Approaches Toward Restoration of Bilinearly Degraded Images

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Abstract—In response to the practical need for restoring images which are degraded by systems which are linear, this paper focuses on the development and application of tools required for this purpose. When the blurring phenomenon can be modeled by a shift-variant bilinear system, the data restoration problem can be most conveniently formulated as a special system of linear equations with nonnegative coefficients, whose solution is required to satisfy constraints like nonnegativity in addition to it being factorable with the factors having a certain characterizing property. Recursive techniques for restoration are first developed when the blurring system is either causal or weakly causal. It is shown how these recursive techniques when applied several times and the solutions superposed can, sometimes, be used to restore images degraded by noncausal blurs. Algorithms based on noniterative and iterative schemes are, subsequently, developed to tackle directly the noncausal blurs. Performances of the various algorithms when applied to noisy images are briefly compared.

I. INTRODUCTION

THE restoration of images which have been degraded is a key problem in image processing. This is normally required following the acquisition of sensor data and prior to detailed image analysis and understanding. Many physical processes responsible for image distortion or degradation may be modeled by systems which are linear (shift-invariant or shift-variant), and various restoration procedures exist in these situations. Very recently, an efficient recursive scheme has been described [1] to recover images corrupted by blurs that may be modeled by linear shift-variant systems. Although a large number of physical processes might be satisfactorily modeled via linear systems, the constraint of linearity is too stringent in many other situations. One method for representing the input/output behavior of nonlinear systems is via the Volterra series [2, pp. 382–386]. The second-order term of the Volterra series expansion is a special case of the input/output representation of a one-dimensional (1-D) bilinear system described by the equation

\[ g(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(x; x_1, x_2) f_1(x_1) f_2(x_2) \, dx_1 \, dx_2 \]  

where \( g(x) \) is the output at \( x \), \( f_i(x_i) \) is the input at \( x = x_i \) for \( i = 1, 2 \), and \( q(x; x_1, x_2) \) specifies the double impulse response (DIR) of the bilinear system at the output coordinate \( x \) due to unit impulses at input coordinates \( x_1 \) and \( x_2 \). The special case, referred to above, of interest here is the single-input (in contrast to two inputs) single-output system described by (for the 1-D case)

\[ g(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(x; x_1, x_2) f(x_1) f(x_2) \, dx_1 \, dx_2. \]

In the optical systems literature [3], the preceding input/output description, which has a natural \( n \times \) generalization for \( n > 1 \) (see [4] for the \( n = 2 \) case), has been referred to as the bilinear representation (although it is a special case of it). To avoid any confusion and horror brevity, we will continue to use that notation in our paper, unless explicitly mentioned otherwise. Some applications that require modeling via (2) include coherent imaging through the turbulent atmosphere [3], imaging by optical systems with time-varying pupils [4], high resolution X-ray imaging system where the object is illuminated by partially coherent waves [5], and imaging on translucent substrates with thin absorbing patterns on its surface and noncoherently illuminated [6].

Several restoration schemes for bilinearly distorted images have been proposed. These extend from suboptimal deterministic restoration methods [5, 7] to nonlinear statistical restoration schemes [8]. The feasibility of applying the iterative technique for finding the inverse operator of a multilinear map in [9] to the nonlinear image restoration problem has been pointed out in [10].

In this paper, several different approaches towards restoring bilinearly degraded images are considered. Section II provides a link to the earlier work described in [1]. The recursive scheme developed in [1] for restoring images degraded by linear shift-variant systems may, in principle, be used to restore bilinearly degraded images, but the computational cost is very high. This paper, therefore, emphasizes the development of more suitable techniques for coping with the bilinear degrading phenomenon. In Section III, a recursive procedure to restore 1-D and 2-D causal or weakly causal bilinearly degraded images is considered. When the constraints of causality or weak causality are violated, direct recursive schemes for restoration do not exist. Although optical systems are, in general, noncausal, it is, however, often possible to ex-

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ploit the advantages of recursion by decomposing a noncausal restoration problem into several causal or weakly causal restoration problems, to each of which the recursive technique is applied and then the results are superposed. Section IV is devoted to the development of a technique for handling general noncausal blurs and a noniterative as well as an iterative implementation are described. To avoid cluttering of the main issues, the proof of the main theorem in this section is given in Appendix A. In Section V, the presence of signal-independent additive noise is taken into account for the recursive method of Section III, while a similar noise analysis for the method of Section IV is included in Appendix B. Important conclusions are drawn in the final section. Inferences from the results of implementation on test cases in addition to theoretical analysis support these conclusions.

II. RESTORATION OF 1-D BILINEARLY DEGRADED IMAGES VIA THE 2-D LINEAR MODEL

The possibility of deducing certain properties of \( n-D \) bilinear systems from the analysis of corresponding \( 2n-D \) linear systems is well known [2, p. 219]. This fact encourages the investigation of the fast recursive restoration scheme for linear shift-variant systems in [1] to the situation under consideration here. The discrete finite counterpart of (2) is

\[
g(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) f(m_1) f(m_2),
\]

where \( \{g(n)\} \) and \( \{f(m)\} \) are, respectively, the output and input sequences, assumed real, \( q(n; m_1, m_2) \) is the discrete system DIR at the output coordinate \( n \) due to unit impulses at the input coordinates \( m = m_1 \) and \( m = m_2 \), and \( N \) is the finite number of equispaced points at which the signal is sampled. Note that \( q(n; m_1, m_2) \) is, in general, complex with [3]

\[
q(n; m_1, m_2) = q^*(n; m_2, m_1),
\]

and also [6]

\[
\text{Re} \{q(n; m_1, m_2)\} \geq 0.
\]

The output sequence \( \{g(n)\} \) satisfies the nonnegativity constraint

\[
g(n) \geq 0,
\]

and the objective is to recover \( \{f(m)\} \) in (3) subject to the additional constraint

\[
f(m) \geq 0.
\]

In order to provide a link with the result in [1], we define here the "causal" restriction of (3) to be

\[
g(n) = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} q(n; m_1, m_2) f(m_1) f(m_2),
\]

which implies that in the causal case

\[
q(n; m_1, m_2) = 0, \quad m_1 > n \text{ and/or } m_2 > n.
\]

Although the restriction imposed in (9) is invalid in many applications, it is often possible to decompose a problem described by (3) into a sum of problems that are describable by (8), each of which, as will be seen later, can be recursively tackled. Recursive solutions, although not generally applicable, have several advantages. The input/output 1-D bilinear system description in (8) is a special case of the following 2-D linear system input/output relationship:

\[
y(n_1, n_2) = \sum_{m_1=0}^{m_2} \sum_{m_2=0}^{m_2} h(n_1, n_2; m_1, m_2) x(m_1, m_2),
\]

where \( h(n_1, n_2; m_1, m_2) \) has been defined as follows:

\[
y(n_1, n_2) = \sum_{m_1=0}^{m_2} \sum_{m_2=0}^{m_2} h(n_1, n_2; m_1, m_2) x(m_1, m_2),
\]

with the proviso

\[
q(n; m_1, m_2) = h(n, n; m_1, m_2),
\]

\[
0 \leq n, m_1, m_2 \leq N - 1
\]

and

\[
g(n) = y(n, n),
\]

\[
0 \leq n \leq N - 1.
\]

Given the sequences \( \{g(n)\} \) and \( \{q(n; m_1, m_2)\} \), the objective is to recover \( \{f(m_1) f(m_2)\} \) (and consequently, the input sequence \( \{x(m_1, m_2)\} \) by finding \( \{x(m_1, m_2)\} \) from (10) after substituting (11) and (13) in (10) and, finally, making use of (12). The input/output description of the 2-D linear system in (10) can be recast into the following system of linear equations:

\[
y = Hx
\]

where the elements of \( y \) and \( x \) have been lexicographically ordered as,

\[
y = [y(0, 0), y(0, 1), \ldots, y(0, N - 1), y(1, 0), \ldots, y(1, N - 1), \ldots, y(N - 1, N - 1)]^T
\]

\[
x = [x(0, 0), x(0, 1), \ldots, x(0, N - 1), x(1, 0), \ldots, x(1, N - 1), \ldots, x(N - 1, N - 1)]^T
\]

while a typical element \( h(n_1, n_2; m_1, m_2) \), \( 0 \leq n_1, n_2, m_1, m_2 \leq N - 1 \), occurs in the \((n_1N + n_2) + 1\)th row and \((m_1N + m_2 + 1)\)th column of the \( N^2 \times N^2 \) matrix \( H \).

