HÖLDER CONTINUITY OF SOBOLEV FUNCTIONS AND RIESZ POTENTIALS

PATRICK J. RABIER

Abstract. Let v be a distribution on $\mathbb{R}^N$ with gradient in L^p for some $1 \leq p < \infty$ and let $\gamma \in (0,1)$ if $p \leq N, \gamma \in [1-N/p, 1)$ if $p > N$. The main result of this paper states that if $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$, then $v \in C^{0,\gamma}(\mathbb{R}^N)$. The trivial case $p > N$ and $\gamma = 1 - N/p$ is Morrey's theorem.

An investigation of the condition $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ produces other special cases that do not restrict p. Some rely on "uniformly local" integrability, more general than its global counterpart. Another can be phrased in terms of the Fourier transform of functions dominating $|\nabla v|^p$.

1. Introduction

Throughout the paper, $1 \leq p < \infty$ unless some limitation is explicitly stated and it is understood that $\mathbb{R}^N$ is the domain of all function spaces with unspecified domain (e.g., $L^p = L^p(\mathbb{R}^N)$).

The space of distributions v on $\mathbb{R}^N$ such that $\nabla v \in (L^p)^N$ is denoted by $D^{1,p}$ ("homogeneous" Sobolev space, also called Beppo-Levi space after Deny and Lions [5]). The mapping $v \to ||\nabla v||_p$ is a seminorm on $D^{1,p}$ and a norm on the quotient space $D^{1,p}/\mathbb{R}$. It is common knowledge that the distributions $v \in D^{1,p}$ are functions in L^p_{loc} and, hence, in the Sobolev space $W^{1,p}_{loc}$ ([12, p. 23]).

A famous theorem of Morrey ([17, p. 83]) asserts that $D^{1,p} \subset C^{0,1-N/p}$ when $p > N$. Although this is usually stated for $W^{1,p}$, the proof for $D^{1,p}$ is the same. It is equally notorious that the functions of $D^{1,p}$ need not even be continuous if $N > 1$ and $p \leq N$. In the remaining case $N = p = 1$, an absolutely continuous but nowhere locally Hölder continuous function was found by Hardy [8] one hundred years ago. Thus, $p > N$ is a necessary and sufficient condition for the Hölder continuity, even locally, of all the functions in $D^{1,p}$.

More is known for $W^{k,p}$ with $k \geq 1$. We limit our comments to $k = 1$. As an easy by-product of Morrey's theorem, $v \in W^{1,p}$ with $p > N$ is in $C^{0,\gamma}$ for every $\gamma \in (0, 1 - N/p]$ ([1, p. 85]). This is false if $v \in D^{1,p}$. Malý [11] observed that if $p > 1$ and $\gamma \in (0, 1 - 1/p)$, every $v \in W^{1,p}$ coincides with a function $v_\gamma \in C^{0,\gamma}$ on the complement of an open subset U_γ of $\mathbb{R}^N$ with arbitrarily small capacity, a Lusin-type property customarily referred to as Hölder quasicontinuity. Swanson [21] showed that Malý's result remains true for every $\gamma \in (0,1)$ and he recently removed the restriction $p \neq 1$ ([22]). These works contain various refinements and extensions. According to Calderón and Zygmund [4], $v \in W^{1,p}$ is C^1 on the

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complement of an open set with arbitrarily small measure, but this does not say anything about global Hölder continuity.

This paper is devoted to the Hölder (quasi)continuity of the functions of $D^{1,p}$. The fact that the space $D^{k,p}$ (partial derivatives of order k in L^p) contains nonconstant polynomials when $k \geq 2$ justifies confining attention to $k = 1$. The general results when $v \in W^{1,p}_{loc}$ (or $W^{k,p}_{loc}$) in Bojarski *et al.* [3, Theorem 1.4] are, of necessity, limited to local Hölder continuity and the specific case of $D^{1,p}$ has not been addressed in the literature[1]. It could possibly be investigated along the line of what has been done for $W^{1,p}$, but this would require finding an adequate substitute for the characterization $W^{1,p} = (I - \Delta)^{-1/2}(L^p)$, crucial to both [11] and [21]. Known integral representations ([16, Chapter 6]) could perhaps provide a starting point, but this remains highly speculative.

As we shall see, a completely different approach not only resolves the issue but also delivers more than what has so far been obtained for $W^{1,p}$. Everything hinges upon exhibiting a close connection between Hölder continuity and Riesz potentials for functions of $D^{1,p}$.

The first part of the main Theorem 4.1 states that if $p \leq N$ and $\gamma \in (0, 1)$, or if $p > N$ and $\gamma \in [1 - N/p, 1)$, every $v \in D^{1,p}$ is Hölder continuous with exponent γ on the complement of an open subset U_γ with arbitrarily small measure. When $p > N$, the restriction $\gamma \geq 1 - N/p$ is necessary (unlike when $v \in W^{1,p}$) as is readily seen on the example $v(x) = (1 + |x|)^{1-\varepsilon-N/p}$ with arbitrarily small $\varepsilon > 0$.

While the above does not disclose the relevance of the Riesz potentials, their role is explicit in the second part of Theorem 4.1, to the effect that if $v \in D^{1,p}$ and $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$, the "exceptional" open set U_γ is empty, so that $v \in C^{0,\gamma}$. This is a generalization of Morrey's theorem, which is recovered when $p > N$ and $\gamma = 1 - N/p$, for then $|x|^{(1-\gamma)p-N} * |\nabla v|^p = 1 * |\nabla v|^p$ is the constant function $||\nabla v||_p^p$. This generalization is equally new when $v \in W^{1,p}$.

If $p \leq N$ and $\gamma \in (0, 1)$ or if $p > N$ and $\gamma \in (1 - N/p, 1)$, that is, if $(1 - \gamma)p \in (0, N)$, the Riesz potential of order $(1 - \gamma)p$ maps L^1 into some Lorentz space $L^{q,\infty}$ (weak L^q) with $q < \infty$, but not into L^∞, so that the condition $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ ensuring that $v \in C^{0,\gamma}$ introduces a restriction on ∇v.

This raises the question of finding more easily verifiable assumptions implying $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$. A first option involves the so-called uniformly local Lebesgue spaces L^q_{uloc}, which lie somewhere between the classical spaces and their local variants. These spaces are well suited to formulate additional integrability conditions on ∇v (Theorem 5.1) or on the higher order derivatives of v (Corollary 5.3) to obtain Hölder continuity properties beyond Morrey's theorem, especially when $p \leq N$.

A second and completely different way to ensure that $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ is to impose conditions on the Fourier transform of tempered distributions $f \in L^1_{loc}$ that dominate $|\nabla v|^p$ (Theorem 5.6). An example is given that can be handled with Theorem 5.1 or Theorem 5.6.

All the results have rather straightforward extensions when, more generally, $\partial_j v \in L^{p_j}$ with $1 \leq p_j < \infty, 1 \leq j \leq N$, but $C^{0,\gamma}$ must then be replaced with a space depending upon two exponents. See Section 6 for a statement of the main theorem in this setting and for further comments.

[1]The sharper form of Lusin's theorem for $D^{1,2}$ in Deny and Lions [5, Theorem 3.1, p. 354] refers to ordinary continuity.

The two key ingredients to the proof of Theorem 4.1 are (i) an elaboration on the Lebesgue differentiation theorem, to the effect that Hölder continuity follows from uniform estimates for the averages $|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(z)|dx$ (Theorem 2.1) and (ii) the remark that for every $0<s<1$ with $s\leq N/p$, these averages are controlled by $C(s,p,N)\rho^{1-s}[(|x|^{sp-N}*|\nabla v|^p)(z)]^{1/p}$ (Theorem 3.3).

