)\frac{I}{2} + (1-\xi(z))\rho \quad (5)$$

Now, let us do a consideration about the channel similarly the done consideration in [13] to statistical treatment of polarization mode dispersion in optical fiber. Suppose that the components stand apart Δz (space units) from each other. Then, we can divide a channel with length z in n equal parts in such way that $n = \frac{z}{\Delta z}$. At the instant t_1 the later pulse $|H, L\rangle$ is entering the channel whereas the earlier pulse $U_{S_1}|H, S\rangle$ is already at the point z_1. After, at instant t_2, the later pulse is at point z_1 and the earlier pulse is at point z_2, moreover, the later pulse develops into $U_{L_1}|H, L\rangle$ whereas the earlier pulse develops into $U_{S_2}U_{S_1}|H, S\rangle$. At each ith-section of the channel, the earlier pulse and the later pulse develop according with unitary evolutions U_{S_i} and U_{L_i} respectively. Therefore, we can write the global channel transformation as

$$\alpha \prod_{i=1}^{n} U_{L_i}|H, L\rangle + \beta \prod_{i=1}^{n} U_{S_i}|H, S\rangle \tag{6}$$

In each section of channel (with length Δz) the probability of $U_{L_i} = U_{S_i}$ is $\xi(\Delta z)$. Then the probability of perfect correction is $(\xi(\Delta z))^n$. Though $U_{L_i} \neq U_{S_i}$ is a rare event, i.e. $\xi(\Delta z) \approx 0$, the number of trials n is quite big, and therefore the hypothesis that small variations occur in some part of the channel needs to be tested. What determines the number of trials is the channel length (and also Δz). If we take $U(\phi_L, \chi_L) = \prod_{i=1}^{n} U_{L_i}$ and $U(\phi_S, \chi_S) = \prod_{i=1}^{n} U_{S_i}$, we can rewrite (6) as

$$\alpha(\cos\phi_L|H\rangle + e^{i\chi_L}\sin\phi_L|V\rangle) + \beta(\cos\phi_S|H\rangle + e^{i\chi_S}\sin\phi_S|V\rangle). \tag{7}$$

The efficiency of the QEC is measured by the fidelity between the input state and the output state of the decoder. This fidelity follows as:

$$\begin{aligned} F^2 &= |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 f & (8)\\ f &= \cos(\phi_S)\cos(\phi_L) + \sin(\phi_S)\sin(\phi_L)\cos(\chi_L - \chi_s). & (9) \end{aligned}$$

Now, define $\phi_S = \phi$, $\phi_L = \phi + \Phi$, $\chi_S = \chi$ and $\chi_L = \chi + X$ to rewrite f as follows:

$$\begin{aligned} f &= \cos(\phi)\cos(\phi + \Phi) + \sin(\phi)\sin(\phi + \Phi)\cos(X) \\ &= \frac{1}{2}\{\cos\Phi(1 + \cos X) + \cos(2\phi + \Phi)(1 - \cos X)\}. \end{aligned} \tag{10}$$

Observe that f is a parameter depending on the channel variations. Moreover, if $X = \Phi = 0$ there is no error and $F^2 = 1$. Furthermore, note that the expected value of F^2 depends of expected value of f, since $E(F^2) = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 E(f)$.

Firstly, we consider that the random variable ϕ has uniform probability distribution, since the associated variation can be arbitrary. As the analyzed function is $\cos(2\phi + \Phi)$, we assume that ϕ is a random variable with uniform probability distribution in $[\frac{-\pi-\Phi}{2}, \frac{\pi-\Phi}{2}]$. In this way, we can write the expected value of f for ϕ by

$$
\begin{aligned}
E_\phi(f) &= \frac{1}{2}\{\cos\Phi(1+\cos X) + (1-\cos X)\frac{1}{\pi}\underbrace{\int_{\frac{-\pi-\Phi}{2}}^{\frac{\pi-\Phi}{2}} \cos(2\phi+\Phi)d\phi\}}_{0} \\
&= \frac{1}{2}\{\cos\Phi(1+\cos X)\}. \qquad (11)
\end{aligned}
$$

