A Switched Systems Approach to Image-Based Localization of Targets that Temporarily Leave the Field of View

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Abstract—Vision sensors provide rich information about the environment and can be utilized for tracking the position of moving objects from a moving camera. This paper presents the development of dwell time conditions to guarantee convergence to an ultimate bound of state estimation errors for a class of vision based observers with intermittent measurements. Bounds are developed on the unstable growth of the estimation errors during the periods when the object being tracked is not visible. A Lyapunov analysis for the switched system is performed to develop an inequality in terms of the duration of time the observer can view the moving object and the duration of time the object is out of the field of view.

I. INTRODUCTION

Advances in imaging technology and estimation techniques have enabled the use of cameras in sensing applications. Vision sensors operate by projecting a 3D scene onto a 2D image plane. The depth information can be recovered through various estimation techniques. Stereoscopy is typically employed, however geometric constraints limit the range and accuracy of the reconstruction; accurately measuring large depths requires a large space between the two cameras, which is infeasible in munitions or mobile vehicles. By moving a single camera, geometric constraints can be met and 3D information can be recovered. This is known as the structure from motion (SfM) problem [1]. The more general trajectory triangulation problem [2] refers to applications where both the camera and features are in motion. Solutions to this problem can be broadly classified as either causal or batch methods.

Typical solutions to the trajectory triangulation problem impose constraints on the class of trajectories of the moving feature. In [2], a batch algorithm is developed to estimate the position of an object moving in either a straight line or conic path. In [3], authors utilize parameter estimation techniques for objects moving in general trajectories. A factorization-based algorithm is applied for a scene with multiple objects moving in straight lines with constant velocities in [4].

Batch methods require multiple images before producing a reconstruction estimate, and therefore are less useful in real-time applications compared to causal or online methods. These methods rely on a dynamic model of the camera and target rather than algebraic and geometric constraints. In [5], a nonlinear observer is developed for an object moving with constant linear velocities. In [6], the results are extended for general motion of a target moving on a ground plane observed by a downward looking camera by use of an unknown input observer. The development in this paper focuses on causal image-based estimation methods under the constraint of intermittent sensing. The error convergence conditions presented in this paper make no assumption on the object motion except that the linear and angular velocities of the target and camera are bounded, and therefore, are applicable to any online method.

Intermittent sensing loss may be caused by failure in the feature tracking algorithm, occlusions in the environment, or even purposeful evasion by the camera. Literature on feedback control with intermittent measurements focuses on modeling sensing loss as a random variable and proving stochastic stability of the system. In [7], an $H_{\infty}$ fuzzy filter is developed to estimate the states of a linear time invariant discrete time system with intermittent sensing loss modeled as a Bernoulli process. Similarly, authors in [8] develop a robust filter for a nonlinear stochastic system with missing measurements. In [9], an $H_{\infty}$ feedback controller is developed and stochastic stability is proven given a condition on the probability of sensing loss is met. In contrast, the development in this paper makes no assumption on the nature of the switching; loss of sensing can occur arbitrarily. Also, a condition is developed to guarantee the convergence of the actual estimation error, not just the expected value of the error.

At each instance when sensing is re-established, the estimator can be reinitialized with the previous state estimate; however, there is no guarantee that the estimation errors will eventually converge, even if the estimator is proven to be stable when the moving object is in view [10]. Switched systems theory provides a framework to analyze the stability of these online estimators as the object moves in and out of the camera field of view (FOV). An average dwell time condition is developed in [11] for linear switched systems with stable and unstable subsystems. Since the subsystems are all linear, state trajectories are exponential, and a condition on the overall time spent in the stable and unstable systems can be developed based on the ratio of maximum unstable eigen-
values and minimum stable eigenvalues. Similar conditions were developed in [12] for nonlinear switched systems with exponentially stable and exponentially unstable subsystems. In this work, the unstable system is not exponentially unstable, and therefore algebraic simplifications cannot be made that yield the less restrictive average dwell time conditions. The contribution of this work is in the development of the dwell time and reverse dwell time requirements for estimator error convergence for uncertain nonlinear dynamics which exhibit finite escape time instabilities.

