Global Asymptotics of the Meixner Polynomials

Xiang-Sheng Wang

York University, Toronto, Canada

(This is a joint work with R. Wong.)
Outline

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The Meixner polynomials

For $\beta > 0$ and $0 < c < 1$, the Meixner polynomials are explicitly given by

$$M_n(z; \beta, c) = \binom{-n}{\beta} \binom{-n}{1 - \frac{1}{c}} = \sum_{k=0}^{n} \frac{(-n)_k(-z)_k}{(\beta)_k k!} \left(1 - \frac{1}{c}\right)^k,$$

where $(a)_0 := 1$ and $(a)_k := a(a + 1) \cdots (a + k - 1)$ for $k \in \mathbb{N}^*$. The Meixner polynomials satisfy the discrete orthogonality condition

$$\sum_{k=0}^{\infty} c^k (\beta)_k \frac{1}{k!} M_m(k; \beta, c) M_n(k; \beta, c) = \frac{c^{-n} n!}{(\beta)_n (1-c)^{\beta}} \delta_{mn}.$$

We are interested in finding large-$n$ behavior of $M_n(z; \beta, c)$. 
Figure 1: The zeros of $M_n(z; \beta, c)$ with $n = 100$, $\beta = 1.5$ and $c = 0.5$. 
Figure 2: The first several zeros of $M_n(z; \beta, c)$ with $n = 100$, $\beta = 1.5$ and $c = 0.5$. 
What have been done?

• Using probabilistic arguments, Maejima and Van Assche have given an asymptotic formula for $M_n(n\alpha; \beta, c)$ when $\alpha < 0$ and $\beta$ is a positive integer. Their result is given in terms of elementary functions.

• Jin and Wong have applied the steepest-descent method for integrals to derive two infinite asymptotic expansions for $M_n(n\alpha; \beta, c)$. One holds uniformly for $0 < \varepsilon \leq \alpha \leq 1 + \varepsilon$, and the other holds uniformly for $1 - \varepsilon \leq \alpha \leq M < \infty$; both expansions involve the parabolic cylinder function and its derivative.

• Recently, Temme uses logarithm transformations to derive two uniform asymptotic formulas for $M_n(n\alpha; \beta, c)$ with $\alpha$ in $[0, \delta]$ and $[-\delta, 0]$ respectively. The gamma function is used to describe asymptotic behavior of the Meixner polynomials near the origin.
What are we going to do?

• In view of Gauss's contiguous relations for hypergeometric functions, we may restrict our study to the case $1 \leq \beta < 2$.

• Fixing any $0 < c < 1$ and $1 \leq \beta < 2$, we intend to investigate the large-$n$ behavior of $M_n(nz - \beta/2; \beta, c)$ for $z$ in the whole complex plane.

• Our results are "global" in the sense that only two asymptotic formulas are needed to cover the whole complex plane.

• Our approach is based on the Deift-Zhou nonlinear steepest-descent method for oscillatory Riemann-Hilbert problems.
The Deift-Zhou nonlinear steepest-descent method

- Deift et al. (CPAM 1999): orthogonal polynomials with respect to exponential weights.
- Baik et al. (Annals of Mathematics Studies 2007): orthogonal polynomials with respect to a general class of discrete weights.
- many other developments and applications · · ·
Local asymptotics and global asymptotics

- Local asymptotics ($a$ and $b$ are turning points, $\delta$ is a small positive number)
  1. negative real line: $(-\infty, -\delta]$ (Maejima and Van Assche)
  2. near the origin: $[-\delta, 0]$ and $[0, \delta]$ (Temme)
  3. saturated interval: $[\delta, a - \delta]$
  4. near left turning point: $[a - \delta, a + \delta]$
  5. oscillatory interval: $[a + \delta, b - \delta]$
  6. near right turning point: $[b - \delta, b + \delta]$
  7. exponential interval: $[b + \delta, \infty)$

- Global asymptotics (Jin and Wong): $[\delta, 1 + \delta]$ and $[1 - \delta, M]$.