Everywhere, χ_E stands for the characteristic function of the subset $E\subset\mathbb{R}^N$. If E is (Lebesgue) measurable, $|E|$ is its (Lebesgue) measure and $||\cdot||_{q,E}$, or simply $||\cdot||_q$ when $E=\mathbb{R}^N$, is the norm of $L^q(E)$. As is customary, q' always refers to the Hölder conjugate of q and $q^*:=Nq/(N-q)$ if $q<N, q^*:=\infty$ otherwise. Also, $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$ and B_r and $\widetilde{B}_r$ denote the open ball with center 0 and radius $r>0$ in $\mathbb{R}^N$ and the exterior of $\overline{B}_r$, respectively. We use $B_r(x)$ and $\widetilde{B}_r(x)$ when the center is a point $x\neq 0$. More specialized notation will be introduced in due time.

2. Lebesgue differentiation theorem and Hölder continuity

Recall that $z\in\mathbb{R}^N$ is a Lebesgue point of $v\in L^1_{loc}$ if there is $v_z\in\mathbb{R}$ such that $\lim_{\rho\to 0^+}|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v_z|dx=0$. The set of such points will be denoted by L_v. Neither L_v nor v_z for $z\in L_v$ is affected by modifying v on a null-set. The Lebesgue differentiation theorem asserts that almost every $z\in\mathbb{R}^N$ is in L_v and that $v_z=v(z)$ for almost every $z\in L_v$. Accordingly, after changing v on a null-set, we may and shall always assume that $v_z=v(z)$ for every $z\in L_v$.

It is trivial that if V is a neighborhood of some point x_0 and if $v\in C^{0,\gamma}(V)$, there are a neighborhood $W\subset V$ of x_0 and constants $C>0$ and $\rho_0>0$ such that $|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(z)|dx\leq C\rho^\gamma$ for every $z\in W$ and every $0<\rho<\rho_0$. The following theorem shows that a precise form of the converse is true.

Theorem 2.1. *Given $v\in L^1_{loc}$, suppose that for some subset $E\subset L_v$ there are $K>0$ and $\gamma>0$ such that*

$$|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(z)|dx\leq K\rho^\gamma, \tag{2.1}$$

for every $z\in E$ and every $\rho>0$. Then, $\overline{E}\subset L_v$ and (2.1) holds for every $z\in\overline{E}$ and every $\rho>0$. Furthermore,

$$|v(y)-v(z)|\leq K_\gamma|y-z|^\gamma, \tag{2.2}$$

for every $y,z\in\overline{E}$ where $K_\gamma:=K(1+2^{N+\gamma})$ and so $v\in C^{0,\gamma}(\overline{E})$.

Proof. As a first step, we prove (2.2) for every $y,z\in E$. Evidently, we may assume that E contains more than one point.

Let then $y,z\in E$ be such that $y\neq z$. For every $\rho>0$,

$$\begin{aligned}|v(y)-v(z)| &= |B_\rho|^{-1}\int_{B_\rho(z)}|v(y)-v(z)|dx\\ &\leq |B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(y)|dx+|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(z)|dx.\end{aligned}$$

Hence, by (2.1),

$$|v(y)-v(z)|\leq|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(y)|dx+K\rho^\gamma.$$

In particular, this holds with $\rho = |y-z|$. With this choice, $B_\rho(z) \subset B_{2\rho}(y)$, so that $|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v(y)|dx \leq 2^N|B_{2\rho}|^{-1}\int_{B_{2\rho}(y)}|v(x)-v(y)|dx$, where $|B_{2\rho}| = 2^N|B_\rho|$ was used. By (2.1), $|B_{2\rho}|^{-1}\int_{B_{2\rho}(y)}|v(x)-v(y)|dx \leq 2^\gamma K\rho^\gamma$. Altogether, $|v(y)-v(z)| \leq (1+2^{N+\gamma})\rho^\gamma = (1+2^{N+\gamma})|y-z|^\gamma$. This proves (2.2).

Next, we show that $\overline{E} \subset L_v$ and that (2.1) continues to hold when $z \in \overline{E}$. Let $(z_n) \subset E$ be a sequence tending to z. With no loss of generality, assume $|z_n - z| < 1$. It follows from the first step that $|v(z_n)| \leq |v(z_1)| + K_\gamma|z_n - z_1|^\gamma \leq |v(z_1)| + 2^\gamma K_\gamma$, so that $(v(z_n))$ is bounded. Let the subsequence $(v(z_{n_k}))$ converge to some number v_z and fix any $\rho > 0$. The sequence $((v - v(z_{n_k}))\chi_{B_\rho(z_{n_k})})$ tends to $(v-v_z)\chi_{B_\rho(z)}$ a.e. (everywhere except on $\partial B_\rho(z)$) and $|(v-v(z_{n_k}))\chi_{B_\rho(z_{n_k})}| \leq (|v| + |v(z_1)| + 2^\gamma K_\gamma)\chi_{B_{\rho+1}(z)} \in L^1$. By dominated convergence, $((v-v(z_{n_k}))\chi_{B_\rho(z_{n_k})})$ tends to $(v-v_z)\chi_{B_\rho(z)}$ in L^1 and so $|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v_z|dx = \lim_{k\to\infty}|B_\rho|^{-1}\int_{B_\rho(z_{n_k})}|v(x)-v(z_{n_k})|dx$. Therefore, it follows from (2.1) with $z = z_{n_k}$ that $|B_\rho|^{-1}\int_{B_\rho(z)}|v(x)-v_z|dx \leq K\rho^\gamma$. Since $\gamma > 0$, this shows that $z \in L_v$. Hence, $v_z = v(z)$ and (2.1) holds.

At this stage, we have found that $\overline{E} \subset L_v$ and that for every $\rho > 0$, (2.1) holds with E replaced with $\overline{E}$. By the first part of the proof, the same thing is true of (2.2). ∎

3. Lebesgue averages and Riesz potentials

There is nothing new to the fact that interesting connections exist between the Riesz potentials and the Lebesgue points of Sobolev functions. For instance, it is shown in [24, p. 115] that if $1 < p < N$ and $v \in W^{1,p}$, the Lebesgue averages of v about z converge whenever $(|x|^{1-N} * |\nabla v|)(z) < \infty$.

In this section, we prove a strong form of an estimate for the Lebesgue averages of v about z in terms of $(|x|^{\alpha-N} * |\nabla v|^p)(z)$ for suitable α depending upon $p \in [1,\infty)$ (Theorem 3.3).

For $\sigma \in \mathbb{R}$, let $L^p_{|x|^\sigma}$ denote the weighted Lebesgue space

$$L^p_{|x|^\sigma} := \{h \in L^1_{loc} : |x|^{\sigma/p}h \in L^p\} = \{h \in L^1_{loc} : |x|^\sigma|h|^p \in L^1\},$$

equipped with the natural norm $||h||_{1,|x|^\sigma} := ||\,|x|^{\sigma/p}h||_p$. Since $|x|^{sp-N}$ is a so-called A_p weight when $0 < s < N$ (A_p weights include $|x|^\delta$ with $-N < \delta < N(p-1)$), the following approximation lemma is a special case of Muckenhoupt and Wheeden [18, Lemma 8] ($p = 1$) and Miller [14, Lemma 2.4] ($1 < p < \infty$). See also Turesson [23, Theorem 2.1.4].