The results in this paper are useful in applications of both passive and active sensing loss. In the passive case, i.e. when the observer has no control of when sensing loss occurs, the observer will continue to produce state estimates when the target is in view. However, since the estimate errors can diverge over multiple sensing losses, the results provide a condition on when the estimates can be trusted. In the active case, i.e. when the observer purposely loses sight of the moving target, the conditions provide guidance on when the sensor can be repositioned to break the line-of-sight and the maximum time before the target needs to be reacquired. For example, various applications exist where mission objectives may be facilitated by allowing the camera to move away from the observed target such that it leaves the FOV. Moreover, for vehicular systems with motion constraints, guidance and control objectives may be severely restricted if the agent has to move so that the target remains in the FOV [13]. For such scenarios, the stability conditions developed in this paper provide sufficient dwell times during which the target needs to be observed or can leave the FOV.

II. KINEMATIC MOTION MODEL

In the following development, () refers to geometric vectors, i.e. members of \( \mathbb{R}^3 \), and is distinguished from its \( \mathbb{R}^3 \) expression in any particular reference frame. Also, \( \frac{d}{dt} \) refers to the rate of change of a vector as viewed by an observer fixed in reference frame \( (\cdot) \). Let \( F_G \) be an inertial reference frame, with arbitrarily chosen origin and Euclidean coordinate system. Let \( F_O \) be a moving reference frame attached to the tracked target with arbitrarily chosen origin and Euclidean coordinate system. Let \( F_C \) be a reference frame attached to the moving camera, with right handed coordinate system origin at the principle point of the camera, the \( e_3 \) axis pointing out along the optical axis of the camera, \( e_1 \) axis aligned with the horizontal axis of the camera, and \( e_2 \) parallel to \( e_3 \times e_1 \), as shown in Fig. 1. Let the vectors \( \vec{r}_q \) and \( \vec{r}_c \) represent the position of a feature point on the target and the principle point of the camera with respect to the origin of the ground frame. The relative position of the feature point with respect to the camera origin is

\[
\vec{r}_{q/c} = \vec{r}_q - \vec{r}_c.
\] 

The relative velocity as viewed by an observer in the ground frame, denoted \( \vec{v}_{q/c} \), is

\[
\vec{v}_{q/c} = \frac{d}{dt} \vec{r}_{q/c} = \frac{d}{dt} \vec{r}_q - \frac{d}{dt} \vec{r}_c.
\] 

To facilitate the subsequent analysis, it is beneficial to relate the rate of change of the relative position as viewed by an observer fixed in the ground frame to the rate of change as viewed by an observer fixed in the camera frame as

\[
G \frac{d}{dt} \vec{r}_{q/c} = \frac{d}{dt} \vec{v}_q - \vec{v}_c - \vec{\omega} \times \vec{r}_{q/c},
\] 

where \( G \vec{\omega} \) is the angular velocity of the camera frame with respect to the ground frame. Equating the right hand sides of (2) and (3), and simplifying yields

\[
C \frac{d}{dt} \vec{r}_{q/c} = \vec{v}_q - \vec{v}_c - \vec{\omega} \times \vec{r}_{q/c}.
\]

In the following derivation, all vectors are expressed in the camera coordinate system, i.e. the basis fixed in \( F_C \). Let

\[
\vec{r}_{q/c} = \begin{bmatrix} X & Y & Z \end{bmatrix}^T \in \mathbb{R}^3, \tag{5}
\]

\[
G \vec{v}_q = \begin{bmatrix} v_{q1} & v_{q2} & v_{q3} \end{bmatrix}^T \in \mathbb{R}^3, \tag{6}
\]

\[
G \vec{v}_c = \begin{bmatrix} v_{c1} & v_{c2} & v_{c3} \end{bmatrix}^T \in \mathbb{R}^3, \tag{7}
\]

\[
G \vec{\omega} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T \in \mathbb{R}^3, \tag{8}
\]

where \( X, Y, Z \in \mathbb{R} \) denote the Euclidean coordinates of the target position relative to the camera position, \( v_{q1}, v_{q2}, v_{q3} \in \mathbb{R} \) denote the linear velocities of the target with respect to the ground frame, \( v_{c1}, v_{c2}, v_{c3} \in \mathbb{R} \) denote the linear velocities of the camera with respect to the ground frame and \( \omega_1, \omega_2, \omega_3 \in \mathbb{R} \) denote the angular velocity of the camera frame with respect to the ground frame. Substituting (5)-(8) into (4) yields

\[
\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} v_{q1} - v_{c1} + \omega_3 Y - \omega_2 Z \\ v_{q2} - v_{c2} + \omega_1 Z - \omega_3 X \\ v_{q3} - v_{c3} + \omega_2 X - \omega_1 Y \end{bmatrix}. \tag{9}
\]