We assume that variations in Φ and X are rare events (they occur with probability $\xi(\Delta z)$) and that, when they occur, Φ and X and modified by $\pm\Delta\Phi$ and $\pm\Delta X$, respectively. Thus, we derive a two-dimensional random walk which starts at point 0 and at each step moves by $\pm\Delta\Phi$ ($\pm\Delta X$). Each step happens with probability $p_{\Delta\Phi}$ for $+\Delta\Phi$ and probability $(1-p_{\Delta\Phi})$ for $-\Delta\Phi$ (and similar to ΔX, that is, probability $p_{\Delta X}$ for $+\Delta X$ and probability $(1-p_{\Delta X})$ for $-\Delta X$). Thus, the discrete random variation in the channel can be expressed by $\Phi = t\Delta\Phi$ and $X = s\Delta X$. Moreover, we have

$$
\begin{aligned}
P_n(k) &= \binom{n}{k}\xi(\Delta z)^k(1-\xi(\Delta z))^{(n-k)} \qquad (12) \\
P_k(p_{\Delta X}, s) &= \frac{k!}{\left(\frac{k+s}{2}\right)!\left(\frac{k-s}{2}\right)!}{p_{\Delta X}}^s(1-p_{\Delta X})^{(k-s)} \qquad (13) \\
P_k(p_{\Delta\Phi}, t) &= \frac{k!}{\left(\frac{k+t}{2}\right)!\left(\frac{k-t}{2}\right)!}{p_{\Delta\Phi}}^t(1-p_{\Delta\Phi})^{(k-t)}. \qquad (14)
\end{aligned}
$$

Where the equation (12) is the probability of occurring k variations in n trials, the equation (13) represents the probability of $\Phi = s\Delta\Phi$ when k variations ($s \leq k$) occur and the equation (14) represents the probability of $X = t\Delta X$ when k variations ($t \leq k$) occur. Thus, after n trials the probability of $\Phi = s\Delta\Phi$ and $X = t\Delta X$ is

$$
P_n(s,t) = \sum_{k=1}^{n} P_n(k)P_k(p_{\Delta X}, s)P_k(p_{\Delta\Phi}, t). \qquad (15)
$$

Where $s, t \in \mathcal{S} = \{-k, -k+2, \dots, k-2, k\}$ and $\sum_{s,t\in\mathcal{S}} P_n(s,t) = 1$. What can we say about the positions Φ and X of the walk after k steps? It is not hard to see that the $E(\Phi) = E(X) = 0$. A similar analysis, assuming independence between Φ and X, derives that $\mathrm{Var}(\Phi) = E(\Phi^2) = k(\Delta\Phi)^2$ and $\mathrm{Var}(X) = E(X^2) = k(\Delta X)^2$. Note that, although X and Φ are assumed to be independent, their variance is function of the number of trials k that occur with probability $P_n(k)$ as in the equation (12).

Firstly, let us analyze the probability given by equation (12). The Taylor's series below is a representation of the degree of depolarization as an infinite sum of terms calculated from the values of its derivatives at the point zero.

$$
\xi(z) = \xi(0) + \xi'(0)z + \xi''(0)\frac{z^2}{2!} + \xi'''(0)\frac{z^3}{3!} + \cdots \qquad (16)
$$

At point $z = 0$ the quantum state is totally pure, therefore, the degree of depolarization is null, that is, $\xi(0) = 0$. Calculating this function for small Δz we can approximate $\xi(\Delta z) \approx \xi'(0)\Delta z$. Moreover, $\xi(\Delta z) \approx 0$ because in a short length of propagation, the effects of light polarization are negligible. So, the probability of k occurrences such that U_{L_i} is different of U_{S_i} to a small variation in Φ and X is

$$\begin{aligned}
\lim_{n\to\infty} P_n(\xi,k) &= \lim_{n\to\infty} \binom{n}{k} \xi(\Delta z)^k (1-\xi(\Delta z))^{(n-k)} \\
&= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\xi'(0)z}{n}\right)^k \left(1-\frac{\xi'(0)z}{n}\right)^{(n-k)} \\
&= \lim_{n\to\infty} \underbrace{\frac{n!}{n^k(n-k)!}}_{1} \frac{(\xi'(0)z)^k}{k!} \overbrace{\left(1-\frac{\xi'(0)z}{n}\right)^n}^{e^{-\xi'(0)z}} \overbrace{\left(1-\frac{\xi'(0)z}{n}\right)^{-k}}^{1} \\
&= \frac{(\xi'(0)z)^k}{k!} e^{-\xi'(0)z}. \qquad (17)
\end{aligned}$$