Note that \( H \) is lower-triangular because of the restriction of causality. In order to satisfy (11), \( h(n_1, n_2; m_1, m_2) \) has been defined as follows:
It was seen in [1] that (14) could be solved recursively for $x$ if and only if $h(n_1, n_2; n_1, n_2) \neq 0$. The consequent constraint imposed, namely, that

$$q(n; n, n) \neq 0,$$

(18)

will be assumed to hold. A restoration algorithm based on an iterative procedure that makes possible the recovery of $x(m_1, m_2)$ in (12) subject to the nonnegativity constraint $f(m) \geq 0$, by solving (14), is fully described in [11].

The proof for convergence of this algorithm is contained in [11]. Although the algorithm uses as a core component the fast technique for 2-D linear shift-variant systems, described in [11], it can be quite slow in restoring bilinearly degraded images because of the large number of iterations that might be necessary to reduce the error in restoration below an acceptable level. Also, the restoration of 2-D bilinearly degraded images requires the use of 4-D linear models. Although this is possible in principle from the results in [11], the space-time computational complexity can be intolerably high. Therefore, in the subsequent sections, alternate methods for attaining the desired objective are explored.

III. Recursive Restoration of Bilinearly Degraded Images

A. 1-D Image Restoration

A direct recursive restoration procedure of 1-D images, degraded by the bilinear system in (8) subject to the constraints in (5), (6), and (7), will be presented in this section. The input/output description of the 1-D bilinear system in (8) can be rewritten as

$$g(n) = q(n; n, n) f^2(n) + \left[ \sum_{m=0}^{n-1} \left\{ q(n; m, n) + q(n; n, m) \right\} f(m) \right] f(n) + \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{n-1} q(n; m_1, m_2) f(m_1) f(m_2).$$

(19)

Using (4), (5), and (7), it is easy to see that

$$q(n; n, n) f^2(n) + \left[ \sum_{m=0}^{n-1} \{ q(n; m, n) \} f(m) \right] f(n) \geq 0.$$  

(20)

From (19) and (20), the inequality in (21) follows.

$$g(n) - \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{n-1} q(n; m_1, m_2) f(m_1) f(m_2) \geq 0.$$  

(21)

Define

$$A_n \triangleq q(n; n, n),$$

(22)

$$B_n \triangleq \sum_{m=0}^{n-1} \{ q(n; m, n) + q(n; n, m) \} f(m),$$

(23)

$$C_n \triangleq g(n) - \sum_{m_1=0}^{n-1} \sum_{m_2=0}^{n-1} q(n; m_1, m_2) f(m_1) f(m_2).$$

(24)

Using (5), (7), (18), and (21), it follows that

$$A_n > 0, \quad B_n \geq 0, \quad \text{and} \quad C_n \geq 0.$$  

(25)

$C_n$ can be obtained by implementing the following recursion:

$$\hat{C}_n^{(0)} = 0$$

(26)

$$\hat{C}_n^{(k)} = \hat{C}_n^{(k-1)} + \sum_{m=0}^{k-2} \{ q(n; m, k - 1) + q(n; k - 1, m) \} f(m) f(k - 1)$$

$$+ q(n; k - 1, k - 1) f^2(k - 1),$$

$$k = 1, 2, \ldots, n.$$  

(27)

$$C_n = g(n) - \hat{C}_n^{(n)}.$$  

(28)

The expression (19) can be rewritten in terms of $A_n$, $B_n$, and $C_n$:

$$A_n f^2(n) + B_n f(n) - C_n = 0.$$  

(29)

It is easy to see that

$$f(0) = \sqrt{C_0/A_0}.$$  

Suppose that the values for $f(m) \geq 0$, $m = 0, 1, \ldots, n - 1$, have been obtained; then, $A_n$, $B_n$, and $C_n$ can be computed by using (22), (23), and (24). Hence, $f(n)$ can be obtained as the nonnegative solution of (29). After the recursion is completed, the deconvolved sequence $\{ f(m) \}_{m=0}^{n-1}$ is uniquely obtained.

B. 2-D Image Restoration

For a 2-D discrete bilinear system, the counterpart of the 1-D input/output description in (8) is

$$y(n_1, n_2) = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} q(n_1, n_2; m_1, m_2, l_1, l_2) x(m_1, m_2) x(l_1, l_2)$$

(30)

where all the notations are self-evident. The counterpart of (29) will be of the form

$$A_{n_1, n_2} f_1^2(n_1, n_2) + B_{n_1, n_2} x(n_1, n_2) - C_{n_1, n_2} = 0.$$
The recursive implementation analogous to (26), (27), and (28) also can be obtained. Since (30) characterizes a quar-
terplane 2-D discrete system, the row-by-row recursion, i.e., \((n_1, n_2) \rightarrow (n_1 + 1, n_2) \rightarrow \cdots \rightarrow (N - 1, n_2) \rightarrow (0, n_2 + 1) \rightarrow (1, n_2 + 1) \rightarrow \cdots\), and the column-by-
column recursion are two of several possibilities for im-
plementing the recursion. The counterpart of (30) for the
nonsymmetric half-plane case is described in \[11\].

It was mentioned earlier that noncausal systems, not
describable by (8) or (30), are common in the optical lit-
erature. The next example illustrates how the algorithm
described in this section can be used to restore images
degraded by special types of noncausal blurs.

**Example 1:** The coma type of blur described in \[1\] is
used to synthesize a 2-D bilinear system DIR. Specifically
\[
q(n_1, n_2; m_1, m_2, l_1, l_2)
= h(n_1, n_2; m_1, m_2) h^*(n_1, n_2; l_1, l_2) \gamma(m_1, m_2, l_1, l_2)
\]  
(31)

where the field’s coherence function,
\[
\gamma(m_1, m_2, l_1, l_2) = \text{sinc} \left( \frac{(m_1 - l_1)}{4} \right) \text{sinc} \left( \frac{(m_2 - l_2)}{4} \right)
\]  
and the system’s coherent impulse response function
\[
h(n_1, n_2; m_1, m_2) = \sqrt{h_{s}(n_1, n_2; m_1, m_2)}.
\]  
(32)

In (32), \(h_{s}(n_1, n_2; m_1, m_2)\) is the point-spread function for
a lens with coma aberration; it has quadrantal symmetry
and its shape in each quadrant is described by
\[
h_{s}(n_1, n_2; m_1, m_2) = [1/(m_1\Delta^2 + m_2\Delta^2)] h_0(x_1, x_2)
\]  
(33)

where \(\Delta\) denotes the sampling distance and
\[
x_1 = (m_1 n_1 + m_2 n_2)/(m_1^2 + m_2^2)
\]  
\[
x_2 = (m_1 n_2 - m_2 n_1)/(m_1^2 + m_2^2)
\]
and regions I and II are defined in Fig. 1. From Fig. 1, it is clear that by constraining $R_0$ to be less than a certain value, it is possible to ensure that the $i$th quadrant input, $i = 1, 2, 3, 4$, does not affect the $j$th quadrant ($i \neq j$) output. This analysis has been done in [1]. Then, each segmented image is blurred bilinearly by using (30) and (31). Fig. 2 shows the original $31 \times 31$ image and Fig. 3(a) shows the blurred image. To simulate the blurred image, $\Delta$ in (33) has been chosen as 0.254. By applying the algorithm in Section III-A to each quadrant, and by superposing those results, the restored image shown in Fig. 3(b) is obtained.

IV. RESTORATION OF NONCAUSAL BILINEARLY DEGRADED IMAGES

A. Noniterative Method

In this section, the problem of restoring an image, degraded by the bilinear system described by (3) subject to the constraints (5), (6), and (7), will be discussed. By using (4) and the realness of $f(m)$, (3) can be rewritten as

$$g(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \text{Re} \left[q(n; m_1, m_2) f(m_1) f(m_2)\right],$$

$$n = 0, 1, \ldots, N - 1.$$  

Without loss of any generality, the bilinear system DIR

$$Q_n \triangleq \begin{bmatrix} q(n; 0, 0) & q(n; 0, 1) & \cdots & q(n; 0, N-1) \\ q(n; 1, 0) & q(n; 1, 1) & \cdots & q(n; 1, N-1) \\ \vdots & \vdots & \ddots & \vdots \\ q(n; N-1, 0) & q(n; N-1, 1) & \cdots & q(n; N-1, N-1) \end{bmatrix}$$

$n = 0, 1, \ldots, N - 1.$

will, henceforth, be assumed to be real. The notations used in this section are introduced next.

Notations:

$f$ (boldface lowercase): a column vector

$b_f$: a square or rectangular matrix

$Q_n$: a matrix $Q$ indexed by $n$

$f_i$: the $i$th element of the vector $f$

$(B)_i$: the $i$th row of the matrix $B$

$\langle f, g \rangle = \Sigma_{i=1}^{N} (f_i)(g_i)$: the Euclidean innerproduct of vectors $f$ and $g$

$R^N$: the $N$-dimensional Euclidean space.