As is common in structure estimation literature, the states of the system are defined as \( z = [x_1, x_2, x_3]^T = [X, Y, Z]^T \in \mathbb{R}^3 \) to facilitate the analysis [14]-[19]. The perspective state dynamics \( \dot{x} = g(t, x) \), where \( g(t, x) : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is
a nonlinear function that nonlinearly depends on the partially measurable states, can be expressed as 
\[
\dot{x}_1 = \frac{\dot{X}}{Z} - \frac{X}{Z^2} \dot{Z}, \\
\dot{x}_2 = \frac{\dot{Y}}{Z} - \frac{Y}{Z^2} \dot{Z}, \\
\dot{x}_3 = -\frac{Z}{Z^2} \dot{Z}, 
\]
(10)
Substituting (9) into (10) and simplifying yields 
\[
\dot{x}_1 = \Omega_1 + f_1 + v_1 x_3 - x_1 v_3 x_3 \\
\dot{x}_2 = \Omega_2 + f_2 + v_2 x_3 - x_2 v_3 x_3 \\
\dot{x}_3 = -v_3 x_3^2 - (x_2 \omega_1 - x_1 \omega_2) x_3 + v_3 x_3^2 
\]
where \( \Omega_1, \Omega_2, f_1, f_2 \in \mathbb{R} \) are defined as 
\[
\Omega_1 = \omega_3 x_2 - \omega_2 - \omega_2 x_1^2 + \omega_1 x_1 x_2, \\
\Omega_2 = \omega_1 - \omega_3 x_1 - \omega_2 x_1 x_2 + \omega_1 x_2^2, \\
f_1 = (v_3 x_3 - v_1) x_3, \\
f_2 = (v_3 x_2 - v_2) x_3, 
\]
(12)
Assumption 1. The state \( x \) is bounded, i.e. \( x \in \mathcal{X} \), where \( \mathcal{X} \subset \mathbb{R}^3 \) is a compact set.

Remark 1. In order for the state estimates to converge to the states while remaining bounded, the states themselves must remain bounded. During periods in which the target is observable, bounds on the states are a result of the physical constraints on the imaging system. For image formation, the target must remain in front of the camera principal point by an arbitrarily small amount, \( \epsilon \in \mathbb{R} \). This provides an arbitrarily small lower bound on \( Z \) and therefore an arbitrarily large upper bound on \( x_3 \). Also, the strict 180° bound on the camera FOV, and the bound on \( Z \) provide an effective bound on \( X \) and \( Y \), therefore bounding \( x_1 \) and \( x_2 \). During the periods in which the target is unobservable, these physical constraints no longer apply. However, Assumption 1 implies that the target does not exhibit finite escape during the unobservable periods. This restricts the relative motion of the target with respect to the camera; the target cannot move behind the camera, even during the unobservable periods, else the state \( x_3 \) will pass through \( \infty \).

Assumption 2. Bounds for the camera and target velocities exist and are known, i.e. the following inequalities are satisfied 
\[
\begin{bmatrix}
|v_1| & |v_2| & |v_3| \\
|v_1| & |v_2| & |v_3| \\
|\omega_1| & |\omega_2| & |\omega_3|
\end{bmatrix}^T \leq \begin{bmatrix}
\bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\
\bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\
\bar{\omega}_1 & \bar{\omega}_2 & \bar{\omega}_3
\end{bmatrix}^T
\]
where \( \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3 \in \mathbb{R} \) are known non-negative constants.

Remark 2. Conservative bounds on the target velocities can easily be established. For example, the velocities of observed vehicular systems can readily be upper bounded with some domain knowledge.