- Global asymptotics (our improved results): $[0, 1]$ and $(-\infty, 0] \cup [1, \infty)$.
Global asymptotics via Riemann-Hilbert problem

- Jacobi polynomials: Wong and Zhang (Tran. AMS 2006)
- Discrete Chebyshev polynomials: Lin and Wong (in preparation)
- many other references · · ·
Riemann-Hilbert problem

• 1D → 2D (Fokas, Its and Kitaev): relate the Meixner polynomials with a $2 \times 2$ matrix-valued function which is the unique solution to an interpolation problem.

• Discrete → Continuous (Baik et al.): change the discrete interpolation problem to a continuous Riemann-Hilbert problem (RHP) whose unique solution can be expressed in terms of the solution to the basic interpolation problem.
Step 1: 1D → 2D

Define

\[
P(z) := \begin{pmatrix}
\pi_n(z) & \sum_{k=0}^{\infty} \frac{\pi_n(k)w(k)}{z-k} \\
\gamma_{n-1}^2 \pi_{n-1}(z) & \sum_{k=0}^{\infty} \frac{\gamma_{n-1}^2 \pi_{n-1}(k)w(k)}{z-k}
\end{pmatrix},
\]

where \(\pi_n(z)\) is the monic Meixner polynomials. For any \(k \in \mathbb{N}\), we have

\[
\text{Res}_{z=k} P_{12}(z) = \pi_n(k)w(k) = P_{11}(k)w(k),
\]

\[
\text{Res}_{z=k} P_{22}(z) = \gamma_{n-1}^2 \pi_{n-1}(k)w(k) = P_{21}(k)w(k).
\]

Thus,

\[
\text{Res}_{z=k} P(z) = \lim_{z \to k} P(z) \begin{pmatrix} 0 & w(z) \\ 0 & 0 \end{pmatrix}.
\]
Step 2: Discrete → Continuous (example)

Suppose

\[ \text{Res } Q(z) = \lim_{z \to 0} Q(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \]

Define

\[ R(z) := \begin{cases} Q(z) \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}, & \text{for any } z \in D(0, 1) \setminus \{0\}; \\ Q(z), & \text{for any } z \in \mathbb{C} \setminus D(0, 1). \end{cases} \]

We then have \( R(z) \) analytic at \( z = 0 \) and

\[ R_+(z) = R_-(z) \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}, \text{ for any } z \in \partial D(0, 1). \]
Turning points and equilibrium measure

- Mhaskar-Rakhmanov-Saff (MRS) numbers (turning points)
  \[ a = \frac{1 - \sqrt{c}}{1 + \sqrt{c}}, \quad b = \frac{1 + \sqrt{c}}{1 - \sqrt{c}}. \]

- Let \( x_i \) be the \( i \)th zeros of \( M_n(nz - \beta/2; \beta, c) \), we have the following asymptotic zero distribution
  \[
  \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}(x) \rightarrow \rho(x) = \begin{cases} 
  1 & x \in [0, a]; \\
  \frac{1}{\pi} \arccos \frac{x(b+a)-2}{x(b-a)} & x \in [a, b]; \\
  0 & \text{otherwise}. 
  \end{cases}
  \]
Figure 3: The zero distribution of \( M_n(nz - \beta/2; \beta, c) \) with \( n = 100, \beta = 1.5 \) and \( c = 0.5 \). In this case the turning points are \( a \approx 0.17157 \) and \( b \approx 5.82843 \).
Local asymptotics: some local Riemann-Hilbert problems