Lemma 3.1. *Assume $0 < s < N$ and let $\theta \in \mathcal{C}_0^\infty$ be a nonnegative and nonincreasing function of $|x|$ such that $\int_{\mathbb{R}^N}\theta = 1$. Set $\theta_n(x) := n^N\theta(nx)$. If $h \in L^p_{|x|^{sp-N}}$, then $\theta_n * h \in L^p_{|x|^{sp-N}}$ and $\theta_n * h \to h$ in $L^p_{|x|^{sp-N}}$.*

As is customary, we use the notation $v(r,\sigma)$ for the expression of a function $v(x)$ in spherical coordinates

Lemma 3.2. *Assume $0 < s < 1$ and $s \leq N/p$. If v is a distribution on $\mathbb{R}^N$ such that $\nabla v \in \left(L^p_{|x|^{sp-N}}\right)^N$, then $v \in W^{1,1}_{loc}$ and there is a constant c_v such that*

$$||v(r,\cdot) - c_v||_{1,\mathbb{S}^{N-1}} \leq Cr^{1-s}\left(\int_{B_r}|x|^{sp-N}|\nabla v|^p\right)^{1/p}, \tag{3.1}$$

for every $r > 0$, *where* $C > 0$ *depends only upon* s, p *and* N.

Proof. Note that $L^p_{|x|^{sp-N}} \subset L^p_{loc}$ since $s \leq N/p$. Thus, $v \in W^{1,p}_{loc} \subset W^{1,1}_{loc}$ by [12, p. 23]. Suppose first that v is smooth. Then, $v(r,\omega) - v(0) = \int_0^r \partial_r v(t,\omega)dt$ where $\partial_r v$ is the radial derivative of v, whence $|v(r,\omega) - v(0)| \leq \int_0^r |\nabla v(t,\omega)|dt = \int_0^r t^{-s+1/p}t^{s-1/p}|\nabla v(t,\omega)|dt$. By Hölder's inequality,

$$|v(r,\omega) - v(0)| \leq Cr^{1-s}\left(\int_0^r t^{sp-1}|\nabla v(t,\omega)|^p dt\right)^{1/p},$$

with $C = 1$ if $p = 1$ and $C = [(1-s)p']^{-1/p'}$ if $p > 1$. Upon integrating over $\mathbb{S}^{N-1}$, another application of Hölder's inequality yields

$$||v(r,\omega) - v(0)||_{1,\mathbb{S}^{N-1}} \leq C|\mathbb{S}^{N-1}|^{1/p'}r^{1-s}\left(\int_{B_r} |x|^{sp-N}|\nabla v|^p\right)^{1/p}$$

and (3.1) follows with $c = v(0)$.

In general, set $v_n := \theta_n * v$ with θ_n the mollifying sequence of Lemma 3.1. Then, v_n is smooth and $v_n \to v$ in $W^{1,1}_{loc}$, so that v_n tends to v in $L^1(r\mathbb{S}^{N-1})$ for every $r > 0$. This amounts to saying that

$$v_n(r,\cdot) \to v(r,\cdot) \text{ in } L^1(\mathbb{S}^{N-1}), \tag{3.2}$$

for every $r > 0$. Furthermore, by Lemma 3.1, $\nabla v_n = \theta_n * \nabla v \to \nabla v$ in $\left(L^p_{|x|^{sp-N}}\right)^N$. In particular,

$$\int_{B_r} |x|^{sp-N}|\nabla v_n|^p \to \int_{B_r} |x|^{sp-N}|\nabla v|^p. \tag{3.3}$$

From the above, (3.1) holds with v and c_v replaced with v_n and $v_n(0)$, respectively. As a result, $|v_n(0)||\mathbb{S}^{N-1}| \leq ||v_n(r,\cdot)||_{1,\mathbb{S}^{N-1}} + Cr^{1-s}\left(\int_{B_r} |x|^{sp-N}|\nabla v_n|^p\right)^{1/p}$. Thus, by (3.2) and (3.3), the sequence $v_n(0)$ is bounded. After passing to a subsequence, assume $v_n(0) \to c_v \in \mathbb{R}$. Then, (3.1) follows from (3.2) and (3.3) by taking the limit in the inequality $||v_n(r,\cdot) - v_n(0)||_{1,\mathbb{S}^{N-1}} \leq Cr^{1-s}\left(\int_{B_r} |x|^{sp-N}|\nabla v_n|^p\right)^{1/p}$. ∎

For the next theorem, recall the blanket hypothesis that a function of L^1_{loc} equals the limit of its Lebesgue averages at every Lebesgue point. Recall also that if f, g are nonnegative and measurable functions on $\mathbb{R}^N$, the classical convolution $f * g$ is defined (possibly infinite) and $f * g = g * f$.

Theorem 3.3. *Assume* $0 < s < 1$ *and* $s \leq N/p$. *There is a constant* $C > 0$ *depending only upon* s, p *and* N *such that, for every* $v \in W^{1,1}_{loc}$, *every* $z \in \mathbb{R}^N$ *and every* $\rho > 0$,

$$|B_\rho|^{-1}\int_{B_\rho(z)} |v(x) - v(z)|dx \leq C\rho^{1-s}[(|x|^{sp-N} * |\nabla v|^p)(z)]^{1/p}. \tag{3.4}$$

Proof. If $(|x|^{sp-N} * |\nabla v|^p)(z) = \infty$, (3.4) is trivial irrespective of C and ρ and irrespective of the choice of $v(z)$ when z is not a Lebesgue point of v. Thus, it suffices to prove the existence of C when $(|x|^{sp-N} * |\nabla v|^p)(z) < \infty$. To do this, it is not restrictive to assume $z = 0$ since the general case follows by changing $v(x)$ into $v(x+z)$. Accordingly, we henceforth assume $z = 0$ and $(|x|^{sp-N} * |\nabla v|^p)(0) < \infty$.

Equivalently, $\nabla v \in \left(L^p_{|x|^{sp-N}}\right)^N$ and then $(|x|^{sp-N} * |\nabla v|^p)(0) = \int_{\mathbb{R}^N} |x|^{sp-N}|\nabla v|^p$. Therefore, by Lemma 3.2, there is a constant c_v such that

$$||v(r,\cdot) - c_v||_{1,\mathbb{S}^{N-1}} \leq Cr^{1-s}[(|x|^{sp-N} * |\nabla v|^p)(0)]^{1/p},$$

where $C > 0$ depends only upon s, p and N. Let $\rho > 0$ be given. Since $\int_{B_\rho} |v(x) - c_v|dx = \int_0^\rho r^{N-1}||v(r,\cdot) - c_v||_{1,\mathbb{S}^{N-1}}dr$, it follows from the above inequality that

$$\int_{B_\rho} |v(x) - c_v|dx \leq C\rho^{N+1-s}|[(|x|^{sp-N} * |\nabla v|^p)(0)]^{1/p},$$

after changing C into $C/(N+1-s)$. As a result,

$$|B_\rho|^{-1}\int_{B_\rho} |v(x) - c_v|dx \leq C\rho^{1-s}[(|x|^{sp-N} * |\nabla v|^p)(0)]^{1/p},$$

for every $\rho > 0$ after changing C into $C|B_1|^{-1}$. Since $s < 1$, this shows that 0 is a Lebesgue point of v, so that $c_v = v(0)$ and the proof is complete. ∎

4. Hölder continuity of Sobolev functions

The Hölder continuity properties of the functions of $D^{1,p}$ that can be inferred from the results of Sections 2 and 3 are summarized in the next theorem. For $1 \leq p < \infty$, we set

$$\Gamma_p := \begin{cases} (0,1) \text{ if } p \leq N, \\ [1 - N/p, 1) \text{ if } p > N. \end{cases} \tag{4.1}$$

Theorem 4.1. *(i) Let $\gamma \in \Gamma_p$. If $v \in D^{1,p}$, there is an open subset U_γ of $\mathbb{R}^N$ with arbitrarily small measure such that $v \in C^{0,\gamma}(\mathbb{R}^N \backslash U_\gamma)$.*
(ii) If, in addition, $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$ for some $\gamma_0 \in \Gamma_p$, then $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma \leq \gamma_0$. More precisely, $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ and there is a constant $C = C(\gamma, p, N) > 0$ such that

$$|v(y) - v(z)| \leq C|y - z|^\gamma || \, |x|^{(1-\gamma)p-N} * |\nabla v|^p||_\infty^{1/p}, \tag{4.2}$$

for every $y, z \in \mathbb{R}^N$.