The $(N \times N)$ nonnegative matrix $Q_n$ is first defined.

With $Q_n$ in (34), (3) can be rewritten as

$$g(n) = f^T Q_n f, \quad n = 0, 1, \ldots, N - 1$$

where

$$f = [f(0), f(1), \cdots, f(N-1)]^T.$$
Let
\[(B)_i \triangleq [(Q)_{i-1}^1, (Q)_{i-2}^1, \cdots, (Q)_{i-N}^1], \quad i = 1, 2, \cdots, N \tag{37}\]
Clearly, \((B)_i\) has 1 row and \(N^2\) columns and it is formed by concatenating sequentially the rows of \((Q)_{i-1}\). Let
\[X \triangleq f f' \tag{38}\]
and
\[x \triangleq [(X)_1, (X)_2, \cdots, (X)_N]' \tag{39}\]
Then, from (35),
\[g(n) = \langle(B)_{n+1}' x \rangle, \quad n = 0, 1, \cdots, N - 1. \tag{40}\]
Letting
\[g \triangleq [g(0), g(1), \cdots, g(N - 1)]', \tag{41}\]
the following matrix–vector representation for (3) is obtained in (42):
\[g = Bx. \tag{42}\]
Therefore, the image restoration problem of interest here requires the finding of the nonnegative vector \(f\) which satisfies (38), (39), and (42).

Consider a symmetric matrix \((Q)_n\) of rank 1, each of whose elements are nonnegative. It is always possible to factor \((Q)_n\) as
\[(Q)_n = d_n d_n', \quad n = 0, 1, \cdots, N - 1, \tag{43}\]
where \(d_n\) is nonnegative (i.e., each element of vector \(d_n\) is nonnegative) [12, pp. 6–8]. \(f, B, X, x,\) and \(g\) are defined as in (36)–(42).

**Fact 1:** Define an \(N \times 1\) vector \(w,\) whose elements are
\[(w)_i = \sqrt{(g)_i}, \quad i = 1, 2, \cdots, N \tag{44}\]
and an \(N \times N\) matrix \(U\)
\[U \triangleq \begin{bmatrix}
    d_0^n \\
    d_1' \\
    \vdots \\
    d_{N-1}'
\end{bmatrix} \tag{45}\]
If \(f\) is a solution of
\[Uf = w, \tag{46}\]
then \(x\) obtained from \(f\) via (38) and (39) will be a solution of (42).

**Proof:** Equation (46) implies that
\[\langle d_{i-1}', f \rangle = (w)_i. \]
On squaring both sides of the preceding equation, and then using successively (44), (43), and (40), one obtains for \(i = 1, 2, \cdots, N,\)
\[(g)_i = g(i - 1) = f' d_{i-1} d_{i-1}' f = f' Q_{i-1} f = \langle(B)_{i}', x \rangle. \]
Therefore, if \(f\) is a solution of (46), then \(x\) is a solution of (42).

Fact 1, when applied to the image restoration problem, requires the solution \(f\) [from (46)] to be nonnegative. For this to be possible it is necessary that \(w\) be nonnegative. Associated with the samples \(g(0), g(1), \cdots, g(N - 1)\) in \(g,\) there exists a unique nonnegative \(w\) whose elements satisfy (44). It should be recognized, however, that a nonnegative \(w\) and a nonnegative \(U\) do not guarantee a nonnegative solution \(f\) in (46). Necessary and sufficient conditions for \(f\) in (46) to be nonnegative are discussed in [13].

When the matrix \((Q)_n\) in (34) is not of rank 1 but satisfies the remaining conditions, \((Q)_n\) will be approximated to the desired form in (43). Two definitions are introduced prior to the enunciation of the theorem describing this approximation.

**Definition 1:** The Euclidean matrix norm \(\|D\|_E\) of a matrix \(D\) is defined as
\[\|D\|_E = \sqrt{\text{Tr} \{D'D\}}, \tag{47}\]
where \(\text{Tr} A\) denotes the trace of a square matrix \(A.\)

**Definition 2:** Let a set of \(N \times N\) matrices \(S\) be defined as
\[S \triangleq \{D : D = dd', \quad d \in R^N\}. \tag{47}\]
Then, by "the best approximation of an \(N \times N\) matrix \(Q\) on \(S,\)" we mean the matrix \(D^*\) satisfying
\[\|Q - D^*\|_E = \min_{D \in S} \|Q - D\|_E. \tag{48}\]

The above two definitions can be explained as follows. Let
\[a = [(Q)_{i1}, (Q)_{i2}, \cdots, (Q)_{iN}]', \tag{44}\]
and
\[b = [(D)_{i1}, (D)_{i2}, \cdots, (D)_{iN}]', \tag{48}\]
where \(D = dd', \quad d \in R^N\). Then, the best approximation \(D^*\) of a matrix \(Q\) on \(S\) requires that the associated vector \(b^*\), which has the form of (48), minimizes the usual Euclidean vector norm of \([a - b]\), i.e.,
\[\|a - b^*\| = \min_{b} \|a - b\|. \tag{49}\]

**Theorem:** Let \(S\) be the set of \(N \times N\) matrices as in (47). Then, the best approximation \(D^*\) of a symmetric nonnegative matrix \(Q\) on \(S\) is given by
\[D^* = \lambda_1 v_1 v_1', \tag{49}\]
where \(\lambda_1\) and \(v_1\) are, respectively, the dominant eigenvalue and the dominant eigenvector of the matrix \(Q.\)

The proof of the theorem is given in the Appendix A. By applying the theorem to \((Q)_n, \quad n = 0, 1, \cdots, N - 1,\) the best approximant on \(S\) can be obtained. Let \(\lambda_{n1}\) and \(v_{n1}\) denote, respectively, the dominant eigenvalue and eigenvector of \((Q)_n.\) Then, by the Perron–Frobenius theorem [12, pp. 6–8], \(\lambda_{n1}\) and \(v_{n1}\) are all nonnegative. Let
\[(E)_{n+1} \triangleq [(D)_1^n, (D)_2^n, \cdots, (D)_N^n], \tag{50}\]
where
\[D_n = d_n d_n', \quad n = 0, 1, \cdots, N - 1 \tag{51}\]
and
\[ d_n = \sqrt{\lambda_n} v_{n1}, \quad n = 0, 1, \cdots, N - 1. \] (52)

Let \( b_n \) and \( c_n \) denote, respectively, \((B)_n^t + 1\) and \((E)_n^t, n = \, 0, 1, \cdots, N - 1\). Then (40) can be rewritten as
\[ g(n) = \langle b_n, x \rangle, \quad n = 0, 1, \cdots, N - 1. \] (53)

By applying the theorem, the approximated version of (53) is given by
\[ g(n) \approx \langle c_n, x \rangle, \quad n = 0, 1, \cdots, N - 1. \] (54)

The left-hand side of (54) can be regarded as an estimated value for
\[ g^*(n) \triangleq \langle c_n, x \rangle, \quad n = 0, 1, \cdots, N - 1. \]

This estimation can be improved by considering the fact that (42) or (53) enables us to compute the inner product value of \( b \) and \( x \) provided that \( b \in \mathbf{R}(B') \), where
\[ \mathbf{R}(B') \triangleq \{ z : z = B'y, y \in \mathbf{R}^N \}. \]

Consider an estimate of \( g^*(n) \) given in (55). Let \( \hat{e}_n \) be the projection of \( e_n \) on \( \mathbf{R}(B') \).
\[ \hat{g}(n) \triangleq \langle \hat{e}_n, x \rangle, \quad n = 0, 1, \cdots, N - 1. \] (55)

Replacing \( g(n) \) in (54) by \( \hat{g}(n) \) from (55), one obtains
\[ \hat{g}(n) \approx \langle \hat{e}_n, x \rangle, \quad n = 0, 1, \cdots, N - 1. \] (56)

Consider next the following error functions:
\[ E_1(x) \triangleq \sum_{n=0}^{N-1} [g^*(n) - g(n)]^2 = \sum_{n=0}^{N-1} [\langle e_n, x \rangle - \langle b_n, x \rangle]^2 = \sum_{n=0}^{N-1} [\langle e_n - b_n, x \rangle]^2 \]

and
\[ E_2(x) \triangleq \sum_{n=0}^{N-1} [g^*(n) - \hat{g}(n)]^2 = \sum_{n=0}^{N-1} [\langle e_n - \hat{e}_n, x \rangle]^2. \]

Since \( x \) is an indeterminate, it may be, sometimes, advantageous to accept \( E_2(x) \) as an error function because of the inequality established next (It should, however, be noted that the preceding argument is heuristic.)
\[ \| e_n - b_n \|^2 = \| \hat{e}_n - b_n \|^2 + \| e_n - \hat{e}_n \|^2 \geq \| e_n - \hat{e}_n \|^2. \]

The above inequality follows from the orthogonality of \( \hat{e}_n - b_n \) and \( e_n - \hat{e}_n \). It is easy to see that
\[ \hat{g}(n) = \langle \hat{e}_n, x \rangle = \langle c_n, \hat{x} \rangle, \] (57)

where \( \hat{x} \in \mathbf{R}(B') \) is the minimum norm solution of (42). By applying to (42) the result stated in the theorem and fact 1, together with (56) and (57), the following algorithm is obtained.