- Local RHP near the turning points $a$ and $b$: Airy parametrix (Deift et al., 1999).
- Local RHP near the interval $(a, b)$: elementary function.
- Local RHP near the origin: gamma function.
Local RHP near the turning points $a$ and $b$

\[
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \\
\begin{pmatrix}
-\omega \text{Ai}(\omega z) & \omega^2 \text{Ai}(\omega^2 z) \\
-i\omega^2 \text{Ai}'(\omega z) & i\omega \text{Ai}'(\omega^2 z)
\end{pmatrix} \quad \text{to} \quad \begin{pmatrix}
\text{Ai}(z) & \omega^2 \text{Ai}(\omega^2 z) \\
i \text{Ai}'(z) & i\omega \text{Ai}'(\omega^2 z)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \quad \text{to} \quad \begin{pmatrix}
-\omega^2 \text{Ai}(\omega^2 z) & -\omega \text{Ai}(\omega z) \\
i\omega \text{Ai}'(\omega^2 z) & -i\omega^2 \text{Ai}'(\omega z)
\end{pmatrix} \quad \text{to} \quad \begin{pmatrix}
\text{Ai}(z) & -\omega \text{Ai}(\omega z) \\
i \text{Ai}'(z) & -i\omega^2 \text{Ai}'(\omega z)
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}
\]

Figure 4: The Airy parametrix and its jump conditions.
Local RHP near the interval \((a, b)\)

\[ J_N(x) = \begin{cases} 
\begin{pmatrix} 
0 & -(1 - x)^{\beta - 1} \\
(1 - x)^{1 - \beta} & 0 
\end{pmatrix}, & \text{for any } x \in (a, 1); \\
\begin{pmatrix} 
0 & -(x - 1)^{\beta - 1} \\
(x - 1)^{1 - \beta} & 0 
\end{pmatrix}, & \text{for any } x \in (1, b). 
\end{cases} \]

\[ N(z) = \begin{pmatrix} 
\frac{(z - 1)^{1 - \beta}}{2} (\sqrt{z - a} + \sqrt{z - b})^{\beta} \\
\frac{(z - a)^{1/4}(z - b)^{1/4}}{(z - a)^{1/4}(z - b)^{1/4}} 
\end{pmatrix} \\
\begin{pmatrix} 
\frac{i(z - 1)^{1 - \beta}}{2} (\sqrt{z - a} + \sqrt{z - b})^{2 - \beta} \\
\frac{(z - a)^{1/4}(z - b)^{1/4}}{(z - a)^{1/4}(z - b)^{1/4}} 
\end{pmatrix}. \]
Local RHP near the origin

(D1) $D(z)$ is analytic in $\mathbb{C} \setminus (-i\infty, i\infty)$;

(D2) $D_+(z) = D_-(z)[1 - e^{\pm 2i\pi(nz - \beta/2)}]$, for any $z \in (-i\infty, i\infty)$;

(D3) for $z \in \mathbb{C} \setminus (-i\infty, i\infty)$, $D(z) = 1 + O(|z|^{-1})$ as $z \to \infty$.

The solution is given by

$$D(z) = \begin{cases} 
\frac{e^{nz}\Gamma(nz - \beta/2 + 1)}{\sqrt{2\pi}(nz)^{nz+(1-\beta)/2}} & \text{Re } z > 0; \\
\frac{\sqrt{2\pi}(-nz)^{-nz+(\beta-1)/2}}{e^{-nz}\Gamma(-nz + \beta/2)} & \text{Re } z < 0.
\end{cases}$$
Local asymptotics: some notations

- The monic Meixner polynomials: \( \pi_n(z) := (\beta)_n(1 - \frac{1}{c})^{-n} M_n(z; \beta, c) \).
- Potential function \( v(z) := -z \log c \) and Lagrange constant \( l := 2 \log \frac{b-a}{4} - 2 \).
- For \( z \in \mathbb{C} \setminus (-\infty, b] \),
  \[ \phi(z) := z \log \frac{\sqrt{bz - 1} + \sqrt{az - 1}}{\sqrt{bz - 1} - \sqrt{az - 1}} - \log \frac{\sqrt{z-a} + \sqrt{z-b}}{\sqrt{z-a} - \sqrt{z-b}}. \]
- For \( z \in \mathbb{C} \setminus (-\infty, 0] \cup [a, \infty) \),
  \[ \tilde{\phi}(z) := z \log \frac{\sqrt{1-az} + \sqrt{1-bz}}{\sqrt{1-az} - \sqrt{1-bz}} - \log \frac{\sqrt{b-z} + \sqrt{a-z}}{\sqrt{b-z} - \sqrt{a-z}}. \]
Local asymptotics: regions of approximation

