Synchronization of master–slave Boolean networks with impulsive effects: Necessary and sufficient criteria

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ABSTRACT

This paper investigates the synchronization of master–slave Boolean networks with impulsive effects. Necessary and sufficient conditions on the synchronization for master–slave Boolean networks with impulsive effects are derived by converting the logical systems into algebraic expression with the method of semi-tensor product of matrices. Some convenient conditions are proposed to judge whether a Boolean network with impulsive effects can be synchronized. Finally, one illustrative example is given to show the effectiveness of the proposed results.

1. Introduction

With the rapid development of systems biology [1], genetic regulatory networks have recently been a newly developing topic and have attracted much attention due to its close relationship with neural networks, protein webs and other biological systems [2–6]. There are various types of genetic regulatory networks proposed in recent years, such as Markov-type genetic networks [7,8], probabilistic Boolean networks [9] and Boolean networks [1].

Boolean network (BN) was firstly proposed by Kauffman for modeling complex and nonlinear biological systems [1], and then developed by Shmulevich et al. [10], Farrow et al. [11]. In a Boolean network, the state of each node has only two states: 1 (active) or 0 (inactive) at each discrete time. Moreover, each node evolves its state according to a Boolean function, which is a logical function. In recent years, Boolean network seems especially suitable for modeling genetic regulation networks and biological systems, as the ON (OFF) state corresponds the transcribed (quiescent) state of a gene. Moreover, BNs can also be used for modeling other cellular processes.

Recently, a new method called semi-tensor product (STP) of matrices is firstly proposed by Cheng and his colleagues [12] which can convert a Boolean network into an algebraic expression. Based on this approach, many interesting results have been obtained in the past few years. Cheng and Qi [13] and Zhao et al. [14] used this method to analyze the controllability and observability of Boolean control networks. Cheng et al. [15] investigated the stability and stabilization of Boolean networks. Then, Li and Sun [16] studied the controllability of probabilistic Boolean control networks and the observability of Boolean control networks with impulsive effects was studied in [17]. In [18], Li and Chu considered the complete synchronization of Boolean networks. In [19], Zhao et al. proposed an aggregation algorithm towards large-scale Boolean network. The global stability at a limit cycle of switched Boolean networks was studied in [20], while the controllability of Boolean control networks with time delays both in states and inputs was studied in [21].

In real world, many evolutionary processes may experience abrupt changes of states at certain time instants. These changes maybe due to changes in the external environment disturbances or the interconnections of subsystems. Moreover, these abrupt changes may occur at prescribed time and/or triggered by specified events along a particular trajectory. Then, to describe systematically on evolutionary of a real process with a short-time disturbance, it is natural to omit the duration of the disturbance and just assume these perturbations to be “instantaneous”, that is, in the pattern of impulses. In some related literatures [22], impulsive controller can be designed to increase the stabilization of complex dynamical networks. However, in some cases, impulses can play a negative role on the issue of synchronization problem for some dynamical networks [23]. There is a large amount of papers investigating the systems with impulsive effects, such as [24–27]. Boolean networks are widely used in biology systems, but
to the best of our knowledge, there are not many literatures concerning about impulsive effects [17,28]. Boolean networks involving impulsive disturbances appear as a natural description of evolution phenomena of some real world problems. Thus, to investigate a Boolean network with impulsive effects is a meaningful and challenging topic in the future research.

In the past decades, the phenomenon of collective behavior has inspired large amount of researchers, such as synchronization analysis and control of complex networks [29–34], consensus in multi-agent systems [35,36] and synchronization of Kauffman networks [37]. The synchronization of Boolean networks, synchronization of two deterministic BNs and synchronization in an array of coupled BNs have been respectively studied in [18,38] and so is the synchronization of multi-valued logical networks in [39]. In [40], we investigated the synchronization in an array of Boolean networks with time delay. Motivated by the above discussions, in this paper, we study the synchronization of master–slave BNs with impulsive effects.

The rest of paper is organized as follows: in Section 2, we present some necessary preliminaries on semi-tensor product of matrices and some matrix expression of logic. In Section 3, we acquire necessary and sufficient criteria for synchronization of master–slave BNs with impulsive effects. In Section 4, one numerical example is given to illustrate the obtained results. Finally, Section 5 presents the conclusion.

2. Preliminaries

In this section, we give an outline of semi-tensor product of matrices, the vector form of Boolean variables and the algebraic representation of Boolean function by Cheng and his colleagues [41].