Algorithm:

Step 1: Obtain \( N \times 1 \) vectors, \( d_n, n = 0, 1, \cdots, N - 1 \), as in (52). Form \( N^2 \times 1 \) vectors, \( e_n, n = 0, 1, \cdots, N - 1 \), by
\[ e_n = [(D_n)_{11}, (D_n)_{12}, \cdots, (D_n)_N]^t, \] (58)

where \( D_n \) is given in (51).

\[ \langle (B)_i, \hat{x} \rangle = \langle g \rangle, \quad i = 1, 2, \cdots, N, \] (60)

where \( (B)_i \) and \( g \) are defined in (37) and (41).

Proof: To prove (60), \( x \) is decomposed as
\[ x = \hat{x} + \hat{x}, \] (61)
where \( \hat{x} \in R(\mathcal{B}') \) and \( \mathcal{N} = N(\mathcal{B}) \), where the null space \( N(\mathcal{B}) \) is defined as

\[ N(\mathcal{B}) \triangleq \{ y : By = 0, y \in R^{N^2} \}. \]

Premultiplying \( \mathcal{B} \) on both sides of (61), one obtains

\[ Bx = B\hat{x} + B\hat{x} = B\hat{x}. \]

Substituting (42) in the preceding equation, one obtains

\[ B\hat{x} = g, \]

which proves the stated fact.

It is clear from facts 1 and 2 how the vector \( \tilde{\mathcal{O}} \) can be constructed from its projection \( \tilde{\mathcal{O}} \) by adding the null component to this projection. More specifically, given \( \tilde{\mathcal{O}} \), construct \( \hat{g} = E\hat{x} \), where \( E \) has been found by applying the theorem to \( \tilde{\mathcal{O}} \), generated from \( B \) as explained earlier. Then fact 1 is applied to construct \( \tilde{\mathcal{O}} = x \), which is of the required factorable form [see (38) and (39)].

The projection of \( \mathcal{O} \) on \( R(E') \) is represented by

\[ \mathcal{B} = (E'E)\mathcal{O}, \]

where \( E' \) denotes the Moore–Penrose inverse of \( E \), since \( E'E \) is idempotent and the projection operator on \( R(E') \) [14]. In order to obtain the system equation in the form of \( g = Ex \) which is the approximated version of (42), the value \( (E)(\mathcal{O}) \) needs to be computed.

\[ (E)(\mathcal{O}) = (E)(E+E)(\mathcal{O}) = (E)(\mathcal{O}), \]

since \( EE'E = E \) [14]. \( (E)(\mathcal{O}) \) represents the innerproduct operation in step 2. Hence, this step includes the orthogonal projection operation on \( R(\mathcal{E}') \). The innerproduct results in (59) may be negative. But since \( \langle e_n, x \rangle \) is an estimate for \( \langle e_n, x \rangle \) and from (38), (39), (51), and (58), \( \langle e_n, x \rangle = (\langle d_n, f \rangle)^2 \geq 0 \), \( \langle e_n, x \rangle \) can be clipped to zero whenever \( \langle e_n, x \rangle \) is negative.

The 2-D bilinear system representation corresponding to (3) is given by

\[ y(n_1, n_2) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} q(n_1, n_2, m_1, m_2, l_1, l_2) \]

\[ x(m_1, m_2) x(l_1, l_2). \]

(62)

Initially, the following form of the 2-D DIR is considered:

\[ q(n_1, n_2, m_1, m_2, l_1, l_2) = q(n_1, m_1, l_1) q(n_2, m_2, l_2). \]

(63)

In the Kähler illumination system [15, pp. 524–526] with an incoherent square source, the 2-D field’s coherence function is of the form,

\[ \gamma(m_1, m_2, l_1, l_2) = \gamma(m_1, l_1) \gamma(m_2, l_2) \]

where \( \gamma(m, l) = C_1 \cdot \text{sinc} \left[ C_2 \cdot (m - l) \right] \) and \( C_1 \) and \( C_2 \) are constants. It is well known [15, pp. 392–395] that the Fraunhofer diffraction pattern for a square aperture results in a product separable form of the coherent impulse response, i.e.,

\[ h(n_1, n_2, m_1, m_2) = h(n_1, m_1) h(n_2, m_2) \]

where \( h(n; m) = C_3 \cdot \text{sinc} \left[ C_4 \cdot (n - m) \right] \) and \( C_3 \) and \( C_4 \) are constants. Hence, the optical microscopic imaging system of the square incoherent source will have a DIR of the form in (63).

When the 2-D DIR is as in (63), then (62) can be rewritten as

\[ y(n_1, n_2) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} q(n_1, m_1, l_1) q(n_2, m_2, l_2) \]

\[ x(m_1, m_2) x(l_1, l_2). \]

\[ = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} q(n_1, m_1, l_1) \]

\[ x(m_1, m_2) x(l_1, l_2). \]

(64)

The minimum norm solution, \( \hat{d}_{n_1, n_2, l_1, l_2}, n_1 = 0, 1, \cdots, N-1, \) can be obtained via the application of step 2 in the 1-D algorithm. Define

\[ q_{n_1, n_1} = q(n_1; i - 1, j - 1), \]

\[ 0 \leq n_2 \leq N - 1, \quad \text{and} \quad 0 \leq i, j \leq N. \]

Then, as in (52), the \( N \times 1 \) vectors, \( d_{n_2}, n_2 = 0, 1, \cdots, N - 1, \) are obtained. With fixed \( n_1 \),

\[ \hat{y}(n_1, n_2) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} d(n_1, m_1, l_1) d(n_2, m_2, l_2) \hat{w}(n_1, m_1, l_1), \]

\[ n_2 = 0, 1, \cdots, N - 1, \]

(65)

where \( d(n; m) \) is the \( m \)th element value of the vector \( d_n \), can be obtained. By substituting (64) for \( \hat{w}(n_1, m_1, l_1) \), (65) can be rewritten as (for \( n_1 = 0, 1, \cdots, N - 1 \))

\[ \hat{y}(n_1, n_2) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \sum_{l_1=0}^{N-1} \sum_{l_2=0}^{N-1} q(n_1, m_1, l_1) \sum_{m_2=0}^{N-1} \sum_{l_2=0}^{N-1} \]

\[ \cdot d(n_2, m_2) d(n_2, l_2) x(m_1, m_2) x(l_1, l_2). \]

(66)

Let

\[ z(m_1, m_2) \triangleq \sum_{m_2=0}^{N-1} d(n_2, m_2) x(m_1, m_2). \]

(67)

Then (66) can be rewritten as

\[ \hat{y}(n_1, n_2) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n_1, m_1, l_1) z(m_1, n_2) z(l_1, n_2). \]

(68)
By applying 1-D algorithm to (68) for the fixed value of $n_2$, $n_2 = 0, 1, \ldots, N - 1$, $z(m_1, n_2)$, $m_1 = 0, 1, \ldots, N - 1$, is obtained. Then, for any fixed $m_1$, the 1-D algorithm is used to compute $x(m_1, m_2)$ in (67). The 2-D algorithm with DIR of the form in (63) can easily be summarized in a flowchart [11].

The partially coherent imaging system characterized by a DIR of the type in (63) is used to construct the examples. This DIR is fully described by

$$q(n; m_1, m_2) = h(n; m_1) h^*(n; m_2) \gamma(m_1, m_2), \quad (69)$$

where $h(n; m)$, the coherent impulse response, is

$$h(n; m) = \begin{cases} \text{sinc} \left[ (n - m)/N_h \right], & |n - m| < N_h \\ 0, & \text{otherwise} \end{cases} \quad (70)$$

and $\gamma(m_1, m_2)$, the field’s coherence function, is

$$\gamma(m_1, m_2) = \text{sinc} \left[ (m_1 - m_2)/N_c \right]. \quad (71)$$

It is easy to see that the conditions

$$N_s \geq N_h - 2 \quad \text{and} \quad N_c \geq 2N_h - 2$$

are sufficient for the DIR in (63) and (69)-(71) to satisfy the nonnegativity condition in (3). To evaluate the restoration performance, the performance factor $c$ has been computed by

$$c = \sqrt{e_b/e_a}, \quad (72)$$

where $e_a$ and $e_b$ denote, respectively, the mean-squared errors after restoration and before restoration. The expressions for $e_a$ and $e_b$ are

$$e_a = \left[ 1/N^2 \right] \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} [u^2(m_1, m_2) - u^2(m_1, m_2)]^2$$

$$e_b = \left[ 1/N^2 \right] \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} [y(m_1, m_2) - u^2(m_1, m_2)]^2,$$

where $y(m_1, m_2)$, $u^2(m_1, m_2)$, and $u^2(m_1, m_2)$ denote, respectively, the intensity of the degraded image, the original image, and the restored image.