To facilitate the subsequent stability analysis, the unknown nonlinear functions in (12) can be bounded as 
\[
f_1 \leq \bar{v}_1 \|x\| + \bar{v}_3 \|x\|^2, \\
f_2 \leq \bar{v}_2 \|x\| + \bar{v}_3 \|x\|^2, \\
\Omega_1 \leq \bar{\omega}_2 + \bar{\omega}_3 \|x\| + (\bar{\omega}_1 + \bar{\omega}_2) \|x\|^2, \\
\Omega_2 \leq \bar{\omega}_1 + \bar{\omega}_3 \|x\| + (\bar{\omega}_1 + \bar{\omega}_2) \|x\|^2. 
\]
(13)-(16)
Substituting (13)-(16) into (11) and bounding the remaining terms yields the following inequalities:
\[
\dot{x}_1 \leq \bar{\omega}_2 + (\bar{v}_1 + \bar{\omega}_3 + \bar{v}_q) \|x\| + (\bar{v}_3 + \bar{\omega}_1 + \bar{\omega}_2 + \bar{v}_q) \|x\|^2, \\
\dot{x}_2 \leq \bar{\omega}_1 + (\bar{v}_1 + \bar{\omega}_3 + \bar{v}_q) \|x\| + (\bar{v}_3 + \bar{\omega}_1 + \bar{\omega}_2 + \bar{v}_q) \|x\|^2, \\
\dot{x}_3 \leq (\bar{v}_3 + \bar{\omega}_1 + \bar{\omega}_2 + \bar{v}_q) \|x\|^2. 
\]
(17)-(19)

III. IMAGING MODEL

Using projective geometry, the image coordinates of the feature point, \( p = \begin{bmatrix} u & v & 1 \end{bmatrix}^T \in \mathbb{R}^3 \), where \( u, v \in \mathbb{R} \), are related to the normalized Euclidean coordinates, \( m = \begin{bmatrix} \frac{X}{Z} & \frac{Y}{Z} & 1 \end{bmatrix}^T \in \mathbb{R}^3 \), by
\[
p = A m,
\]
where \( A \in \mathbb{R}^{3 \times 3} \) is the known, invertible camera intrinsic parameter matrix [20]. Since \( A \) is invertible, the states \( x_1 \) and \( x_2 \) are measurable.

IV. STRUCTURE ESTIMATION OBJECTIVE

To quantify the structure estimation objective, let the state estimate error, \( e \in \mathbb{R}^3 \), be defined as 
\[
e = x - \hat{x},
\]
where \( \hat{x} \in \mathbb{R}^3 \) denotes the state estimate determined from an observer. The evolution of \( e \) is defined by the family of systems 
\[
\dot{e} = f_p(t, x, \hat{x})
\]
(21)
where \( f_p : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \), \( p \in \{s, u\} \), \( s \) is an index referring to the system in which the target is observable and \( u \) is an index referring to the system in which the target is unobservable. When the target is in view, the states \( x_1 \) and \( x_2 \) are measurable, and the closed-loop error dynamics are given by 
\[
f_s = g(t, x) - \hat{x},
\]
where \( \hat{x} \) is defined by some observer. However, when the target is out of the camera FOV, the state estimates cannot be updated (i.e., \( \hat{x} = 0 \), and the error dynamics are given by 
\[
f_u = g(t, x).
\]
Assumption 3. An observer for the state $x$ has been developed such that, when the states $x_1$ and $x_2$ are measurable, the state estimation error is exponentially convergent, i.e. $||e(t)|| \leq k||e(t_0)|| \exp[-\lambda_{on}(t-t_0)]$ for some positive constants $\lambda_{on}, k \in \mathbb{R}$.

Remark 3. Exponentially convergent observers for image-based structure estimation are available from results such as [6], [18], [21]–[23].

V. STABILITY ANALYSIS

In the following development, the switching signal $\sigma : [0, \infty) \rightarrow \{s, u\}$ indicates the active subsystem. Also, let $t_{on}^n \in \mathbb{R}$ denote the time of the $n^{th}$ instance at which the target enters the camera FOV and $t_{off}^n \in \mathbb{R}$ denote the time of the $n^{th}$ instance at which the target exits the camera FOV, where $n \in \mathbb{N}$. The dwell time in the $n^{th}$ instance of subsystem $s$ and $u$ is denoted by $\Delta t_{on}^n \triangleq t_{on}^n - t_{off}^n \in \mathbb{R}$ and $\Delta t_{off}^n \triangleq t_{off}^{n+1} - t_{on}^{n} \in \mathbb{R}$, respectively. Finally, $\Delta t_{on}^n \triangleq \inf_{n \in \mathbb{N}} \Delta t_{on}^n \in \mathbb{R}$ and $\Delta t_{off}^n \triangleq \sup_{n \in \mathbb{N}} \Delta t_{off}^n \in \mathbb{R}$ denote the minimum dwell time in subsystem $s$ and maximum dwell time in subsystem $u$, respectively, for all $n$.