Definition 1 (Cheng and Qi [41]).

Let \( X \) be a row vector of dimension \( np \) and \( Y \) be a column vector of dimension \( p \). Then we can split \( X \) into \( p \) equal-size blocks as \( X^1, X^2, \ldots, X^p \), which are \( 1 \times n \) rows. Define the STP of \( X \) and \( Y \) as

\[
\begin{align*}
X \bowtie Y &= \sum_{i=1}^{p} X_i^i Y_i^i \in \mathbb{R}^n, \\
Y^T \bowtie X^T &= \sum_{i=1}^{p} Y_i^i X_i^i \in \mathbb{R}^n.
\end{align*}
\]

Let \( A \in M_{m \times n} \) and \( B \in M_{p \times q} \). If either \( n \) is a factor of \( p \), saying \( nt = p \) and denote it as \( A \prec_i B \), or \( p \) is a factor of \( n \), saying \( ny = p \) and denote it as \( A \succ_i B \), then we define the STP of \( A \) and \( B \), denoted by \( C = A \bowtie B \), as follows: \( C \) consists of \( m \times q \) blocks as \( C = (C_l) \) and each block is

\[
C_l = A_{l} \bowtie B_{j}, \quad l = 1, \ldots, m; j = 1, \ldots, q.
\]

where \( A_l \) is the \( l \)-th row of \( A \) and \( B_j \) is the \( j \)-th column of \( B \).

We use some simple numerical examples to describe it.

Example 2.

1. Let \( A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ -2 & -1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), then we have

\[
A \bowtie B = \begin{bmatrix} 1 \cdot 2 \cdot 1 & 1 \cdot 2 \cdot 1 & -2 \\ 1 \cdot 2 \cdot 3 & 1 \cdot 2 \cdot 3 & -2 \\ -2 \cdot 3 \cdot 2 & -2 \cdot 3 \cdot 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & -4 & -5 \\ 8 & 11 & -8 & -11 \end{bmatrix}.
\]

Definition 3 (Cheng et al. [12]). For an \( n \times m \) matrix \( A \) and a \( p \times q \) matrix \( B \), then the STP of \( A \) and \( B \) is

\[
A \bowtie B = (A \oplus I_{pq}) (B \otimes I_{pq})
\]

where \( I = lcm(m, p) \). If \( m = p \), then \( A \bowtie B = AB \), so we obtain the standard matrix product. Thus, the STP of matrices is a generalization of the conventional matrix product. In this paper, we can omit “\( \bowtie \)” for convenience and the matrix product is assumed to be the semi-tensor product in [12,41,42].

Definition 4 (Cheng and Qi [41]). An \( mn \times mn \) matrix \( W_{mn} \) is called a swap matrix, if it is constructed by the way; label its columns by \( (11, 12, \ldots, 1n, \ldots, m1, \ldots, mn) \) and similarly label its rows by \((11, 21, \ldots, m1, \ldots, 1n, 2n, \ldots, mn)\). Then its element in the position \((i, j, (i, j))\) is assigned as

\[
w_{i,j,(i,j)} = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}
\]

When \( m = n \), we denote \( W_{mn} \) by \( W_{nn} \) or \( W_{n} \).

Example 5. The swap matrix \( W_{[3,2]} \) can be constructed by Definition 4. According to Eq. (1), we have

\[
W_{[3,2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]

Lemma 6 (Cheng and Qi [41]). Let \( X \in \Delta_m \) and \( Y \in \Delta_n \) be two columns. Then we have

\[
W_{mn} \bowtie X \bowtie Y = Y \bowtie X, \quad W_{mn} \bowtie X \bowtie X = X \bowtie Y.
\]

2.1. Matrix expression of logic

In this subsection, we recall the matrix expression of logic. We firstly introduce some notations.

1. Define a delta set as \( \Delta_n = (\delta_{ij}^n) \), where \( \delta_{ij}^n \) is the \( i \)-th column of the identity matrix \( I_n \).

2. We denote the \( i \)-th column of matrix \( A \) by \( \text{Col}(A) \) and denote the set of columns of matrix \( A \) by \( \text{Col}(A) \). We also denote the \( i \)-th row of matrix \( A \) by \( \text{Row}(A) \) and denote the set of rows of matrix \( A \) by \( \text{Row}(A) \).