**Example 2:** The algorithm described in (62)-(68) is implemented on an image generated by convolving a given image with a system characterized by a DIR specified in (63) and (69)-(71). Fig. 2 shows the 31 $\times$ 31 original image. Fig. 5(a) is the degraded image when $N_h = 4$, $N_s = 8$, and $N_c = 32$, representing the relatively coherent case [16]. The algorithm is applied to obtain the restored image, shown in Fig. 5(b). The calculated performance factor is 22.40.

**Example 3:** The image in Fig. 2 has been blurred by the system having the same DIR as in example 2 but with a different value for $N_s$. Here, to represent the partially coherent case [16], $N_s$ has been assigned the value 8. The degraded image is shown in Fig. 6(a) and the restored image in Fig. 6(b). The performance factor is 2.51. The restoration is not as good as in the previous example because the degrading system belongs to the partially coherent class and not the relatively coherent class.

**B. Iterative Method**

In this section, an iterative method to obtain the solution $\{f(m)\}_{m=0}^{N-1}$ of (3), starting from the approximate so-
lution \( \{ \hat{f}(m) \}_{m=0}^{N-1} \), obtained via the application of the algorithm in Section IV-A, will be discussed. Let \( \{ f^{(k)}(m) \}_{m=0}^{N-1} \) denote values of \( \{ f(m) \}_{m=0}^{N-1} \) at the kth iteration. Initially, define
\[
f^{(0)}(m) \triangleq \hat{f}(m), \quad m = 0, 1, \cdots, N - 1,
\]
and call this, for convenience later, as the zeroth iteration. Let
\[
\Delta f^{(k)}(m) \triangleq f(m) - f^{(k)}(m), \quad m = 0, 1, \cdots, N - 1, \quad k = 0, 1, 2, \cdots.
\]
Then, (3) can be rewritten as
\[
g(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) [f^{(k)}(m_1)] f^{(k)}(m_2)
+ \Delta f^{(k)}(m_1)] [f^{(k)}(m_2) + \Delta f^{(k)}(m_2)],
\]
\[
n = 0, 1, \cdots, N - 1, \quad k = 0, 1, 2, \cdots.
\]
Assuming that the second-order terms involving the differentials \( \Delta f^{(k)}(m) \) are negligible, the preceding expression, using (4), can be rewritten as
\[
g^{(k)}(n) \triangleq \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} \text{Re} [q(n; m_1, m_2)] [f^{(k)}(m_1) f^{(k)}(m_2)]
+ 2f^{(k)}(m_1) \Delta f^{(k)}(m_2)],
\]
\[
n = 0, 1, \cdots, N - 1, \quad k = 0, 1, 2, \cdots. \quad (73)
\]
Noting that the DIR was assumed to be real in this section, we define,
\[
g^{(k)}(n) \triangleq \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) f^{(k)}(m_1) f^{(k)}(m_2),
\]
and
\[
r^{(k)}(n) \triangleq g^{(k)}(n) - g(n).
\]
Then, by replacing \( g(n) \) in \( r^{(k)}(n) \) by the right-hand side of (73), one obtains
\[
r^{(k)}(n) \approx - \sum_{m_2=0}^{N-1} \left[ 2 \sum_{m_1=0}^{N-1} q(n; m_1, m_2) f^{(k)}(m_1) \right] \Delta f^{(k)}(m_2).
\]
Let
\[
w^{(k)}(n; m_2) \triangleq 2 \sum_{m_1=0}^{N-1} q(n; m_1, m_2) f^{(k)}(m_1), \quad (75)
\]
Then, (75) can be rewritten as
\[
r^{(k)}(n) \approx - \sum_{m_2=0}^{N-1} w^{(k)}(n; m_2) \Delta f^{(k)}(m_2), \quad (76)
\]
After solving for \( \Delta f^{(k)}(m_2) \) in (76), \( f^{(k+1)}(m) \) will be obtained as follows:
\[
f^{(k+1)}(m) = \begin{cases} f^{(k)}(m) + \Delta f^{(k)}(m), & f^{(k)}(m) + \Delta f^{(k)}(m) \geq 0; \\ 0, & \text{otherwise} \end{cases} \quad (77)
\]
where \( \Delta f^{(k)}(m) \) is obtained by inverting the system of equations in (76). For a chosen error bound \( \varepsilon > 0 \), the iterative procedure is continued until \( \| r^{(k)} \| < \varepsilon \), where
\[
r^{(k)} \triangleq [r^{(k)}(0), r^{(k)}(1), \cdots, r^{(k)}(N-1)]^T.
\]
\( \{ f^{(k)}(m) \}_{m=0}^{N-1} \) will converge to the true solution \( \{ f(m) \}_{m=0}^{N-1} \) provided that certain convergence conditions to be described are satisfied.

Let \( P \) denote the following system of equations:
\[
r(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) f^{(m_1)} f^{(m_2)} - g(n), \quad n = 0, 1, \cdots, N - 1. \quad (78)
\]
It is easy to see that, with \( f \) defined as in (36), the Frechet derivative [17, p. 658], \( P'(f^{(k)}) \), of \( P(f) \) at \( f = f^{(k)} \), can be characterized by the Jacobian matrix \( U^{(k)} \) whose elements satisfy
\[
u^{(k)}_{(i+1), (j+1)} = w^{(k)}(i; j), \quad i, j = 0, 1, \cdots, N - 1, \quad k = 0, 1, 2, \cdots, \quad (79)
\]
where \( w^{(k)}(i; j) \) is defined in (75). The second derivatives, \( P''(f) \), of \( P(f) \), can be characterized by a Hessian matrix \( A_f \) whose elements are
\[
a_{n(m_1+1), (m_2+1)} = [\partial^2 r(n)] / [\partial f^{(m_1)} \partial f^{(m_2)}]
= q(n; m_1, m_2) + q(n; m_2, m_1),
\]
\[
n, m_1, m_2 = 0, 1, \cdots, N - 1. \quad (80)
\]
Then, the above iteration, except for the clipping (sometimes referred to as positivity [181]) operation in (77), can be rewritten as
\[
f^{(0)} = \hat{f} \quad (81)
\]
\[
f^{(k+1)} = f^{(k)} - [P'(f^{(k)})]^{-1} P(f^{(k)}), \quad k = 0, 1, 2, \cdots. \quad (82)
\]
The iteration described in (81) and (82) is the well-known Newton’s method for solving \( P(f) = 0 \). The convergence of Newton’s method and its application to concrete functional equations are discussed in [17, ch. XVIII]. Let \( \Omega_0 \) be defined as
\[
\Omega_0 = \{ f : \| f - f^{(0)} \| \leq \rho, f \in R^N \}.
\]
Then, the followings are the sufficient conditions for the iteration (81) and (82) to converge to the unique solution, \( f^* \in \Omega_0 \), of \( P(f) = 0 \):
8. a continuous \( P^*(f) \) exists in \( f \in \Omega_0 \)
ii) \( [P'(f^{(0)})]^{-1} \) exists
iii) \( h = K_\eta \leq 0.5 \), where \( K \) and \( \eta \) are satisfying
Fig. 7. Iterative image restoration.

\[ \|P'(f^{(0)})\|^{-1}P(f^{(0)})\| \leq \eta \]
and \[ \|P'(f^{(0)})\|^{-1}P''(f)\| \leq K, \]

where \( f \in \Omega_0 \).

iv) \[ 1 - \sqrt{1 - 2h}/K \leq r \leq [1 + \sqrt{1 - 2h}]/K. \]

It is easy to see, from (80), that i) can be satisfied for all \( r \geq 0 \), and that if \( r \) is chosen as, for example, \( r = 1/K \), then i) and iv) are satisfied. Hence, the conditions ii) and iii) will be the sufficient conditions for the iteration (81) and (82) to converge to the unique solution \( f^* \in \Omega_0 \) of \( P(f) = 0 \), where \( P \) is the bilinear operation characterized by (78). In (77), the positivity operation is applied. It has been shown [18] that the iteration with the positivity constraint will converge if the iteration without the positivity constraint converges.