Theorem 1. The switched system generated by (21) and switching signal $\sigma$ is asymptotically regulated to a ball of arbitrary size provided that switching signal and the initial condition satisfy the following conditions:

$$ \Delta t_{off}^\max \leq \frac{\pi}{2\beta}, $$

$$ \Delta t_{on}^\min \geq -\frac{1}{\lambda_s} \ln \frac{1}{\mu^2}, $$

$$ \frac{1 - \mu^2 \exp(-\lambda_s \Delta t_{on}^\min)}{2 \mu \exp(-\lambda_s \Delta t_{on}^\min)} \geq \tan(\beta \Delta t_{off}^\max), $$

$$ c_2 ||e(0)||^2 < \bar{d}, $$

where the constants $\beta$, $\lambda_s$, $\mu$, $\bar{d}$, $c_2 \in \mathbb{R}$ are known positive bounding constants.

Proof: The existence of an exponentially tracking state observer implies the existence of a Lyapunov function $V_s : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ that satisfies

$$ c_1 ||e||^2 \leq V_s(t,e) \leq c_2 ||e||^2 $$

$$ \frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial e}(\dot{e}) \leq -c_3 ||e||^2 $$

for some positive scalar constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$, during the periods in which the target is observable [24]. From (26) and (27) it is clear that

$$ \dot{V}_s \leq -\lambda_s V_s $$

when the target is in view, where $\lambda_s = \frac{c_3}{c_2}$.

Define a Lyapunov-like function $V_u : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$ V_u = c_5 e^T e $$

where $c_5 \in \mathbb{R}$ is bounded by $c_1 \leq c_5 \leq c_2$. From (26) and (29) it is clear that

$$ V_p \leq \frac{c_2}{c_1} V_q, \forall p, q \in \{s, u\}, p \neq q. $$

During the period when the target is out of the FOV, the time derivative of $V_u$ can be upper bounded as

$$ \dot{V}_u \leq 2c_5 \left( c_6 ||e|| + c_7 ||e||^2 + c_8 ||e||^3 \right), $$

where (17)-(19) were used and $c_6, c_7, c_8 \in \mathbb{R}$ denote known positive constants based on the upper bounds on the camera and target velocities and an upper bound on $||\dot{e}||$ from Assumption 1. From (29), $||e||$ can be upper bounded by $\sqrt{\frac{V_c}{c_5}}$, resulting in

$$ \dot{V}_u \leq \beta \left( e^T e + 1 \right) $$

where $\beta$ was introduced in (22).

Define the function $W : [0, \infty) \rightarrow \mathbb{R}$ such that $W(t) \triangleq V_{\sigma(t)}(t,e(t))$. From (28) and (32)

$$ \dot{W}(t) \leq \begin{cases} -\lambda_s W(t) & t \in [t_{on}^n, t_{off}^n) \\ \beta \left( W(t)^2 + 1 \right) & t \in [t_{off}^n, t_{on}^{n+1}) \end{cases}, \forall n. $$

The second inequality in (33) indicates that $W$ can grow unbounded in finite time when the target is unobservable. However, from the first inequality, $W$ is regulated to zero when the target is observable. This suggests that if the target is observed for a long enough duration, and the target is out of the FOV for a short enough duration, the net change in $W$ will be negative over a cycle where observability is lost and regained, and consequently the estimation error will decrease. A representative illustration of the evolution of $W$ if these conditions are satisfied is depicted in Fig. 2.
Utilizing the Gronwall-Bellman inequality, (33) can be integrated, yielding
\[ W(t) \leq \begin{cases} W_s & t \in [t_{on}^{n}, t_{off}^{n}], \\ W_n(t) & t \in [t_{n+1}^{off}, t_{n+1}^{on}], \forall n \end{cases} \tag{34} \]
where the functions \( W_n^{on} : [t_{on}^{n}, t_{off}^{n}] \rightarrow \mathbb{R} \) and \( W_n^{off} : [t_{n+1}^{off}, t_{n+1}^{on}] \rightarrow \mathbb{R} \) are defined as
\[ W_n^{on}(t) \triangleq W_n \exp(-\lambda_s (t - t_{on}^{n})), \tag{35} \]
\[ W_n^{off}(t) \triangleq \tan \left( \beta (t - t_{off}^{n}) + \arctan \left( W_n^{off} \right) \right), \tag{36} \]
\( W_n^{on} \) denotes \( W(t_{on}^{n}) \) and \( W_n^{off} \) denotes \( W(t_{off}^{n}) \). From (30), the discontinuities in \( W \) are related as
\[ W \left( t_{off}^{n} \right) \leq \mu W \left( t_{on}^{n} \right) - \mu A + B W_n, \tag{38} \]
with \( \mu = \frac{\pi}{2 \beta} \in \mathbb{R} \). Therefore, the change in \( W \) over a cycle of losing and regaining observability is
\[ W_{n+1}^{on} \leq \mu \tan \left( \beta \Delta t_{off}^{n} + \arctan \left( \mu W_n^{on} e^{-\lambda_s \Delta t_{on}^{n}} \right) \right), \tag{37} \]
for all \( n \), which can be rewritten using a trigonometric identity as
\[ W_{n+1}^{on} \leq \mu \frac{1 - AB W_n^{on}}{1 - AB W_n}, \tag{39} \]
where \( A = \tan \left( \beta \Delta t_{off}^{max} \right) \),
\[ B = \mu \exp \left( -\lambda_s \Delta t_{on}^{min} \right). \]