3. An \( n \times m \) matrix \( L \) is called a logical matrix if the columns of \( L \), denoted by \( \text{Col}(L) \), are of the form \( \delta_{ij}^n \). That is to say, \( \text{Col}(L) \subseteq \Delta_n \).

4. Denote the set of \( n \times s \) logical matrices by \( \mathcal{L}_{n \times s} \).

For simplicity, it can be denoted as \( L = [\delta_{11}^n, \delta_{12}^n, \ldots, \delta_{1s}^n] \).

Secondly, we introduce the vector form of logical variables. We use “\( 1 \)” and “\( 0 \)” to represent the logical values “True” and “False”, respectively or \( D = \{1, 0\} \). To get matrix expression, we use the vector \( \delta_{ij}^n \in \Delta_n \) and \( \delta_{ij}^n \in \Delta_n \) respectively to identify the variables \( 1 \) and \( 0 \). Then, using STP of matrices, a logical function with \( n \) variables \( f : D^n \rightarrow D \) can be expressed into the corresponding algebraic form as follows:

Lemma 7 (Cheng and Qi [41]). Let \( f : (\Delta_2)^n \rightarrow \Delta_2 \) be a Boolean function. Then there exists an unique matrix \( F \in \mathcal{L}_{2 \times 2^n} \) such that \( f(\sigma_1, \sigma_2, \ldots, \sigma_n) = F \bowtie \sigma_1 \bowtie \sigma_2 \bowtie \ldots \bowtie \sigma_n \) for every \((\sigma_1, \sigma_2, \ldots, \sigma_n) \in (\Delta_2)^n\). \( F \) is called the structure matrix of \( f \).
Here, we present the structure matrices of some basic logical operators (Negation: \( \neg \), \( M_a = \delta_2[2,1] \); Conjunction: \( \wedge \), \( M_2 = \delta_2[1,2,2,2] \); Disjunction: \( \vee \), \( M_4 = \delta_2[1,1,1,2] \)); Conditional: \( \rightarrow \), \( M_1 = \delta_2[1,2,1,1] \); Biconditional: \( \leftrightarrow \), \( M_3 = \delta_2[1,2,2,1] \)); the dummy matrix: \( Ed = \delta_2[1,2,1,2] \), which has the following property: for any two logical variables \( u \) and \( v \), we have \( Edu = u \), or \( Edv = v \). The group power-reducing matrix: \( \Phi_n = \delta_{2n} \begin{pmatrix} 1 \ & 2^0 \ & 2^1 \ & \ldots \ & 2^n \ & 2^{n+1} \ & \ldots \ & 2^{2n} \ & 2^{2n+1} \ & 2^{2n+2} \end{pmatrix} \), satisfies \( \sigma \Gamma \Sigma = \Phi_n \sigma \), for \( \forall \sigma \in \Delta_2^n \), which will be used in what follows. To see the definition of a structure matrix and its concerning results, it refers to [41] for more details. A toolbox for all the related computations is available at http://lsc.amss.ac.cn/dcheng/.

3. Main results

In this section, we will consider synchronization of master–slave BNs with impulsive effects.

Consider the following master–slave BNs with impulsive effects, which the impulsive sequence is given as follows: 
\[
0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots, k \in Z^+ \cup \{0\}, i = 1, 2, \ldots, n.
\]
\[
\begin{align*}
(1) & \quad x_i(t+1) = f_i(x_i(t), x_{i+x}(t), x_{i+2x}(t), \ldots, x_{i+(n-1)x}(t)), \quad k \in Z^+ \cup \{0\}, \\
\quad & \quad y_i(t+1) = g_i(x_i(t), y_{i+y}(t), y_{i+2y}(t), \ldots, y_{i+(n-1)y}(t)), \quad k \in Z^+ \cup \{0\}, \\
\end{align*}
\]
where \( f_i, g_i \) are the internal states, and \( x_i, y_i \) are the output states. Let \( \tau_i \) satisfy \( \tau_i > 0 \) and \( \tau_i < \tau_{i+1} \) for all \( i = 1, 2, \ldots, n \). Define \( z(t) = x(t)y(t) \in \Delta_2^{2n} \), then for \( t_{k-1} \leq t < t_k - 1 \), one has
\[
\begin{align*}
(2) & \quad z(t+1) = x(t+1)y(t+1) \\
\quad & \quad = F(t)x(t)y(t) \\
\quad & \quad = (F \otimes G)(\Phi_n \otimes L_{2^n})(x(t)y(t)) \\
\quad & \quad = \Theta(t),
\end{align*}
\]
where \( \Theta = (F \otimes G)(\Phi_n \otimes L_{2^n}) \). For \( t = t_k \), we have
\[
\begin{align*}
(3) & \quad z(t_k) = x(t_k)y(t_k) \\
\quad & \quad = F(t_k)x(t_k-1)y(t_k-1) \\
\quad & \quad = \left( F \otimes G \right)(\Phi_n \otimes L_{2^n})(x(t_k-1)y(t_k-1)) \\
\quad & \quad = \Xi z(t_k-1),
\end{align*}
\]
where \( \Xi = (F \otimes G)(\Phi_n \otimes L_{2^n}) \).