Proof. (i) If $0 < \alpha < N$, it is well known that the convolution with $|x|^{\alpha-N}$ (Riesz potential) maps L^1 into the Lorentz space $L^{N/(N-\alpha),\infty}$ (weak $L^{N/(N-\alpha)}$; see [20, p. 120]). Not only does that imply $(|x|^{\alpha-N} * |\nabla v|^p)(z) < \infty$ for a.e. $z \in \mathbb{R}^N$, but also that the set $\{z \in \mathbb{R}^N : (|x|^{\alpha-N} * |\nabla v|^p)(z) > d\}$ has arbitrarily small measure if d is large enough. This property remains true if $\alpha = N$, for $1 * |\nabla v|^p = ||\nabla v||_p^p$ is constant.

It follows that given $\varepsilon > 0$, the set $\Sigma_+^{s,d} := \{z \in \mathbb{R}^N : (|x|^{sp-N} * |\nabla v|^p)(z) > d\}$ has measure $|\Sigma_+^{s,d}| < \varepsilon$ if $0 < sp \leq N$ and d is large enough.

If $p \leq N$, the condition $0 < sp \leq N$ holds for every $s \in (0,1)$ and, if $p > N$, it implies $s < 1$. Thus, if $p \leq N$ and $s \in (0,1)$, or if $p > N$ and $s \in (0, N/p]$, there is $d > 0$ such that $|\Sigma_+^{s,d}| < \varepsilon$, which is henceforth assumed.

Every s just specified above satisfies $0 < s < 1$ and $s \leq N/p$. Thus, by Theorem 3.3, there is a constant $C = C(s,p,N) > 0$ such that

$$|B_\rho|^{-1}\int_{B_\rho(z)} |v(x) - v(z)|dx \leq Cd^{1/p}\rho^{1-s}, \tag{4.3}$$

for every $\rho > 0$ and every $z \in \Sigma_-^{s,d} := \mathbb{R}^N \backslash \Sigma_+^{s,d}$ and then, by Theorem 2.1, $v \in C^{0,1-s}(\overline{\Sigma_-^{s,d}})$ and $U^{s,d} := \mathbb{R}^N \backslash \overline{\Sigma_-^{s,d}} \subset \Sigma_+^{s,d}$ has measure $|U^{s,d}| < \varepsilon$.

If $p \leq N$, then s is arbitrary in $(0,1)$, so that $\gamma := 1 - s$ is arbitrary in $(0,1)$. If $p > N$, then s is arbitrary in $(0, N/p]$, so that $\gamma := 1-s$ is arbitrary in $[1-N/p, 1)$. In other words, $v \in C^{0,\gamma}(\overline{\Sigma_-^{1-\gamma,d}})$ for every $\gamma \in \Gamma_p$. This proves (i) with $U_\gamma = U^{1-\gamma,d}$.

(ii) If $0 < \alpha_0 \leq N$ and $|x|^{\alpha_0 - N} * f \in L^\infty$ for some nonnegative $f \in L^1$, then $|x|^{\alpha-N} * f \in L^\infty$ for every $\alpha \in [\alpha_0, N]$. To see this, write $(|x|^{\alpha-N} * f)(z) = \int_{|x|\leq 1} |x|^{\alpha-N} f(z-x)dx + \int_{|x|>1} |x|^{\alpha-N} f(z-x)dx$. Since $\alpha \leq N$, the second integral is majorized by $||f||_1$ irrespective of z and, since $\alpha \geq \alpha_0$, the first integral is majorized by $\int_{|x|\leq 1} |x|^{\alpha_0-N} f(z-x)dx \leq (|x|^{\alpha_0-N} * f)(z)$. Hence, $|x|^{\alpha-N} * f \leq |x|^{\alpha_0-N} * f + ||f||_1 \in L^\infty$.

With $\alpha_0 = (1-\gamma_0)p$, and $f = |\nabla v|^p \in L^1$, it thus follows from the assumption $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$ that $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ for every $\gamma \in \Gamma_p$ with $\gamma \leq \gamma_0$. If so, $\Sigma_+^{1-\gamma,d}$ above has measure 0 if $d \geq || \, |x|^{(1-\gamma)p-N} * |\nabla v|^p||_\infty$, whence $U_\gamma = U^{1-\gamma,d} = \emptyset$.

To prove (4.2), recall that the constant C in (4.3) depends only upon s, p and N and, hence, only upon γ, p and N when $s = 1 - \gamma$. In turn, $Cd^{1/p}$ in (4.3) is the constant K in (2.1) and the constant K_γ in (2.2) is $K(1 + 2^{N+\gamma})$. Thus, from the above and from (2.2),

$$|v(y) - v(z)| \leq Cd^{1/p}(1 + 2^{N+\gamma})|y - z|^\gamma,$$

for every $d \geq || \, |x|^{(1-\gamma)p-N} * |\nabla v|^p||_\infty$ and every $y, z \in \mathbb{R}^N$. The choice $d = || \, |x|^{(1-\gamma)p-N} * |\nabla v|^p||_\infty$ yields (4.2) after changing $C(1 + 2^{N+\gamma})$ into C (still depending only upon γ, p and N). ∎

If follows from part (i) of the theorem that if $v \in D^{1,p}$ with $1 \leq p \leq N$, then $|v(x)|$ grows slower than any given positive power of $|x|$ on the complement of an open set with arbitrarily small measure. This is consistent with known pointwise behavior of the functions of $D^{1,p}$. Mizuta [15] has shown that (i) $v \in D^{1,N}$ with $N > 1$ grows slower than $(\log |x|)^{1-1/N}$ outside a set "$(1, N)$-thin at infinity" and (ii) $v \in D^{1,p}$ with $1 < p < N$ tends to a constant along almost every line perpendicular to a hyperplane (similar to Fefferman [6] when v is C^1) and along almost every ray from the origin (or any other point). There seems to be no result sharper than ours when $p = 1 < N$ or when $1 < p < N$ and $|x| \to \infty$ without x moving along a line.

When $p > N$ and $\gamma = 1 - N/p$, (4.2) is the classical inequality $|v(y) - v(z)| \leq C|y - z|^{1-N/p}||\nabla v||_p$ with $C = C(p, N)$. More generally, if $0 < \alpha \leq N$, then $D^{1,p}_\alpha := \{v \in D^{1,p} : |x|^{\alpha-N} * |\nabla v|^p \in L^\infty\}$ is a vector subspace of $D^{1,p}$([2]) ($D^{1,p}_N = D^{1,p}$). Furthermore, $|| \, |x|^{\alpha-N} * |\nabla v|^p||_\infty^{1/p}$ is a seminorm on $D^{1,p}_\alpha$ and a norm on $D^{1,p}_\alpha/\mathbb{R}$. The triangle inequality follows at once the remark, due to König [10], that $(a + b)^p = \inf_{0<t<1} t^{1-p}a^p + (1 - t)^{1-p}b^p$ when $a, b \geq 0$. Thus, (4.2) reflects the continuity of the embedding of $D^{1,p}_{(1-\gamma)p}/\mathbb{R}$ into $C^{0,\gamma}/\mathbb{R}$ equipped with the norm $\sup_{y\neq z} |v(y) - v(z)|/|y - z|^\gamma$.