Figure 5: Local asymptotic regions.
Local asymptotics: saturated region

For \( z \in \Omega_\pm^1 \), we have

\[
\pi_n(nz - \beta/2) \sim -2 \sin(n\pi z - \beta \pi/2)(-n)^n e^{n\nu(z)/2 + nl/2 - n\tilde{\phi}(z)} \\
\times z^{(1-\beta)/2} \left( \frac{\sqrt{b-z} + \sqrt{a-z}}{2} \right)^\beta \\
\times \frac{1}{(a - z)^{1/4}(b - z)^{1/4}}.
\]
Local asymptotics: oscillatory region

Let $z = \frac{b-a}{2} \cos u + \frac{b+a}{2} = -\frac{b-a}{2} \cos \tilde{u} + \frac{b+a}{2}$. We have

$$
\pi_n(nz - \beta/2) \sim 2 \cos[n\pi z - \beta \pi/2 + \pi/4 + \beta \tilde{u}/2 \mp in\phi(z)](-n)^n e^{nv(z)/2 + nl/2} \\
\times z^{(1-\beta)/2} (\frac{b-a}{4})^{\beta/2} \\
\times (z-a)^{1/4} (b-z)^{1/4}
$$

for $z \in \Omega^2_{\pm}$, and

$$
\pi_n(nz - \beta/2) \sim 2 \cos[\pi/4 - \beta u/2 \mp in\phi(z)]n^n e^{nv(z)/2 + nl/2} \\
\times z^{(1-\beta)/2} (\frac{b-a}{4})^{\beta/2} \\
\times (z-a)^{1/4} (b-z)^{1/4}
$$

for $z \in \Omega^3_{\pm}$.
Local asymptotics: exponential region

For \( z \in \Omega^4 \cup \Omega^\infty \), we have

\[
\pi_n(nz - \beta/2) \sim n^n e^{nv(z)/2+nl/2-n\phi(z)} \\
\times z^{(1-\beta)/2} \left(\frac{\sqrt{z-a}+\sqrt{z-b}}{2}\right)^\beta \\
\times \left(\frac{1}{(z-a)^{1/4}(z-b)^{1/4}}\right).
\]
Local asymptotics: near the origin

For \( z \in \Omega^0_l \), we have

\[
\pi_n(nz - \beta/2) \sim D(z)n^ne^{nv(z)/2+nl/2-n\phi(z)}
\]
\[
\times \left( -z \right)^{(1-\beta)/2} \left( \frac{\sqrt{b-z}+\sqrt{a-z}}{2} \right)^\beta
\]
\[
\times \frac{\left( \sqrt{b-z}+\sqrt{a-z} \right)^\beta}{(b - z)^{1/4}(a - z)^{1/4}}.
\]

For \( z \in \Omega^0_{r, \pm} \), we have

\[
\pi_n(nz - \beta/2) \sim -2 \sin(n\pi z - \beta\pi/2)D(z)(-n)^n e^{nv(z)/2+nl/2-n\tilde{\phi}(z)}
\]
\[
\times \left( -z \right)^{(1-\beta)/2} \left( \frac{\sqrt{b-z}+\sqrt{a-z}}{2} \right)^\beta
\]
\[
\times \frac{\left( \sqrt{b-z}+\sqrt{a-z} \right)^\beta}{(a - z)^{1/4}(b - z)^{1/4}}.
\]
Local asymptotics: near left turning point