Example 4: The above iteration was applied to the degraded image in examples 2 and 3. When \( k = 3 \), the restored image shown in Fig. 7 was obtained for both cases. The performance factors were, respectively, 710.15 and 65.26.

V. IMAGE RESTORATION OF THE NOISY BLURRED IMAGE

The blurred image with signal-independent additive noise can be described by

\[ g(n) = \sum_{m_1=0}^{n} \sum_{m_2=0}^{n} q(n; m_1, m_2) f(m_1, m_2) + v(n) \]

where \( v(n) \) is taken to be a zero mean white Gaussian noise. The restored image is obtainable by applying the algorithm in Section III-A. However, due to the presence of noise, the nonnegativity condition in (21) may not be satisfied. Then, the recursive solution of (29) might lead to either two negative solutions, or two complex solutions instead of the desired nonnegative solution along with a negative one. The algorithm in Section III-A will be adapted for this case.

Let

\[ C_k^{(i)} = g(k) - C_k^{(i)}, \quad i = 0, 1, \cdots, k, \]

\[ k = 0, 1, 2, \cdots, \]

where \( C_k^{(i)} \) is given in (27). Suppose that nonnegative values for \( f(0), f(1), \cdots, f(n-1) \) have been obtained and

\[ C_k^{(i)} \geq 0, \quad i = 1, 2, \cdots, n, \quad k = i, i + 1, \cdots, N - 1. \]  \hspace{1cm} (84)

Recall that \( A_n \geq 0 \) and \( B_n \geq 0 \) due to the nonnegativity of \( \text{Re} \{q(n; m_1, m_2)\} \). Also, (84) implies that

\[ C_n^{(n)} \geq 0. \]

Solving (29) for \( f(n) \) will result in one nonnegative solution, say, \( s_1 \), and one negative solution \( s_2 \). Let

\[ f^{(1)}(n) = s_1. \]

Using \( f^{(1)}(n) \) in (85), \( C_k^{(n+1)} \), \( k = n + 1, n + 2, \cdots, N - 1 \), can be computed by the scheme detailed in Section III. If

\[ C_k^{(n+1)} \geq 0, \quad k = n + 1, n + 2, \cdots, N - 1, \]  \hspace{1cm} (86)

then

\[ f(n) = f^{(1)}(n), \]

and one can proceed with the algorithm to solve for \( f(n + 1) \).

If one or more of \( C_k^{(n+1)} \) turn out to be negative, say,

\[ C_k^{(n+1)} < 0, \quad k = n + 1, n + 2, \cdots, C_k^{(n+1)} < 0, \]

then choose \( C_l^{(n+1)} \) as

\[ C_l^{(n+1)} = \min \{ C_l^{(n+1)}, C_{l+1}^{(n+1)}, \cdots, C_k^{(n+1)} \}. \]

For \( C_l^{(n+1)} \) to be nonnegative, \( f(n) \) has to satisfy

\[ C_l^{(n)} - \left[ q(l; n, n) f^2(n) + \sum_{m=0}^{n-1} \{ q(l; n, m) + q(l; m, n) \} f(m) f(n) \right] \geq 0 \]

where \( C_l^{(n)} \geq 0 \) by (84). Let \( 0 < \alpha \leq 1 \). Then, (87) can be rewritten as

\[ q(l; n, n) f^2(n) + \sum_{m=0}^{n-1} \{ q(l; n, m) + q(l; m, n) \} f(m) f(n) - \alpha C_l^{(n)} = 0. \]

Choosing the proper value for \( \alpha \) and solving (88) for \( f(n) \) will result in \( f^{(2)}(n) \). By using this \( f^{(2)}(n) \), the nonnegativity of \( C_k^{(n+1)} \), \( k = n + 1, n + 2, \cdots, N - 1 \), can be re-checked. Note that

\[ C_l^{(n+1)} = (1 - \alpha) C_l^{(n)} \geq 0. \]

If all the inequalities in (86) are satisfied by \( f^{(2)}(n) \), then set

\[ f(n) = f^{(2)}(n) \]

and proceed with the algorithm to obtain \( f(n + 1) \). If any one or more of inequalities in (86) are not satisfied by \( f^{(2)}(n) \), then, by repeating the above procedure until all the inequalities in (86) can be satisfied, one can obtain a desired nonnegative \( f(n) \). It is easy to see that the number of negative \( C_k^{(n+1)} \) will be reduced by at least one when-
ever the above procedure is repeated. Hence, by repeating the above procedure a finite number of times, one can obtain a nonnegative $f(n)$ satisfying (86).

**Example 5:** In the simulation, white Gaussian noise which yielded an SNR of 30 dB was added to the blurred image of Fig. 3(a). The resulting noisy image is shown in Fig. 8(a). By choosing $\alpha = 0.5$, the restored image in Fig. 8(b) was obtained. The performance factor was 5.97.

In Appendix B, a noise analysis is reported for the restoration procedure developed in Section IV to handle noncausal degrading phenomena.

**VI. CONCLUSIONS**

From the analysis carried out in Section V and Appendix B, it can be concluded that the recursive technique described in Section III performs well on noisy images, when the noise is additive and signal independent. In many cases, noncausal blurs can be handled by applying these recursive techniques separately and superposing the results. It is recommended that in those cases, this strategy be used because of superior performance in the presence of noise over the direct methods developed to handle noncausal blurs. The procedure developed in Section IV to tackle noncausal blurs works ideally when the illumination in the optical imaging system is completely coherent (because, in this case, the conditions in fact 1 are satisfied), and works well in the relatively coherent case in the presence as well as in the absence of signal independent additive noise. This complements the results in [7], which are known to perform well in the relatively incoherent case. The results in [16] work well in the relatively coherent case, provided the image is of low contrast. This restriction is not necessary to apply satisfactorily the procedure developed in Section IV. Although the algorithm presented in [8] also performs satisfactorily in the relatively coherent case, the second-order statistics of the original image must be available. The image statistics need not be known a priori to apply the method of Section IV, as substantiated in Appendix B. The scopes for generalizing the procedure based on the suboptimal 2-D Kalman filtering ideas, applied in [19] to restore images in the presence of signal independent as well as signal dependent noise when the blur is linear shift-variant, could be investigated in the case of bilinear blurs with noise present.

**APPENDIX A**

**Proof of Theorem**

To prove the theorem in Section IV-A, the following definition A and lemmas A1, A2 are necessary.

**Definition A:** A matrix $E$ is said to be idempotent if $E^2 = E$.

**Lemma A1:** Let $Q$ be an $N \times N$ symmetric matrix with distinct eigenvalues $\lambda_i$ of multiplicities $m_i, i = 1, 2, \ldots, r$. Then, there exist the idempotents $E_i, i = 1, 2, \ldots, r$ having the properties

1) $E_iE_j = 0$, if $i \neq j$,
2) $\sum_{i=1}^{r} E_i = I_N$, and
3) $Q = \sum_{i=1}^{r} \lambda_i E_i$,

where $I_N$ denotes the $N \times N$ identity matrix. Moreover, the $E_i$'s, $i = 1, 2, \ldots, r$ are uniquely determined by

$$E_i = \sum_{j=1}^{m_i} v_jv_j^T,$$

where $v_j, j = 1, 2, \ldots, m_i$, are the orthonormal eigenvectors of $Q$ corresponding to $\lambda_i, i = 1, 2, \ldots, r$.