By a theorem of McShane [13, Corollary 1], if $E \subset \mathbb{R}^N$ is any subset and if $v \in C^{0,\gamma}(E)$ with $0 < \gamma \leq 1$, then v can be extended to all of $\mathbb{R}^N$ as a $C^{0,\gamma}$ function. Therefore, it is plain that Theorem 4.1 (i) is equivalent to the following Lusin-type property:

[2] If $\alpha \leq 0$ or $\alpha > N$, it is not hard to check that $D^{1,p}_\alpha = \mathbb{R}$.

Corollary 4.2. *Let $\gamma \in \Gamma_p$. If $v \in D^{1,p}$, there is $v_\gamma \in C^{0,\gamma}$ such that $v = v_\gamma$ outside an open subset U_γ with arbitrarily small measure.*

Remark 4.1. *We do not know whether v_γ can be taken in $C^{0,\gamma} \cap D^{1,p}$ and, if so, whether $||\nabla(v - v_\gamma)||_p$ can be made arbitrarily small. The analogous question when $v \in W^{1,p}$ has been settled in the affirmative; see the references given in the Introduction.*

5. More on Hölder continuity on $\mathbb{R}^N$

Part (ii) of Theorem 4.1 raises the issue of finding extra conditions on $v \in D^{1,p}$ ensuring that $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$. We give a first answer in (the proof of) Theorem 5.1 below. To begin with, we introduce the "uniformly local" Lebesgue spaces $L^p_{uloc} := \{f \in L^p_{loc} : \sup_{z\in\mathbb{R}^N} ||f||_{p,B_1(z)} < \infty\}$, much larger than L^p when $p < \infty$. These spaces have some of the features of the L^p spaces on subsets of finite measure, such as $L^{p_2}_{uloc} \subset L^{p_1}_{uloc}$ if $p_2 \geq p_1$. They were first used by Kato [9] in 1975 in connection with some PDE questions and have become popular in some circles, but we are not aware that they have previously played a role in pure analysis. We shall actually need the more general uniformly local Lorentz spaces

$$L^{p,q}_{uloc} := \{f \in L^{p,q}_{loc} : \sup_{z\in\mathbb{R}^N} ||f||_{p,q,B_1(z)} < \infty\},$$

where $1 \leq q \leq p < \infty$ (which will suffice for our purposes) and where $||f||_{p,q,B_1(z)}$ is the norm of f in $L^{p,q}(B_1(z))$. Here and in what follows, the norm of $f \in L^{p,q}(E)$ is understood as $\left(\int_0^\infty [t^{1/p} f^*(t)]^q dt/t\right)^{1/q}$ where f^* is the decreasing rearrangement of f. The assumption $q \leq p$ ensures that this is indeed a norm [2, p. 218].

Naturally, along with the spaces $L^{p,q}_{uloc}$ we may define

$$D^{1,(p,q)}_{uloc} := \{v \in L^1_{loc} : \nabla v \in (L^{p,q}_{uloc})^N\},$$

with the same restrictions on p and q as above.

Theorem 5.1. *(i) Let $v \in D^{1,p} \cap D^{1,(p_1,p)}_{uloc}$ for some $p_1 \in (N,\infty), p_1 \geq p$. Then, $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma \leq 1 - N/p_1$. More precisely, $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ and the inequality (4.2) holds.*
(ii) In particular, if $v \in D^{1,p} \cap D^{1,q}_{uloc}$ for some $q > \max\{p, N\}$, then $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma < 1 - N/q$.

Proof. (i) As a preamble, observe that if $\alpha \leq N$ and $f \in L^1$, then $|x|^{\alpha-N} * f \in L^\infty$ if and only if $|x|^{\alpha-N}\chi_{B_1} * f \in L^\infty$ because $|x|^{\alpha-N}\chi_{\widetilde{B}_1} \in L^\infty$, whence $|x|^{\alpha-N}\chi_{\widetilde{B}_1} * f \in L^\infty$.

Also, if B is any open ball and $f \geq 0$ is measurable, then $f \in L^{p_1,p}(B)$ if and only if $f^p \in L^{p_1/p,1}(B)$ and $||f||_{(p_1,p),B} = ||f^p||_{(p_1/p,1),B}$. This follows at once from the definition of the norms and from $(f^p)^* = (f^*)^p$. In particular, $f \in L^{p_1,p}_{uloc}$ if and only if $f^p \in L^{p_1/p,1}_{uloc}$.

From the above with $f = |\nabla v|$, it follows that $|\nabla v|^p \in L^1 \cap L^{p_1/p,1}_{uloc}$. With $\alpha := Np/p_1 \in (0, N]$, this reads $|\nabla v|^p \in L^1 \cap L^{N/\alpha,1}_{uloc}$. Now, $|x|^{\alpha-N} \in L^{N/(N-\alpha),\infty}(B_1)$ $(= L^\infty(B_1)$ if $\alpha = N)$, the associated space of $L^{N/\alpha,1}(B_1)$. Hence, by Hölder's inequality in Lorentz spaces ([2, p. 60]) and since $\sup_{\zeta\in\mathbb{R}^N} ||\,|\nabla v|^p||_{N/\alpha,1B_1(\zeta)} =$

$\sup_{\zeta\in\mathbb{R}^N} ||\,|\nabla v|\,||_{p_1,p,B_1(\zeta)} < \infty$,

$$\begin{aligned}(|x|^{\alpha-N}\chi_{B_1} * |\nabla v|^p)(z) &= \int_{B_1} |x|^{\alpha-N}|\nabla v(z-x)|^p dx \\ &\leq ||\,|x|^{\alpha-N}||_{N/(N-\alpha),\infty,B_1}||\,|\nabla v|^p(z-\cdot)||_{N/\alpha,1,B_1} \\ &= ||\,|x|^{\alpha-N}||_{N/(N-\alpha),\infty,B_1}||\,|\nabla v|^p||_{N/\alpha,1B_1(z)} \\ &\leq C||\,|x|^{\alpha-N}||_{N/(N-\alpha),\infty,B_1} \sup_{\zeta\in\mathbb{R}^N} ||\,|\nabla v|\,||_{p_1,p,B_1(\zeta)} < \infty,\end{aligned}$$

for every $z \in \mathbb{R}^N$. Thus, $|x|^{\alpha-N}\chi_{B_1} * f \in L^\infty$ and so $|x|^{\alpha-N} * f \in L^\infty$. To complete the proof, set $\gamma_0 = 1-N/p_1 \in \Gamma_p$ (see (4.1) and recall $p_1 \geq p$), so that $\alpha = (1-\gamma_0)p$ and use Theorem 4.1 (ii).