Let \( \tilde{F}(z) := \left[ -\frac{3}{2} n \phi(z) \right]^{2/3} \), we have for \( z \in \Omega^a \),

\[
\pi_n(nz - \beta/2) \sim (-n)^n \sqrt{\pi} e^{nv(z)/2+nl/2} \times \left\{ \left[ \cos(n\pi z - \beta \pi/2) \operatorname{Ai}(\tilde{F}(z)) - \sin(n\pi z - \beta \pi/2) \operatorname{Bi}(\tilde{F}(z)) \right] \right.
\]
\[
\times \left\{ \left[ \cos(n\pi z - \beta \pi/2) \operatorname{Ai}'(\tilde{F}(z)) - \sin(n\pi z - \beta \pi/2) \operatorname{Bi}'(\tilde{F}(z)) \right] \right.
\]
\[
\times \frac{\left( \sqrt{b-z} + \sqrt{a-z} \right)^\beta + \left( \sqrt{b-z} - \sqrt{a-z} \right)^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\tilde{F}(z)^{-1/4}}
\]
\[
+ \left[ \cos(n\pi z - \beta \pi/2) \operatorname{Ai}'(\tilde{F}(z)) - \sin(n\pi z - \beta \pi/2) \operatorname{Bi}'(\tilde{F}(z)) \right] \right.
\]
\[
\times \left\{ \left[ \cos(n\pi z - \beta \pi/2) \operatorname{Ai}'(\tilde{F}(z)) - \sin(n\pi z - \beta \pi/2) \operatorname{Bi}'(\tilde{F}(z)) \right] \right.
\]
\[
\times \frac{\left( \sqrt{b-z} + \sqrt{a-z} \right)^\beta - \left( \sqrt{b-z} - \sqrt{a-z} \right)^\beta}{z^{(\beta-1)/2}(b-z)^{1/4}(a-z)^{1/4}\tilde{F}(z)^{1/4}} \right\}.
\]
Local asymptotics: near right turning point

Let $F(z) := \left[\frac{3}{2}n\phi(z)\right]^{2/3}$, we have for $z \in \Omega^b$,

$$
\pi_n(nz - \beta/2) \sim n^n \sqrt{\pi} e^{nv(z)/2 + nl/2} \left\{ \frac{(\sqrt{z-a} + \sqrt{z-b})^{\beta}}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4}} \right.
$$

$$
\left. \times \left[ \frac{(\sqrt{z-a} + \sqrt{z-b})^{\beta} + (\sqrt{z-a} - \sqrt{z-b})^{\beta}}{2} \right.ight.
$$

$$
\left. \left. \left. \frac{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4} Ai(F(z))}{(\sqrt{z-a} + \sqrt{z-b})^{\beta} - (\sqrt{z-a} - \sqrt{z-b})^{\beta}} \right. \right.
$$

$$
\left. \left. \left. - \frac{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4} Ai'(F(z))}{2} \right. \right. \right. \right.
$$

$$
\right\}.
$$
Local asymptotics: regions of approximation

Figure 6: Local asymptotic regions.
Global asymptotics: regions of approximation

Figure 7: Global asymptotic regions.
Global asymptotics: outside the rectangle

For $\Re z \notin [0, 1]$ or $\Im z \notin [-\delta, \delta]$,

$$
\pi_n(nz - \beta/2) \sim n^n \sqrt{\pi} D(z) e^{nv(z)/2 + nl/2} \times \left\{ \frac{(\sqrt{z-a} + \sqrt{z-b})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{-1/4}} \text{Ai}(F(z)) \right.
$$

$$
- \frac{(\sqrt{z-a} + \sqrt{z-b})^\beta}{z^{(\beta-1)/2}(z-a)^{1/4}(z-b)^{1/4}F(z)^{1/4}} \text{Ai}'(F(z)) \left. \right\}.
$$
Global asymptotics: inside the rectangle

For $\text{Re} \ z \in (0, 1)$ and $\text{Im} \ z \in (-\delta, \delta)$,

$$
\pi_n(nz - \beta/2) \sim (-n)^n \sqrt{\pi} D(z) e^{nv(z)/2 + nl/2} \times \left\{ \frac{\cos(n\pi z - \beta\pi/2) \text{Ai}(\tilde{F}(z)) - \sin(n\pi z - \beta\pi/2) \text{Bi}(\tilde{F}(z))}{z^{(\beta-1)/2}(b - z)^{1/4}(a - z)^{1/4} \tilde{F}(z)^{-1/4}} \right. \\
\left. + \frac{\cos(n\pi z - \beta\pi/2) \text{Ai}'(\tilde{F}(z)) - \sin(n\pi z - \beta\pi/2) \text{Bi}'(\tilde{F}(z))}{z^{(\beta-1)/2}(b - z)^{1/4}(a - z)^{1/4} \tilde{F}(z)^{1/4}} \right\}.
$$
Numerical computation