The Poisson distribution is the discrete probability distribution that expresses the probability of a number of events (local birefringence variation between the earlier pulse and later pulse) occurring in a fixed channel length, provided that these events occur with an average rate, $z\xi'(0)$, and independently of the time since the last event occurred. In other words, k is the number of occurrences of an event and $z\xi'(0)$ is a positive real number, equal to the expected number of channel variations between the later pulse and earlier pulse that occur during a given channel length z. This limit describes the law of rare events, since each of the individual Bernoulli events rarely triggers. The name may be misleading because the total count of success events in a Poisson process needs not to be rare if the channel length is very long, on the other hand, the parameter $z\xi'(0)$ is not small. Though a channel variation between the earlier pulse and the later pulse is a very rare event, the number of trials grows enough with the channel length. Therefore, the expected value of f is calculated by

$$E(f) = \frac{e^{-\xi'(0)z}}{2} \sum_{k=1}^{\infty} \frac{(\xi'(0)z)^k}{k!} \sum_{s,t=-k,-k+2,\dots}^{k} E_\phi(f) P_k(p_{\Delta X}, s) P_k(p_{\Delta\Phi}, t). \qquad (18)$$

When the channel grows the number of trials grows too. We consider the parameter $z\xi'(0)$ to be very big, in such way that the number of trials is approximately infinity. In addition, we consider that each step $\Delta\Phi(\Delta X)$ happens with probability 1/2. Using the analysis presented in the Appendix A, we can be write the expected value to f as follows:

$$E(f) = \frac{1}{2}\left(e^{-(1-\cos(\Delta\Phi))\xi'(0)z} + e^{-(1-\cos(\Delta\Phi)\cos(\Delta X))\xi'(0)z}\right). \qquad (19)$$

When the earlier and later pulses do not have different local variations between them in any sections of the channel ($\Delta\Phi = \Delta X = 0$), the expected value of f

is always 1 for any channel length z. However, when the variations exist, even small ones, the expected value of f decreases exponentially with the channel length z. Hence, the expected value of the fidelity (F^2) has a minimum value of $|\alpha|^4 + |\beta|^4$ to a sufficiently long channel.

The expected value of f is illustrated in the Figure 2. The parameters $\Delta\Phi$ and ΔX are considered equals and belong to the interval $[0, 0.1\pi]$. This interval is small because we consider only small variations in the channel between the short and long pulses. The hypothesis for this consideration as follows: The pulses $U_S|H, S\rangle$ and $U_L|H, L\rangle$ are separated for one small distance Δz, which implies in the small period of time Δt. Therefore, the random variation in this period of time was considered small.

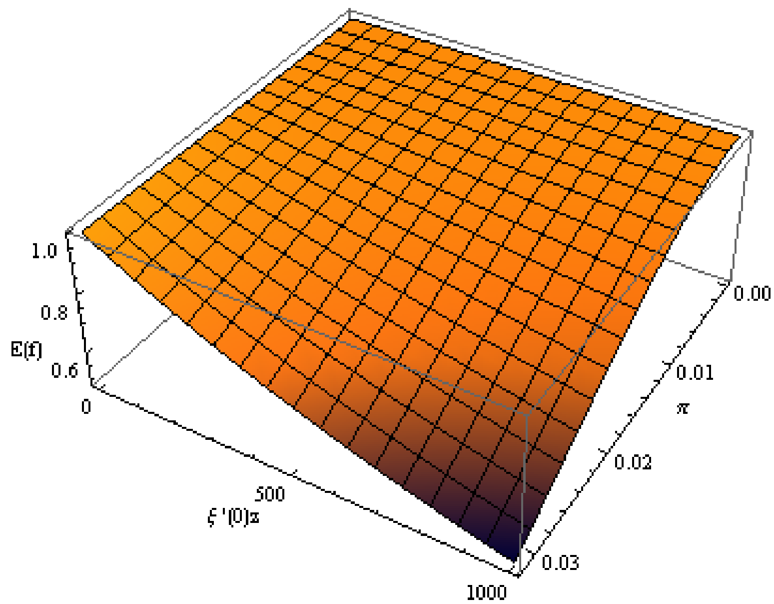

Figure 2: The value $E(f)$ in function of $\xi'(0)z$ and $\Delta\Phi = \Delta X$.

Another parameter in the expected value of f is $\xi'(0)z$ which belongs to the interval $[0, 1000]$ in the Figure 2. In [11], a simple quantum model for light depolarization shows that $\xi(z)$ increases exponentially with z. In [16] some results were simulated for the behavior of the degree of polarization in classical light. In general, the degree of polarization as function of propagation distance along optical fiber is the unity for initial light (totally polarized) and decreases to zero when the distance of propagation increases. However, the shape of the function changes with the variation of some parameters (coupling coefficient between the orthogonal fields and spectral width source). In [16], the degree of polarization showed small variation at $z = 0$ when the coupling coefficient was high and the spectral width source was short. What shows that the design of input pulse in the encoder is relevant to the high-performance of the QEC.