(ii) Since $q > p$, it follows that $L^p(B) \cap L^q(B) \hookrightarrow L^{p_1,p}(B)$ for every ball B and every $p_1 \in [p, q)$ (by a direct verification, or since $L^{p_1,p}(B)$ is an interpolation space between $L^p(B)$ and $L^q(B)$ when $p_1 \in (p, q)$). Furthermore, the norm of the embedding depends only upon the radius of B. Thus, $\sup_{z\in\mathbb{R}^N} ||\nabla v||_{p_1,p,B_1(z)} \leq ||\nabla v||_p + \sup_{z\in\mathbb{R}^N} ||\nabla v||_{q,B_1(z)} < \infty$, so that $v \in D^{1,(p_1,p)}_{uloc}$ and the result follows from (i) since p_1 can be chosen arbitrarily close to q. ∎

If $p_1 = p$ in Theorem 5.1 (i), the assumption is simply $v \in D^{1,p}$ with $p > N$ and the result follows from Morrey's theorem. This is not the case if $p_1 > p$ since, obviously, $v \in D^{1,p} \cap D^{1,(p_1,p)}_{uloc}$ does not imply $v \in D^{1,p_1}$. Of course, Morrey's theorem still yields the weaker $v \in C^{0,\gamma}_{loc}$ for every $\gamma \in \Gamma_p$ with $\gamma \leq 1 - N/p_1$.

A simple special case of (ii) arises when $v \in D^{1,p}$ and $\nabla v \in (L^q + L^r)^N$ where $r \geq q > N$ and $q > p$. Indeed, both L^q and L^r are contained in L^q_{uloc} and so $v \in D^{1,p} \cap D^{1,q}_{uloc}$. If $r > q$, Morrey's theorem is not applicable.

Next, we show that the uniformly local integrability properties of the higher order derivatives of v have an impact on the Hölder continuity of $v \in D^{1,p}$. We need the following lemma.

Lemma 5.2. *If $h \in L^1_{uloc}$ and $\nabla h \in (L^q_{uloc})^N$ for some $1 \leq q \leq \infty$, then $h \in L^{q_1}_{uloc}$ for every $q_1 \leq q^* := Nq/(N-q)$ if $q < N$ and for every $q_1 < \infty$ if $q \geq N$.*

Proof. Assume first $q < N$ and let $B \subset \mathbb{R}^N$ denote any ball with radius 1. Since $\nabla h \in (L^q_{uloc})^N$, it follows that $\nabla h \in (L^q(B))^N$ and, hence, that $h \in W^{1,q}(B)$. By the Poincaré-Wirtinger inequality, $||h-\overline{h}||_{q^*,B} \leq C||\nabla h||_{q,B}$ where $\overline{h} := |B_1|^{-1}\int_B h$ and $C > 0$ is a constant independent of h and B. As a result,

$$||h||_{q^*,B} \leq |B_1|^{-(1/N+1/q')}||h||_{1,B} + C||\nabla h||_{q,B}.$$

By letting $B = B_1(z)$ and by taking the supremum over $z \in \mathbb{R}^N$, it follows that $h \in L^{q^*}_{uloc}$ with $||h||_{q^*,uloc} \leq |B_1|^{-(1/N+1/q')}||h||_{1,uloc} + C||\nabla h||_{q,uloc}$. Since $L^{q^*}_{uloc} \subset L^{q_1}_{uloc}$ when $q_1 \leq q^*$, the lemma is proved.

If now $q \geq N$, then $\nabla h \in (L^r_{uloc})^N$ with $r < N$ arbitrarily close to N and the result follows from the first part with q replaced with r. ∎

Irrespective of q, Lemma 5.2 fails if $q_1 = \infty$ and it is trivially false in the non-uniformly local setting.

If $1 \leq q \leq \infty$ and $0 \leq j \leq N$, we set

$$q^{*j} := \begin{cases} Nq/(N-jq) \text{ if } q < N/j, \\ \infty \text{ if } q \geq N/j. \end{cases}$$

Note $q^{*0} = q, q^{*1} = q^*$ and $q^{*N} = \infty$. Also, $q^{*(j+1)} = (q^{*j})^* = (q^*)^{*j}$ if $j \leq N-1$.

Below, we confine attention to the case $p \leq N$, when no Hölder continuity is true without further assumptions.

Corollary 5.3. *Suppose that $v \in D^{1,p}$ with $1 \leq p \leq N$ and that for some integer $1 \leq k \leq N+1$,*
(i) $\nabla^j v \in (L^1_{uloc})^{N^j}$ if $1 \leq j \leq k-1$ (always true if $k = 2$ since $v \in D^{1,p}$),
(ii) $\nabla^k v \in (L^q_{uloc})^{N^k}$ with $q > N/k$ if $1 \leq k \leq N$, or $q = 1$ if $k = N+1$.
Then, $q^{*(k-1)} > N$ and $v \in C^{0,\gamma}$ for every $\gamma \in (0, 1 - N/q^{*(k-1)})$.

Proof. That $q^{*(k-1)} > N$ follows from $q > N/k$ if $1 \leq k \leq N$ and from $1^{*N} = \infty$ if $q = 1$ and $k = N+1$. Since $\Gamma_p = (0,1)$ when $p \leq N$, the case $k = 1$ is exactly Theorem 5.1 (ii). If $2 \leq k \leq N$, then $q^* > N/(k-1)$ and, by (i) and (ii) and Lemma 5.2 with h any partial derivative of v of order $k-1$, it follows that $\nabla^{k-1} v \in (L^{q_1}_{uloc})^{N^{k-1}}$ with $q_1 = q^*$ if $q^* < \infty$ or $q_1 < \infty$ arbitrarily large if $q^* = \infty$. This reduces the problem to the case when k is replaced with $k-1$ and q is replaced with $q_1 = q^*$ if $q^* < \infty$ or $q_1 < \infty$ arbitrarily large (in particular, $q_1^* = \infty$) if $q^* = \infty$. After $k-1$ steps, the problem is reduced to the case $k = 1$ already settled, with q replaced with $q_{k-1} := q^{*(k-1)}$ if $q^{*(k-1)} < \infty$ or with q_{k-1} arbitrarily large if $q^{*(k-1)} = \infty$. In the latter case, $v \in C^{0,\gamma}$ for every $\gamma \in (0, 1 - N/q_{k-1})$ and so for every $\gamma \in (0,1) = (0, q^{*(k-1)})$ since q_{k-1} is arbitrarily large.

Assume now $k = N+1$, so that $\nabla^{N+1} v \in (L^1_{uloc})^{N^k}$. By (i) and (ii) and Lemma 5.2 with $q = 1$ and h any partial derivative of v of order N, it follows that $\nabla^N v \in (L^{N/(N-1)}_{uloc})^{N^{k-1}}$. This reduces the problem to the case $k = N$ already settled, with q replaced with $q_1 = 1^* = N/(N-1) > 1$. ∎

The variant of Corollary 5.3 when $p > N$ is that if $q^{*(k-1)} > p$, then $v \in C^{0,\gamma}$ for every $\gamma \in [1 - N/p, 1 - N/q^{*(k-1)})$. Even when the uniformly local integrability conditions are replaced with (stronger) classical ones, a proof of Corollary 5.3 bypassing the uniformly local spaces is generally not possible, or requires using unnecessarily convoluted ad-hoc arguments. We give two examples.

A special case of Corollary 5.3 arises when $\partial_1^{i_1} \cdots \partial_N^{i_N} v \in L^{p_{i_1 \cdots i_N}}$ with $1 \leq p_{i_1 \cdots i_N} \leq \infty$ for every $2 \leq i_1 + \cdots + i_N \leq N+1$. If so, $v \in C^{0,\gamma}$ for every $\gamma \in (0,1)$. It is safe to say that no proof can be based on more or less classical embedding theorems. In fact, if $N = 1$, the assumptions are $v' \in L^1$ and $v'' \in L^q$ for some $1 \leq q \leq \infty$. If so, a direct proof -including $\gamma = 1$- is an exercise, yet not a completely trivial one. It becomes significantly more challenging if $v'' \in L^q$ is replaced with $v'' \in L^{q_1} + L^{q_2}$ with $1 \leq q_1, q_2 \leq \infty$, a case also covered by Corollary 5.3.