**Lemma A2:** If $E$ is idempotent, rank $E = \text{Tr} E$.

**Proof of Theorem:** Suppose that the rank of $Q$ is $r$. Let the positive eigenvalues of $Q$ be ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r,$$

where $p$ is the number of positive eigenvalue. Order the negative eigenvalues of $Q$ as

$$\lambda_{p+1} \geq \lambda_{p+2} \geq \cdots \geq \lambda_r,$$

Then

$$\lambda_{r+1} = \cdots = \lambda_N = 0.$$

Let $v_i$ denote the orthonormalized eigenvectors of $Q$ corresponding to $\lambda_i, i = 1, 2, \ldots, N$. Then $v_1, \ldots, v_N$ form an orthonormal basis of $R^N$. Let $d$ be the vector which minimizes

$$\|Q - dd^T\|_F.$$

There $d$ can be expressed as a linear combination of $v_i, i = 1, 2, \ldots, N$.

$$d = \sum_{i=1}^{N} a_i v_i.$$
It is easy to see that
\[ \|Q - dd'\|^2_E = \text{Tr}[(Q - dd')(Q - dd')] \]
\[ = \text{Tr}[Q^2] - 
\text{Tr}[Q(dd')] - \text{Tr}[(dd')Q]
+ \text{Tr}[dd'^2]
= \text{Tr}[Q^2] - 2 \text{Tr}[Q(dd')] + \text{Tr}[(dd')^2]. \]

(A.1)

since
\[ \text{Tr}[Q(dd')] = \text{Tr}[(dd')Q] \quad \text{and} \quad Q = Q'. \]

By using definition A and lemma A1, the first term of (A.1) will become
\[ \text{Tr}[Q^2] = \text{Tr}\left[ \sum_{i=1}^r \lambda_i E_i \right]^2 = \text{Tr}\left[ \sum_{i=1}^r \lambda_i^2 E_i \right]. \]

Hence, by using lemma A2,
\[ \text{Tr}[Q^2] = \text{Tr}\left[ \sum_{i=1}^r \lambda_i^2 E_i \right] = \sum_{i=1}^r \lambda_i^2. \quad \text{(A.2)} \]

The second term in (A.1) will become
\[ \text{Tr}[Q(dd')] = \text{Tr}\left[ \left( \sum_{i=1}^r \lambda_i v_i v_i' \right) \left( \sum_{j=1}^N a_j v_j \right) \left( \sum_{k=1}^N a_k v_k' \right) \right] \]
\[ = \text{Tr}\left[ \left( \sum_{i=1}^r \lambda_i a_i v_i \right) \left( \sum_{k=1}^N a_k v_k' \right) \right] \]
\[ = \sum_{i=1}^r \sum_{k=1}^N \lambda_i a_i a_k \text{Tr}[v_i v_k'] \]
\[ = \sum_{i=1}^r \lambda_i a_i^2, \quad \text{(A.3)} \]

since
\[ \text{Tr}[v_i v_k'] = \langle v_i, v_k' \rangle = \begin{cases} 1, & i = k \\ 0, & \text{otherwise}. \end{cases} \]

For the third term of (A.1),
\[ \text{Tr}[(dd')^2] = \text{Tr}\left[ \left( \sum_{i=1}^N a_i v_i \right) \left( \sum_{j=1}^N a_j v_j' \right) \left( \sum_{k=1}^N a_k v_k \right) \right] \]
\[ = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N a_i a_j a_k \text{Tr}[v_i v_j v_k'] \]
\[ = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N a_i^2 a_j \text{Tr}[v_i v_j' v_k] \]
\[ = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N a_i^2 a_j^2 \quad \text{(A.4)} \]

By substituting (A.2), (A.3), and (A.4) into (A.1), one obtains
\[ \|Q - dd'\|^2_E = \sum_{i=1}^r (\lambda_i - a_i^2)^2 \]
\[ + \left[ \sum_{i=r+1}^N a_i^4 + \sum_{j=1}^p \sum_{k=1}^p a_j^2 a_k^2 (1 - \delta_{jk}) \right], \]

where
\[ \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & \text{otherwise}. \end{cases} \]

Let
\[ \epsilon_1 = \sum_{i=1}^r (\lambda_i - a_i^2)^2 \geq 0 \]
\[ \epsilon_2 = \sum_{i=r+1}^N a_i^4 + \sum_{j=1}^p \sum_{k=1}^p a_j^2 a_k^2 (1 - \delta_{jk}) \geq 0. \]

Then
\[ \|Q - dd'\|^2_E = \epsilon_1 + \epsilon_2. \]

\( \epsilon_1 \) can be rewritten as
\[ \epsilon_1 = \sum_{i=1}^p (\lambda_i - a_i^2)^2 + \sum_{i=p+1}^r (\lambda_i - a_i^2)^2. \]

Let
\[ a = [a_1^2, a_2^2, \ldots, a_N^2]' \]

By varying \( a \) from \([0, 0, \ldots, 0]'\), one can set the optimum \( a^* \) which minimizes \( \epsilon_1 + \epsilon_2 \). Starting from \([0, 0, \ldots, 0]'\), \( \epsilon_2 \) is a nondecreasing function of \( a \) as \( \|a\| \) increases. However, \( \epsilon_1 \) is decreasing monotonically as the \( i \)th entry, \( 1 \leq i \leq p \), of \( a \) increases while the others are kept constant, provided that
\[ 0 \leq a_i^2 \leq \lambda_i, \quad i = 1, 2, \ldots, p \]
\[ a_i^2 = 0, \quad i \geq p + 1. \]

Hence, the optimum \( a^* \) will satisfy the above conditions. Based on these facts, it will be shown that
\[ a^* = [\lambda_1, 0, 0, \ldots, 0]' \]

will be the optimum choice minimizing \( \|Q - dd'\|^2_E \). Let
\[ \hat{a} = [\hat{a}_1^2, \hat{a}_2^2, \ldots, \hat{a}_p^2, 0, \ldots, 0]' \]
and \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \), \( i = 1, 2 \), denote, respectively, values of \( \epsilon_i \) at \( a = a^* \) and \( a = \hat{a} \). It is easy to see that
\[ \hat{\epsilon}_1 + \hat{\epsilon}_2 > \epsilon_1^* + \epsilon_2^* \]
if
\[ \hat{a} = [\hat{a}_1^2, 0, \ldots, 0]', \]
where \( 0 \leq \hat{a}_1^2 < \lambda_1 \). Suppose that \( 0 \leq \hat{a}_i^2 \leq \lambda_i, i = 1, 2, \ldots, p \), and that at least one of \( \hat{a}_i^2 \neq 0, i = 2, \ldots, p \). Then,
\[ \hat{\epsilon}_1 + \hat{\epsilon}_2 = \sum_{i=1}^p (\lambda_i - \hat{a}_i^2)^2 + \sum_{j=1}^p \sum_{k=1}^p \hat{a}_j^2 \hat{a}_k^2 (1 - \delta_{jk}) \]
where
\[ \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & \text{otherwise}. \end{cases} \]
and
\[(\hat{e}_1 + \hat{e}_2) - (e^*_1 + e^*_2) = \lambda_i^2 - 2 \sum_{i=1}^{P} \lambda_i \hat{a}_i^2 + \left[ \sum_{i=1}^{P} \hat{a}_i^2 \right] \geq 0, \]

The equality holds when \( \hat{a}_i = 0, \ i \not\in L \), where
\[ L = \{i: 1 \leq i \leq P, \ \lambda_i = \lambda_1\}. \]

Since
\[ \lambda_i = \sum_{i \in L} \hat{a}_i^2 \quad \text{and} \quad \hat{a}_i = 0, \ i \not\in L, \]

it is easy to see that
\[ \hat{e}_1 + \hat{e}_2 = e^*_1 + e^*_2, \]

where the equality holds when the following conditions hold:
\[ \lambda_1 = \sum_{i \in L} \hat{a}_i^2 \quad \text{and} \quad \hat{a}_i = 0, \ i \not\in L. \] (A.5)

Hence, from the above, it can be concluded that \( a^* \) is the only optimum choice for \( a \) if the multiplicity of \( \lambda_1 \) is 1. Even if it is not 1, still \( a^* \) is an optimum choice among the \( \hat{a}_i \)'s satisfying (A.5), in which case
\[ \hat{e}_1 + \hat{e}_2 = e^*_1 + e^*_2. \]

Therefore, irrespective of the multiplicity of the dominant eigenvalue,
\[ \hat{a}_1 = \sqrt{\lambda_1}, \ \hat{a}_2 = \hat{a}_3 = \cdots = \hat{a}_N = 0 \]

will give an optimal vector \( d \) minimizing \( \|Q - dd'\|_E \). But
\[ dd' = (\sqrt{\lambda_1} v_1) (\sqrt{\lambda_1} v_1^T) = \lambda_1 v_1 v_1^T. \]

Hence, the theorem in Section IV-A has been proved.

**APPENDIX B**

**NOISE ANALYSIS FOR THE NONCAUSAL CASE**

The noise added image \( \bar{g}(n) \), degraded by the bilinear system in (3), is assumed to be modeled by
\[ \bar{g}(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) f(m_1) f(m_2) + r(n), \]
\[ n = 0, 1, \cdots, N - 1, \] (A.6)

where \( r(n) \) represents signal-independent white Gaussian noise with zero mean and known variance \( \sigma_r^2 \). Let \( g(n) \) be
\[ g(n) = \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} q(n; m_1, m_2) f(m_1) f(m_2), \]
\[ n = 0, 1, \cdots, N - 1. \] (A.7)

Applying steps 1)-3) in the algorithm to \( g(n) \) in (A.7), the vector \( z \), defined by
\[ (z)_i \triangleq (w)_i^2, \ i = 1, 2, \cdots, N \] (A.8)

with \( w \) defined in (59), can be expressed as
\[ z = EB^T g, \]

where \( E, B, \) and \( g \) are given in (50), (37), and (41), respectively. Let
\[ H \triangleq EB^T. \]

Then
\[ z = Hg. \]

Applying steps 1)-3) in the algorithm to \( \bar{g}(n) \) in (A.6), the vector \( \bar{z} \), which is the noisy counterpart of \( z \), can be obtained by
\[ \bar{z} = H \bar{g}, \]

where \( \bar{g} = [\bar{g}(0), \bar{g}(1), \cdots, \bar{g}(N - 1)]' \).