4 Conclusion

We analyzed the performance of a quantum error correction setup that does not use ancillas for polarized-encoded qubits. We considered variations in the channel when the birefringence has a local variation after the passaging of the short pulse and before the passaging of the long pulse. We saw that, in this setup, the expected value of the fidelity between the input quantum state in the encoder and the corrected quantum state decreases exponentially when the length of the channel increases.

In this work, we remark some parameters which influence the performance of the QEC without ancillas. The parameter, $\xi'(0)$ can be estimated by analyzing the variation of the Stokes parameters. The parameters $\Delta\Phi$ and ΔX must be estimated experimentally. This was not done in this work. Therefore, we suggest an analysis of the experimental setup to verify that the expected value $E(f)$ has a behavior similar (or equal) to the curve proposed in equation (19). Furthermore, the distance Δz, that is, a parameter of the QEC to synchronize the encoder and decoder, is also useful to establish the performance of the QEC setup. This distance Δz can be designed to avoid interference between long and short pulses caused by effect of dispersions (polarization-mode dispersion, chromatic dispersion etc).

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A Details about the equation 19

This appendix presents the details about the development of equation (18) into equation (19). Firstly, let us develop the following expression:

$$\begin{aligned}
& \sum_{s,t=-k,-k+2,\ldots}^{k} E_\phi(f) P_k(p_{\Delta X}, s) P_k(p_{\Delta\Phi}, t) \qquad (20)\\
= & \frac{1}{2} \sum_{s,t=-k,-k+2,\ldots}^{k} \cos(t\Delta\Phi)\{1+\cos(s\Delta\Phi)\} P_k(p_{\Delta X}, s) P_k(p_{\Delta\Phi}, t)\\
= & \frac{1}{2} \left\{ \sum_{t=-k,-k+2,\ldots}^{k} \cos(t\Delta\Phi) P_k(p_{\Delta\Phi}, t) \left(1 + \sum_{s=-k,-k+2,\ldots}^{k} \cos(s\Delta X) P_k(p_{\Delta X}, s) \right) \right\}
\end{aligned}$$

Considering $p_{\Delta\Phi} = p_{\Delta X} = 1/2$, consider:

$$\begin{aligned}
M_k(\delta) &= \sum_{r=-k,-k+2,\ldots}^{k} \cos(r\delta) P_k\left(1/2, r\right)\\
&= \sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{\cos(r\delta)}{2^k}\\
&= \sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{e^{ir\delta}+e^{-ir\delta}}{2^{k+1}}\\
&= \frac{1}{2}\left(\sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{e^{ir\delta}}{2^k} + \sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{e^{-ir\delta}}{2^k} \right)\\
&= \frac{1}{2}\left(\sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{e^{i\delta\left(\frac{k+r}{2}\right)} e^{-i\delta\left(\frac{k-r}{2}\right)}}{2^k} \right)\\
&+ \frac{1}{2}\left(\sum_{r=-k,-k+2,\ldots}^{k} \frac{k!}{\left(\frac{k+r}{2}\right)!\left(\frac{k-r}{2}\right)!} \frac{e^{-i\delta\left(\frac{k+r}{2}\right)} e^{i\delta\left(\frac{k-r}{2}\right)}}{2^k} \right)\\
&= \frac{1}{2}\left(\left(\frac{e^{i\delta}+e^{-i\delta}}{2}\right)^k + \left(\frac{e^{-i\delta}+e^{i\delta}}{2}\right)^k \right)\\
&= \left(\cos(\delta)\right)^k. \qquad (21)
\end{aligned}$$

Now, we write $\lambda = \xi'(0)z$ to simplify notation. Thus, the expression (20) can be written by

$$
\begin{aligned}
E(f) &= \frac{e^{-\lambda}}{2}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}M_k(\Delta\Phi)(1+M_k(\Delta X)) \\
&= \frac{1}{2}e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\cos^k(\Delta\Phi)(1+\cos^k(\Delta X)) \\
&= \frac{1}{2}e^{-\lambda}\left(\sum_{k=0}^{\infty}\frac{(\lambda\cos(\Delta\Phi))^k}{k!}+\sum_{k=0}^{\infty}\frac{(\lambda\cos(\Delta\Phi)\cos(\Delta X))^k}{k!}\right) \\
&= \frac{1}{2}\left(e^{-(1-\cos(\Delta\Phi))\lambda}+e^{-(1-\cos(\Delta\Phi)\cos(\Delta X))\lambda}\right). \qquad (22)
\end{aligned}
$$