Thus, we convert the master–slave BNs with impulsive effects into the following network:
\[
\begin{align*}
(4) & \quad z(t+1) = \Theta z(t), \quad t_{k-1} \leq t < t_k - 1, \\
(5) & \quad z(t_k) = \Xi z(t_k-1), \quad k \in Z^+ \cup \{0\}.
\end{align*}
\]

In [12], it has been proved that by letting \( z(t) = x(t)y(t) \), we get a bijective mapping \( \Delta_2^n \rightarrow \Delta_2^{2n} \), and system (2) is equivalent to system (4). Then, we have the following synchronization condition for system (2).

Theorem 11. System (2) is said to be synchronized if and only if there is an impulsive time \( t_k \) such that \( Col_i[\ell_i] \subseteq S, Col_i[\Theta] \in S \) and \( Col_i[\Xi] \in \mathcal{S} \), where \( S = \{ \delta_{2^n}^\alpha | \alpha = (i-1)2^n + i, i = 1, 2, \ldots, 2^n \} \), \( C = (\beta i \beta = (j-1)2^n + j, j = 1, 2, \ldots, 2^n \} \), and
\[
L_k = \begin{pmatrix}
\Theta & \Xi^{\delta_0^0 \ldots \delta_0^{i-1}} \Xi & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} \\
\Xi^{\delta_0^0 \ldots \delta_0^{i-1}} & \Xi & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} \\
\Xi^{\delta_0^0 \ldots \delta_0^{i-1}} & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} & \Xi
\end{pmatrix}
\]
for \( k = t_k + j, 1 \leq j \leq t_k - t_k - 1, 1 \).

Thus, we have \( z(k) = L_k z(0) \), where
\[
L_k = \begin{pmatrix}
\Theta & \Xi^{\delta_0^0 \ldots \delta_0^{i-1}} \Xi & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} \\
\Xi^{\delta_0^0 \ldots \delta_0^{i-1}} & \Xi & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} \\
\Xi^{\delta_0^0 \ldots \delta_0^{i-1}} & \Xi^{\delta_0^1 \ldots \delta_0^{i-1}} & \Xi
\end{pmatrix}
\]
for \( k = t_k + j, 1 \leq j \leq t_k - t_k - 1, 1 \).

Remark 8. The master–slave Boolean networks can be seen as two layer networks. The master Boolean network is the first layer network, while the slave one is the second layer network. The multilayer networks have extensive applications in social contangions and biological contangions, while Boolean networks can be used to model genetic regulation networks and biological systems. Thus, multilayer Boolean networks also deserve further study about the stability, synchronization, controllability, and so on.

Now, we define synchronization of the master–slave BNs with impulsive effects.

Definition 9. System (2) (equivalently system (3)) is said to be synchronized if for any initial states \( x(0) = (x_1(0), x_2(0), \ldots, x_n(0)) \in D^n \) and \( y(0) = (y_1(0), y_2(0), \ldots, y_n(0)) \in D^n \), there exists a positive integer \( k \), such that \( x_i(t) = y_i(t) \) for \( t \geq k, 1 \leq i \leq n \).

Remark 10. In this definition, the value of \( k \) depends on the initial states. Since the set \( \{1,0\} \) is a finite set, we can always choose a constant \( k \) big enough which is independent of the initial states.
Boolean networks to more complicated BNs with impulses.