The elements of the sequence \( \{ W_n^{on} \} \) are upper bounded as
\[ W_n^{on} \leq z_n, \forall n \]
where the sequence \( \{ z_n \} \) is defined as
\[ z_0 = W_0^{on}, \]
\[ z_{n+1} = \mu \frac{A + B z_n}{1 - AB z_n}. \tag{40} \]

Since the elements of the sequence \( \{ W_n^{on} \} \) are lower bounded by zero due to the definition of \( W \), the squeeze theorem can be used to show that \( \{ W_n^{on} \} \) converges to a ball upper bounded by \( \lim z_n \). The sequence \( \{ z_n \} \) will converge if it is bounded and monotonic. The following two conditions arise from the requirement that \( z_n \) remain upper bounded over every iteration from \( n \) to \( n+1 \):
\[ \Delta t_{off}^{max} < \frac{\pi}{2 \beta}, \tag{40} \]
\[ A B z_n < 1. \tag{41} \]

For decaying convergence, the sequence is monotonically decreasing for all \( n \) if \( z_{n+1} \leq z_n \), resulting in the condition
\[ A B z_n^2 - (1 - \mu B) z_n + \mu A \leq 0. \tag{42} \]
Since \( A \) and \( B \) are positive for all positive values of \( \Delta t_{on}^{min} \) and \( \Delta t_{off}^{max} \), the inequality in (42) can only be satisfied for various values of \( z_n \) if \( 1 - \mu B \geq 0 \), resulting in the condition
\[ \frac{1}{\lambda_s} \ln \frac{1}{\mu^2} \leq \Delta t_{on}^{min}. \tag{43} \]
Note that since \( \mu \geq 1 \), the left hand side of (43) is always greater than or equal to zero.

Since the left hand side of (42) is a convex parabola, the condition
\[ d \leq z_n < \bar{d}, \]
must also be satisfied in addition to the condition in (43), to satisfy the inequality in (42), where \( d \) and \( \bar{d} \) are solutions to the quadratic equation
\[ A B z_n^2 - (1 - \mu B) z_n + \mu A = 0 \tag{44} \]
and are given by
\[ d = \frac{1 - \mu B - \sqrt{(1 - \mu B)^2 - 4 \mu A^2 B}}{2AB}, \tag{45} \]
\[ \bar{d} = \frac{1 - \mu B + \sqrt{(1 - \mu B)^2 - 4 \mu A^2 B}}{2AB}. \tag{46} \]
The roots, \( d \) and \( \bar{d} \), are real and distinct if
\[ \frac{(1 - \mu B)^2 - 4 \mu A^2 B}{2 \mu \exp \left( -\lambda_s \Delta t_{on}^{min} \right)} > \tan \left( \beta \Delta t_{off}^{max} \right). \tag{47} \]
If the conditions in (40), (43) and (47) are satisfied and if the initial conditions satisfy (41), the sequence is monotonically decreasing. Since the function \( \phi : \mathbb{R} \rightarrow \mathbb{R}, \phi(z) \triangleq \mu \frac{A + B z_n}{1 - AB z_n} \), where \( z \) is a dummy variable representing the argument of \( \phi \), is an increasing function on the interval \( (-\infty, \frac{1}{\lambda_s}) \), and \( d \) and \( \bar{d} \) are both upper bounded by \( \frac{1}{\lambda_s} \).