A much simpler special case of Corollary 5.3 is $N > 1, k = 2$ and $\partial^2_{ij} v \in L^q$ with $q > N/2$. If so, $v \in C^{0,\gamma}$ for every $\gamma \in (0, 1 - N/q^*)$. A direct proof is still problematic. If $q > N$, then ∇v is uniformly (even Hölder) continuous and so, being in $(L^p)^N$, it must be bounded. Thus, $\nabla v \in (L^p \cap L^\infty)^N \subset (L^r)^N$ for every $N < r \leq \infty$, so that $v \in C^{0,\gamma}$ for every $\gamma \in (0,1]$. If $q < N$, there is a constant vector V such that $\nabla v - V \in L^{q^*}$. Thus, $V \in (L^p \cap L^{q^*})^N$ and so $V = 0$ since V is constant. As a result, $\nabla v \in (L^p \cap L^{q^*})^N$. Since $q^* > N$ (recall $q > N/2$), it follows that $v \in C^{0,\gamma}$ for every $0 < \gamma \leq 1 - N/q^*$. However, if $q = N$, there seems to be no classical argument yielding $v \in C^{0,\gamma}$ for any $\gamma \in (0, 1]$, let alone for every $\gamma \in (0, 1)$.

A different path to sufficient conditions for $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$ when $v \in D^{1,p}$ and $\gamma_0 \in \Gamma_p$ goes by Fourier transform. In what follows, $\mathcal{S}'$ denotes the space of tempered distributions (dual of the Schwartz space $\mathcal{S}$) and we use either one of the standard "hat" or $\mathcal{F}$ notations for the Fourier transform.

If $\alpha > 0$, we define the Bessel kernel G_α to be the inverse Fourier transform of $(1 + |\xi|^2)^{-\alpha/2}$. There are various other normalizations in the literature. These minor differences have no impact on the basic properties of the kernels. The following lemma is obviously not new. We just relate it to the relevant properties of convolution and Fourier transform.

Lemma 5.4. *For every $f \in \mathcal{S}'$, the convolution $G_\alpha * f$ is well defined in $\mathcal{S}'$ and $\mathcal{F}(G_\alpha * f) = \widehat{G}_\alpha \widehat{f}$.*

Proof. By [19, p. 247 and p. 268], it suffices to show that G_α is a rapidly decaying distribution (space O'_C in [19]) which, in turn, is equivalent to showing that $G_\alpha * \varphi \in \mathcal{S}$ whenever $\varphi \in C_0^\infty$ ([19, p. 244]). This easily follows from the exponential decay of $G_\alpha(x)$ as $|x| \to \infty$ ([20, p. 132]). ∎

Lemma 5.5. *If $\gamma_0 \in \overset{\circ}{\Gamma}_p$ (interior of the interval Γ_p; see (4.1)) and $v \in D^{1,p}$, the conditions $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$ and $G_{(1-\gamma_0)p} * |\nabla v|^p \in L^\infty$ are equivalent.*

Proof. In the proof of Theorem 5.1, we used the remark that $|x|^{(1-\gamma_0)p-N} * |\nabla v|^p \in L^\infty$ if and only if $|x|^{(1-\gamma_0)p-N}\chi_{B_1} * |\nabla v|^p \in L^\infty$. In this statement, χ_{B_1} may of course be replaced with χ_{B_r} for any $r > 0$.

By [20, p. 132], $G_{(1-\gamma_0)p}(x)/|x|^{(1-\gamma_0)p-N} = c + o(1)$ as $0 \neq x \to 0$, where $c > 0$ is a constant (the condition $\gamma_0 \in \overset{\circ}{\Gamma}_p$ amounts to $0 < (1 - \gamma_0)p < N$) and so $|x|^{(1-\gamma_0)p-N}/G_{(1-\gamma_0)p}(x) = (c + o(1))^{-1} = c^{-1} + o(1)$. Thus, if $r > 0$ is small enough, $|x|^{(1-\gamma_0)p-N}$ and $G_{(1-\gamma_0)p}$ are dominated by a constant multiple of one another on B_r. It follows that $|x|^{(1-\gamma_0)p-N}\chi_{B_r} * |\nabla v|^p \in L^\infty$ if and only if $G_{(1-\gamma_0)p}\chi_{B_r} * |\nabla v|^p \in L^\infty$, which happens if and only if $G_{(1-\gamma_0)p} * |\nabla v|^p \in L^\infty$ since $G_{(1-\gamma_0)p}\chi_{\widetilde{B}_r} \in L^\infty$. ∎

We now spell out a special case of Theorem 4.1 (ii) that may be phrased in terms of Fourier transform.

Theorem 5.6. *Let $\gamma_0 \in \overset{\circ}{\Gamma}_p$ be given. Suppose that $v \in D^{1,p}$ and that $|\nabla v|^p \leq f$ where $f \in L^1_{loc} \cap \mathcal{S}'$ and $(1 + |\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f}$ is a finite Borel measure (i.e., $(1 + |\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f} \in L^1$ if $\widehat{f} \in L^1_{loc}$). Then, $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma \leq \gamma_0$. More precisely, $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$ and the inequality (4.2) holds.*

Proof. By Theorem 4.1 (ii) and Lemma 5.5, it suffices to show that $G_{(1-\gamma_0)p} * |\nabla v|^p \in L^\infty$. Since $G_{(1-\gamma_0)p} > 0$ and $|\nabla v|^p \leq f$, this holds if $G_{(1-\gamma_0)p} * f \in L^\infty$, which follows from Lemma 5.4 and the assumption that $(1 + |\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f}$ is a finite Borel measure. ∎

Even locally, Theorem 5.6 does not follow from Morrey's theorem since the assumptions on f do not ensure that $v \in D^{1,q}_{loc}$ for any $q > p$.

Naturally, $(1+|\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f}$ can only be a finite measure if $\widehat{f}$ is a measure, but it does not have to be finite (if it is, $f \in L^\infty$ and the result is trivial). Also, it is important not to confine attention to $f = |\nabla v|^p$ alone. Indeed, since Fourier transform does not preserve ordering in any way, the fact that $(1+|\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f}$ is a finite Borel measure for some $f \geq |\nabla v|^p$ does not imply the same thing when $f = |\nabla v|^p$. A very simple related example with $N = 1$ and $\gamma_0 = 1$ (albeit ruled out above), is that $\widehat{1} = \delta$ is a finite measure and $1 \geq \chi_{(-1,1)}$, but $\mathcal{F}(\chi_{(-1,1)})(\xi) = -i \sin 2\pi\xi/\pi\xi \notin L^1$ is not a finite measure.

We complete this section with a concrete example. It can be handled with Theorem 5.1 or with Theorem 5.6.

Let $v \in D^{1,p}$ and suppose that there are $x_1, ..., x_m \in \mathbb{R}^N$, not necessarily distinct, such that

$$|\nabla v(x)|^p \leq A_0 + \sum_{j=1}^{m} A_j |x - x_j|^{-\beta_j} \text{ a.e.,} \tag{5.1}$$

with constants $A_0 \geq 0$ and $A_j > 0$ and $0 < \beta_j < \min\{p, N\}$ for $1 \leq j \leq m$. If $m > 0$, (5.1) does not imply $v \in D^{1,q}$ for any $q \neq p$ since $|x - x_j|^{-\beta_j}$ is in no Lebesgue space. We claim that $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma < 1 - \beta/p$ where $\beta := \max_{1 \leq j \leq m} \beta_j$.