Corollary 14. Let \( \delta = \delta_z(\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( d = \delta_z(\beta_1, \beta_2, \ldots, \beta_n) \). Since \( \Theta = (F \otimes G)(\Phi_n \otimes I_{2^n}) \), we denote \( \Theta = \delta_{\Theta}(\eta_1, \eta_2, \ldots, \eta_n) \). \( \Phi_n \) is the group power-reducing matrix and \( \Phi_n = \delta_{\Phi_n}(1, 2^n + 1, 2^n + 2, \ldots, 2^n + 2^n - 1, 2^n) \). As \( \Theta \) is defined above, we firstly calculate \( \Phi_n \otimes I_{2^n} \) that

\[
\Phi_n \otimes I_{2^n} = \delta_{\Phi_n}(1, 2^n + 1, 2^n + 2, \ldots, 2^n + 2^n - 1, 2^n).
\]

Remark 12. Theorem 11 presents a necessary and sufficient criterion for synchronization of master–slave BNs with impulsive effects. In [12], it is pointed out that system (3) is equal to system (4). From the proof of Theorem 11, we can see that the synchronization problem of the master–slave BNs with impulsive effects is converted to the problem of states being equal to the system state in system (4) from any initial states to a constrained set: \( S = \{ \delta_{\Theta}(\alpha = (i-1)2^n + j) : i = 1, 2, \ldots, 2^n \} \) after finite time steps.

Remark 13. References [38,39] have investigated the synchronization problem for general Boolean networks without impulsive effects. However, in many real-world systems, states of genes and the evolutionary process can be affected by sudden environmental changes, or abrupt changes. In fact, in some cases, impulses can play a negative role in the issue of synchronization for some dynamical networks. Thus, Boolean networks with impulsive effects are more general than Boolean networks without impulses. And Boolean networks with impulsive effects can capture more features of genetic regulatory networks or biological systems. Theorem 10 presents a necessary and sufficient condition for synchronization of master–slave BNs with impulsive effects that extend the general Boolean networks to more complicated BNs with impulses.

From the proof of Theorem 11, we can also present a necessary synchronization condition for a given master–slave Boolean network with impulsive effects as follows.

Corollary 14. If system (2) is said to be synchronized with its structure matrices being \( F, G, \hat{F} \) and \( \hat{G} \), the following conditions must be satisfied:

\[
\begin{align*}
\Theta \in S & \cup \in C, \\
\Xi \cup \in C,
\end{align*}
\]

where

\[
\begin{align*}
\Theta &= (F \otimes G)(\Phi_n \otimes I_{2^n}), \\
\Xi &= (\hat{F} \otimes \hat{G})(\Phi_n \otimes I_{2^n}), \\
S &= \delta_{\Theta}(\alpha = (i-1)2^n + j) : i = 1, 2, \ldots, 2^n, \\
C &= (\beta \beta = (j-1)2^n + j) : j = 1, 2, \ldots, 2^n.
\end{align*}
\]

In the following, we will present a necessary synchronization condition based on the original matrix property of the structure matrices \( F, G, \hat{F} \) and \( \hat{G} \) of the master–slave BNs without calculating the matrices \( \Theta \) and \( \Xi \) to judge whether a given master–slave BN with impulsive effects can be synchronized or not.

Let \( F = \delta_z(\alpha_1, \alpha_2, \ldots, \alpha_n) \), \( G = \delta_z(\beta_1, \beta_2, \ldots, \beta_n) \). Since \( \Theta = (F \otimes G)(\Phi_n \otimes I_{2^n}) \), we denote \( \Theta = \delta_{\Theta}(\eta_1, \eta_2, \ldots, \eta_n) \). \( \Phi_n \) is the group power-reducing matrix and \( \Phi_n = \delta_{\Phi_n}(1, 2^n + 1, 2^n + 2, \ldots, 2^n + 2^n - 1, 2^n) \). As \( \Theta \) is defined above, we firstly calculate \( \Phi_n \otimes I_{2^n} \) that

\[
\Phi_n \otimes I_{2^n} = \delta_{\Phi_n}(1, 2^n + 1, 2^n + 2, \ldots, 2^n + 2^n - 1, 2^n).
\]

Remark 16. Corollary 15 gives a necessary synchronization condition for a master–slave BNs with impulsive effects. By Corollary 15, for a given master–slave BNs with impulsive effects, we can obtain its four structure matrices \( F, G, \hat{F} \) and \( \hat{G} \). Then by Corollary 15, we can easily conclude that if one of the conditions in (11) is not satisfied, then we know that this given master–slave BNs cannot be synchronized. Thus, Corollary 15 presents a convenient method to judge whether a given master–slave BNs with impulsive effects can be synchronized or not.