Consequently, if the initial condition \( z_0 \) is in the interval \([\bar{d}, d]\), the sequence is lower bounded and monotonically decreasing, and therefore converges. The limit of the sequence is given by
\[ L \triangleq \lim_{n \to \infty} z_n. \tag{49} \]

Using the definition of the sequence in (49)
\[ \lim_{n \to \infty} z_{n+1} = \mu \frac{A + B L}{1 - AB L} \lim_{n \to \infty} z_n \]
\[ \Rightarrow L = \mu \frac{A + B L}{1 - AB L} \]
which results in an equation similar to (44) with solutions
\[ L = \frac{1 - \mu B \pm \sqrt{(1 - \mu B)^2 - 4 \mu A^2 B}}{2AB}. \tag{48} \]
However, since \( z_n \) monotonically decreases in the interval \([d, \bar{d}]\), if \( z_0 \in [d, \bar{d}) \), the sequence \( \{z_n\} \) converges to the lesser solution, i.e. \( d \), and not \( \bar{d} \).

A similar procedure can be used to show that \( z_n \) monotonically increases outside the interval \([d, \bar{d}]\). Again, since \( \phi \) is an increasing function,
\[
z_n \in [0, \bar{d}] \implies \phi(z_n) \leq \phi(d) \implies z_{n+1} \leq d.
\]

Therefore, if \( z_0 \in [0, \bar{d}] \), \( \{z_n\} \) monotonically increases and is bounded by \( d \). Applying the limit as above, it can be shown that \( \{z_n\} \) then converges to \( d \). Thus, if \( z_0 \in [0, \bar{d}] \) (and therefore automatically satisfies (41)), the elements of the sequence \( \{z_n\} \) continue to satisfy (41) and the sequence converges to \( d \). Consequently, the sequence \( \{W_n^{on}\} \) converges via the squeeze theorem to the ball
\[
0 \leq \lim_{n \to \infty} W_n^{on} \leq d.
\]

From (34)-(36) it is clear that \( W(t) \leq W_n^{on} \), \( \forall t \in [n \Delta t, (n+1) \Delta t) \), \( \forall n \) if conditions (40), (43) and (47) are satisfied.

Therefore,
\[
\limsup_{t \to \infty} W(t) \leq d.
\]

Using (26), (29) and the definition of \( W \), the estimation error converges to the ball
\[
\limsup_{t \to \infty} \|e(t)\|^2 \leq \frac{d}{c_1}.
\]

\[ \text{Remark 4.} \] The stability conditions in (23) and (24) are functions of the error decay rate, \( \lambda_s \). The implication of increasing the decay rate of the Lyapunov-like function, i.e. increasing the observer gains, is that the target has to remain in the FOV for less time. The size of the ultimate bound can also be decreased, either by increasing the dwell time in the observable region or increasing \( \lambda_s \). However, this is only effective up to a limit. Re-examining (45) and using L’Hôpital’s rule, in the limit as \( B \to 0 \) (i.e. \( \lambda_s \to \infty \) or \( \Delta t_{min}^{on} \to 0 \) \( \Delta t_{max}^{on} \to 0 \) \( \bar{d} \to \mu_A \), which is equivalent to the growth in \( W \) during the period in which the target is out of the camera FOV in the case when the estimation error is initially zero. Similarly, from (46), \( \bar{d} \to \infty \) as \( B \to \infty \), allowing an arbitrarily large initial error.

\[ \text{VI. Conclusion} \]

Sufficient observation conditions are developed to guarantee convergence to an ultimate bound of the position estimation error from a vision based observer with intermittent sensing. Evaluation of the conditions only require conservative bounds on the target and camera velocities, bounds on the target range and bearing, and bounds on the dwell times in which the target is in and out of the FOV. Solutions to the dynamics, \( \bar{W} \), of a Lyapunov-like function during both the period when the target is in view and the period when it is out of the FOV were utilized to bound the decay and growth of the estimation errors and therefore relate dwell times to a decay in \( W \). Future goals include the use of less conservative stability conditions to determine less conservative dwell time conditions.

\[ \text{REFERENCES} \]