By Theorem 4.1 (ii), it suffices to show that $|x|^{(1-\gamma)p-N} * |\nabla v|^p \in L^\infty$, but this is not just a trivial consequence of (5.1) because $|x|^{(1-\gamma)p-N} * 1 = \infty$ while $|x|^{(1-\gamma)p-N} * |x - x_j|^{-\beta_j}$ is never in L^∞ (it is infinite if $\gamma \leq 1 - \beta_j/p$ and remains unbounded near x_j if $\gamma > 1 - \beta_j/p$).

However, the result follows from Theorem 5.1 (ii). To see this, set $f_0(x) := 1$ and $f_j(x) := |x - x_j|^{-\beta_j}, 1 \leq j \leq N$, so that (5.1) reads

$$|\nabla v|^p \leq \sum_{j=0}^{m} A_j f_j. \tag{5.2}$$

If $1 \leq s < N/\beta$ and $0 \leq j \leq m$, then $f_j \in L^s_{loc}$ and $f_0 = 1 \in L^s_{uloc}$. If $1 \leq j \leq m$, then f_j is the sum of $|x - x_j|^{-\beta_j}\chi_{\widetilde{B}_1(x_j)} \in L^\infty \subset L^s_{uloc}$ and of $|x - x_j|^{-\beta_j}\chi_{B_1(x_j)} \in L^s \subset L^s_{uloc}$, so that $f_j \in L^s_{uloc}$. Therefore, by (5.2), $v \in D^{1,ps}_{uloc}$ for every $1 \leq s < N/\beta$. Equivalently, $v \in D^{1,q}_{uloc}$ for every $1 \leq q < Np/\beta$.

Now, $Np/\beta > \max\{p, N\}$ since $\beta < \min\{p, N\}$. Hence, by Theorem 5.1 (ii), $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma < 1 - N/q$ and $\max\{p, N\} < q < Np/\beta$. By picking q arbitrarily close to Np/β, it follows that $v \in C^{0,\gamma}$ for every $\gamma \in \Gamma_p$ with $\gamma < 1 - \beta/p$, as claimed.

For another proof, choose $\gamma_0 \in \mathring{\Gamma}_p$ such that $\gamma_0 < 1 - \beta/p$. This is possible since $\beta < \min\{p, N\}$ and γ_0 can be arbitrarily close to $1 - \beta/p$. Thus, by (5.2) and Theorem 5.6, it suffices to show that $(1+|\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f_j}$ is a finite Borel measure for every $0 \leq j \leq m$. This is trivial if $j = 0$ since $(1+|\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f_0} = \delta$ (Dirac delta).

If $1 \leq j \leq m$, then $\widehat{f_j}(\xi) = e^{-2\pi i x_j \cdot \xi}\mathcal{F}(|x|^{-\beta_j})(\xi)$ is a function and $|\widehat{f_j}| = |\mathcal{F}(|x|^{-\beta_j})|$ is a constant multiple of $|\xi|^{\beta_j - N}$. In addition, $(1+|\xi|^2)^{-(1-\gamma_0)p/2}|\xi|^{\beta_j-N} \in L^1_{loc}$ since $\beta_j < N$ and integrability at infinity follows from $\beta_j - N - (1-\gamma_0)p \leq \beta - N - (1-\gamma_0)p < -N$ since $\gamma_0 < 1 - \beta/p$. Thus, $(1+|\xi|^2)^{-(1-\gamma_0)p/2}\widehat{f_j} \in L^1$.

6. A generalization

For completeness, we briefly address the more general case when $\partial_j v \in L^{p_j}$ with $1 \leq p_j < \infty, 1 \leq j \leq N$ rather than just $v \in D^{1,p}$. The results take a slightly different form. Most notably, the Hölder spaces must be replaced with more general spaces $C^{0,(\gamma^-,\gamma^+)}$ depending upon two exponents $0 < \gamma^- \leq \gamma^+$, defined by

$$C^{0,(\gamma^-,\gamma^+)} := \left\{ u \in C^0 : \sup_{y \neq z} \frac{|u(y) - u(z)|}{\max\{|y-z|^{\gamma^-}, |y-z|^{\gamma^+}\}} < \infty \right\}.$$

Thus, if $u \in C^{0,(\gamma^-,\gamma^+)}$, then $u \in C^{0,\gamma^-}_{loc}$ and $|u(y) - u(z)| = O(|y-z|^{\gamma^+})$ when $|y-z| \to \infty$. If $S \subset \mathbb{R}^N$ is any subset, the space $C^{0,(\gamma^-,\gamma^+)}(S)$ is defined in the obvious way.

The generalization of Theorem 4.1 reads as follows.

Theorem 6.1. *If $\gamma_j \in (0,1), 1 \leq j \leq N$, set $\gamma^- := \min_j \gamma_j$ and $\gamma^+ := \max_j \gamma_j$.*
(i) If $\partial_j v \in L^{p_j}$ with $1 \leq p_j < \infty$ and if $\gamma_j \in \Gamma_{p_j}$ (see (4.1)), $1 \leq j \leq N$, there is an open subset U_{γ^-,γ^+} of $\mathbb{R}^N$ with arbitrarily small measure such that $v \in C^{0,(\gamma^-,\gamma^+)}(\mathbb{R}^N \backslash U_{\gamma^-,\gamma^+})$.
(ii) If, in addition, $|x|^{(1-\gamma_{0,j})p_j - N} * |\partial_j v|^{p_j} \in L^\infty$ for some $\gamma_{0j} \in \Gamma_{p_j}, 1 \leq j \leq N$, then $v \in C^{0,(\gamma^-,\gamma^+)}$ whenever $\gamma_j \in \Gamma_{p_j}$ and $\gamma_j \leq \gamma_{0j}, 1 \leq j \leq N$. More precisely, $|x|^{(1-\gamma_j)p_j - N} * |\partial_j v|^{p_j} \in L^\infty$ and there is a constant $C = C(\gamma_1, ..., \gamma_N, p_1, ..., p_N, N) > 0$ such that

$$|v(y) - v(z)| \leq C \left(\sum_{j=1}^N || \, |x|^{(1-\gamma_j)p_j - N} * |\partial_j v|^{p_j} ||_\infty^{1/p_j} \right) \max\{|y-z|^{\gamma^-}, |y-z|^{\gamma^+}\},$$

for every $y, z \in \mathbb{R}^N$.

The proof requires only minor modifications of Theorem 4.1 (and concomitant modifications of Theorem 2.1 and Theorem 3.3) and is omitted.

In part (ii) of Theorem 6.1, $\gamma_1 = \cdots = \gamma_N = \gamma \in \cap_{j=1}^N \Gamma_{p_j} \cap (0, \gamma_{0,j}]$ is possible if the intersection is not empty and, if so, $v \in C^{0,\gamma}$. Note also that if $p_j > N, 1 \leq j \leq N$, the choice $\gamma_{0j} := 1 - N/p_j$ is always admissible in (ii), whence $v \in C^{0,(\gamma_0^-,\gamma_0^+)}$ where $\gamma_0^- := 1 - N/\min_j p_j$ and $\gamma_0^+ := 1 - N/\max_j p_j$. Evidently, the case $p_j = p > N$ is Morrey's theorem.

There is no difficulty in extending Corollary 4.2 (use [13, Theorem 2] with $\max\{\rho^{\gamma^-}, \rho^{\gamma^+}\}$ replaced by its concave hull $\lambda(\rho)$ and note that $\max\{\rho^{\gamma^-}, \rho^{\gamma^+}\} \leq \lambda(\rho) \leq C \max\{\rho^{\gamma^-}, \rho^{\gamma^+}\}$ for some constant C) and, in suitable form, the results of Section 5.

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Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: rabier@imap.pitt.edu