4. Numerical example

In this section, we present a numerical example to illustrate our main theoretical results.

Example 17. Let us consider the following master–slave BNs with each BN having two nodes and the impulsive time sequences are
0 < t_0 < t_1 < 4 < t_2 = 6 < \cdots < t_k = 2(k + 1) < \cdots, \ k \in \mathbb{Z}^+ \cup \{0\} \ as \ follows:

\[
\begin{align*}
  x_1(t+1) &= -x_1(t), \\
  x_2(t+1) &= x_1(t) \land (-x_2(t)), \quad t_k - 1 \leq t < t_k - 1, \\
  x_1(t_k) &= x_1(t_k - 1), \\
  x_2(t_k) &= x_2(t_k - 1), \quad k \in \mathbb{Z}^+ \cup \{0\},
\end{align*}
\]

and

\[
\begin{align*}
  y_1(t+1) &= -y_1(t), \\
  y_2(t+1) &= y_1(t) \land (-y_2(t)), \quad t_k - 1 \leq t < t_k - 1, \\
  y_1(t_k) &= x_1(t_k - 1), \\
  y_2(t_k) &= x_2(t_k - 1), \quad k \in \mathbb{Z}^+ \cup \{0\},
\end{align*}
\]

Denote \( x(t) = x_1(t) \land x_2(t) \) and \( y(t) = y_1(t) \land y_2(t) \), by using the semi-tensor product of matrices, we can obtain the following algebraic form:

\[
\begin{align*}
  x(t^1_0) &= Fx(t), \\
  x(t^k_0) &= \hat{F}x(t^k_0 - 1), \quad t_k - 1 \leq t < t_k - 1, \\
  y(t) &= Gx(t^k_0), \quad t_k - 1 \leq t < t_k - 1, \\
  y(t^k_0) &= \hat{G}x(t^k_0 - 1)y(t^k_0 - 1), \quad k \in \mathbb{Z}^+ \cup \{0\},
\end{align*}
\]

where

\[
\begin{align*}
  F &= \delta_4[1, 2, 3, 1], \\
  \hat{F} &= \delta_4[1, 2, 3, 4], \\
  G &= \delta_4[1, 2, 3, 1, 4, 3, 2, 1, 4, 3, 2, 1, 4, 3, 2, 1], \\
  \hat{G} &= \{[11111] \otimes I_2\} = \delta_4[1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4].
\end{align*}
\]

Meanwhile, we can calculate that

\[
\begin{align*}
  \Theta &= \delta_{16}[16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1], \\
  \Xi &= \delta_{16}[1, 1, 1, 1, 6, 6, 6, 6, 11, 11, 11, 11, 16, 16, 16], \\
  \Xi &\Theta = \delta_{16}[16, 16, 16, 16, 16, 11, 11, 11, 11, 6, 6, 6, 6, 1, 1, 1, 1].
\end{align*}
\]

From Eq. (13), we can obtain that \( \text{Col}_{1, 6, 11, 16}(\Xi) \) and \( \text{Col}_{1, 6, 11, 16}(\Theta) \) are in the set of \( C \) and \( \text{Col}(\Xi \Theta) \subseteq C, \) where \( C = \{\delta_{16}[1, 6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]\}. \)

Thus, by Theorem 11, we can obtain that these master–slave BNs can be synchronized. We define the total synchronization errors as

\[
E(t) = |x(t) - y(t)| + |x_2(t) - y_2(t)|.
\]

\textbf{Fig. 1} shows the time evolution of the states \( x_1(t), x_2(t), y_1(t) \) and \( y_2(t) \) of the master–slave BNs (12) and \textbf{Fig. 2} shows the total synchronization error \( E(t) \).

5. Conclusion

In this paper, we have studied the synchronization for master–slave Boolean networks with impulsive effects. First, using the method of semi-tensor product of matrices and the matrix expression of logic, we converted the logical Boolean networks with impulsive effects into impulsive discrete-time algebraic systems. Then a necessary and sufficient condition was proposed for the synchronization of BNs with impulsive effects. Finally, one illustrative example is given to show the validity of our theoretical results. Some interesting and meaningful topics in the near future are to design an appropriate impulsive sequence such that master–slave Boolean networks can be synchronized and to investigate synchronization of multilayer Boolean networks with (or without) impulsive effects.

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