Figure 8: The true figure and approximate figure of $\pi_n(nz - \beta/2)$ for $n = 100$, $\beta = 1.5$ and $c = 0.5$. Here the turning points are $a \approx 0.17157$ and $b \approx 5.82843$. 
Table 1: The true values and approximate values of $\pi_n (nz - \beta/2)$ for $n = 100$, $\beta = 1.5$ and $c = 0.5$. Here the turning points are $a \approx 0.17157$ and $b \approx 5.82843$. 

<table>
<thead>
<tr>
<th>$z$</th>
<th>True value</th>
<th>Appr. value (local)</th>
<th>Appr. value (global)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z = -1$</td>
<td>$1.99529 \times 10^{233}$</td>
<td>$1.99473 \times 10^{233}$</td>
<td>$1.99501 \times 10^{233}$</td>
</tr>
<tr>
<td>$z = -0.001$</td>
<td>$8.36624 \times 10^{187}$</td>
<td>$8.35137 \times 10^{187}$</td>
<td>$8.35263 \times 10^{187}$</td>
</tr>
<tr>
<td>$z = 0.001$</td>
<td>$3.07930 \times 10^{187}$</td>
<td>$3.07272 \times 10^{187}$</td>
<td>$3.07602 \times 10^{187}$</td>
</tr>
<tr>
<td>$z = 0.05$</td>
<td>$-2.51701 \times 10^{180}$</td>
<td>$-2.51507 \times 10^{180}$</td>
<td>$-2.51523 \times 10^{180}$</td>
</tr>
<tr>
<td>$z = 0.171$</td>
<td>$-9.12697 \times 10^{174}$</td>
<td>$-9.12530 \times 10^{174}$</td>
<td>$-9.11951 \times 10^{174}$</td>
</tr>
<tr>
<td>$z = 0.172$</td>
<td>$-1.22035 \times 10^{175}$</td>
<td>$-1.22003 \times 10^{175}$</td>
<td>$-1.21926 \times 10^{175}$</td>
</tr>
<tr>
<td>$z = 2$</td>
<td>$-4.71541 \times 10^{201}$</td>
<td>$-4.70772 \times 10^{201}$</td>
<td>$-4.71179 \times 10^{201}$</td>
</tr>
<tr>
<td>$z = 5.828$</td>
<td>$2.78146 \times 10^{259}$</td>
<td>$2.78231 \times 10^{259}$</td>
<td>$2.78225 \times 10^{259}$</td>
</tr>
<tr>
<td>$z = 5.829$</td>
<td>$2.86933 \times 10^{259}$</td>
<td>$2.87018 \times 10^{259}$</td>
<td>$2.87046 \times 10^{259}$</td>
</tr>
<tr>
<td>$z = 100$</td>
<td>$2.16586 \times 10^{399}$</td>
<td>$2.16586 \times 10^{399}$</td>
<td>$2.16586 \times 10^{399}$</td>
</tr>
</tbody>
</table>
Future work

• Global asymptotics for a general class of discrete weight.

• The critical case when the turning point and the end point coalesce with each other.
Some pioneer works

• Local asymptotics for a general class of discrete weight with finite nodes (Baik et.al., 2007)

• Global asymptotics of the Krawtchouck polynomials (Dai-Wong, 2007)

• Global asymptotics for a general class of discrete weight with infinite nodes (Ou-Wong, 2010)

• Global asymptotics via recurrence relations (Wang-Wong, 2002; Li-Wong)

• Global asymptotics of discrete Chebyshev polynomials (Pan-Wong; Lin-Wong)

• …
Thank you!