From (A.6) and (A.7), \( \bar{z} \) can be rewritten as
\[ \bar{z} = Hg + Hr. \] (A.9)

Clearly, \( \bar{z} \) contains not only the desired term \( z \) but also the term \( Hr \) representing the linear operation on the noise \( r \). Hence, instead of solving
\[ z = Ex, \] (A.10)

it is required to solve
\[ \bar{z} = Ex + \bar{r}, \] (A.11)

where
\[ \bar{r} = Hr, \]
and \( x \) is of the factorable form shown in (38) and (39). For the noise-free case, \( x \) satisfying (A.10) in a factorable form has been obtained by using (12) following the solution of an \( N \times N \) system of linear equations
\[ w = Uf, \] (A.12)

where \( U \) and \( f \) are given in (45) and (36). For the noisy case, the equation corresponding to (A.12) will be
\[ \bar{w} = Uf + \bar{r} \] (A.13)

because of the presence of the noise term \( \bar{r} \) in \( \bar{z} \). An approximate solution for (A.13) can be obtained by
\[ \bar{f} = U^{-1} \bar{w}. \] (A.14)

But even in the case of nonsingular \( U \), \( \bar{f} \) obtained by (A.14) may greatly differ from
\[ f = U^{-1} w, \]
due to the noise amplification of the small singular modes. This will be explained further below.

Suppose that \( U \) is nonsingular. Let \( u_i \) and \( v_i, \ i = 1, 2, \]
... $N$, be, respectively, the eigenvectors of $UU'$ and $U'U$ corresponding to the singular values $\lambda_i$ of $U$. Then,

$$ f = U^{-1} \overline{w} $$

$$ = \sum_{i=1}^{N} [1/\lambda_i] \langle u_i, \overline{w} \rangle v_i. \quad (A.15) $$

By using (A.13), the expression (A.15) can be rewritten as

$$ f = f + U^{-1} \overline{r} $$

$$ = \sum_{i=1}^{N} \langle v_i, f \rangle v_i + \sum_{i=1}^{N} [1/\lambda_i] \langle u_i, \overline{r} \rangle v_i. \quad (A.16) $$

Let

$$ f_i \triangleq \langle v_i, f \rangle v_i. $$

For the case that

$$ |\langle u_i, \overline{r} \rangle/\lambda_i| \ll |\langle v_i, f \rangle|, \quad i = 1, 2, \cdots, N, \quad (A.17) $$

the solution $\overline{f}$ obtained by (A.14) will be a good approximation of $f$. However, if any of the inequalities in (A.17) is not satisfied, that is,

$$ |\langle u_i, \overline{r} \rangle/\lambda_i| \geq |\langle v_i, f \rangle| $$

for some $i$, (A.18) then the error term $\langle u_i, \overline{r} \rangle v_i/\lambda_i$ contributes more than $f_i$ does in $\overline{f}$. The inequality (A.18) can be rewritten as

$$ \lambda_i \leq |\langle u_i, \overline{r} \rangle/\langle v_i, f \rangle|. \quad (A.19) $$

Hence, if $U$ has small singular values, then $\overline{f}$ obtained by (A.14) will result in a large deviation from the solution $f$ of (A.13).

To avoid this undesired noise amplification, Rushforth et al. [20] and Maeda and Murata [21] replaced $1/\lambda_i$ in (A.15) by a certain function of $\lambda_i$, like $\lambda_i^3/(\alpha + \lambda_i^4)$, so that the inverse of the small singular values do not become large. By choosing the proper value of $\alpha$, a large value in $1/\lambda_i$ can be regularized to a small value in $\lambda_i^3/(\alpha + \lambda_i^4)$. By replacing $1/\lambda_i$ in (A.15) by $\lambda_i^3/(\alpha + \lambda_i^4)$, the expression (A.15) can be rewritten as

$$ \overline{f} = \sum_{i=1}^{N} [\lambda_i^3/(\alpha + \lambda_i^4)] \langle u_i, \overline{w} \rangle v_i. \quad (A.20) $$

The optimal value of $\alpha$ can be chosen so that the absolute value of

$$ C(\alpha) \triangleq \|\overline{w} - U\overline{f}\|^2 - \|\overline{r}\|^2 \quad (A.21) $$

is minimized. To obtain the optimal $\alpha$, it is necessary to express $U\overline{f}$ in terms of $\overline{w}$

$$ U\overline{f} = \sum_{i=1}^{N} \lambda_i \langle v_i, \overline{f} \rangle u_i $$

$$ = \sum_{i=1}^{N} \lambda_i \langle v_i, \sum_{j=1}^{N} [\lambda_j^3/(\alpha + \lambda_j^4)] \langle u_j, \overline{w} \rangle v_j \rangle u_i $$

$$ = \sum_{i=1}^{N} [\lambda_i^3/(\alpha + \lambda_i^4)] \langle u_i, \overline{w} \rangle u_i. \quad (A.22) $$

Since $\overline{w}$ can be rewritten as

$$ \overline{w} = \sum_{i=1}^{N} \langle u_i, \overline{w} \rangle u_i, \quad (A.23) $$

one can obtain, from (A.22) and (A.23),

$$ \|\overline{w} - U\overline{f}\|^2 = \left\| \sum_{i=1}^{N} \langle u_i, \overline{w} \rangle u_i - \sum_{i=1}^{N} [\lambda_i^3/(\alpha + \lambda_i^4)] \langle u_i, \overline{r} \rangle u_i \right\|^2 $$

$$ = \left\| \sum_{i=1}^{N} [\alpha/(\alpha + \lambda_i^4)] \langle u_i, \overline{w} \rangle u_i \right\|^2 $$

$$ = \left\| \sum_{i=1}^{N} [\alpha/(\alpha + \lambda_i^4)] \langle u_i, \overline{r} \rangle u_i \right\|^2 $$

$$ = \sum_{i=1}^{N} [\alpha/(\alpha + \lambda_i^4)]^2 \|u_i, \overline{r}\|^2. \quad (A.24) $$

It is easy to see that $\alpha/(\alpha + \lambda_i^4)$ is a monotonically increasing function of $\alpha \geq 0$. Hence, $\|\overline{w} - U\overline{f}\|^2$ is also a monotonically increasing function of $\alpha \geq 0$. Therefore, (A.21) has one nonnegative real zero. This value of $\alpha$, after it is found, will be chosen as the optimum $\alpha$.

Before finding the optimum value of $\alpha$, $\|\overline{r}\|^2$ has to be determined. Since the variance of $r$ is given as $\sigma_r^2$, the covariance matrix $C_r$ of $\overline{f} = Hr$ is given by

$$ C_r = H C H', \quad (A.25) $$

where $C_r$ is a diagonal matrix with $\sigma_r^2$ in its diagonal entry. The $i$th diagonal entry $\sigma_r^2$ of $C_r$ will be the variance of the $i$th entry of $\overline{r}$. Since the operation

$$ (\overline{w})_i = \sqrt{(\overline{r})_i}, \quad i = 1, 2, \cdots, N \quad (A.26) $$

is nonlinear, it is difficult to obtain the statistics of $\overline{r}$ from those of $\overline{f}$. It will be assumed that the noise $r(n)$ is relatively small in comparison to $g(n)$, $n = 0, 1, \cdots, N - 1$. Then it is easy to see that each entry of $\overline{r}$ is relatively small in comparison to that of $\overline{z}$ [see (A.10) and (A.11)] because

$$ E = B \quad \text{and} \quad H = EB^t = I. $$

By taking the first two terms of the Taylor series expansion of

$$ (\overline{w})_i = \sqrt{(\overline{z})_i} = [(\overline{z})_i + (\overline{\theta})_i)], \quad (A.27) $$

one will obtain the linear approximation of the nonlinear operation in (A.26).

$$ (\overline{w})_i = \sqrt{(\overline{z})_i [1 + (\overline{\theta})_i/(\overline{z})_i]^2} $$

$$ = \sqrt{(\overline{z})_i} + (\overline{\theta})_i/[2\sqrt{(\overline{z})_i}], \quad i = 1, 2, \cdots, N. $$

(A.28)
Fig. 9. Noisy image restoration for relatively coherent case. (a) Noisy degraded image. (b) Restored image.

Hence, from (A.12), (A.13), and (A.28),
\[
(\tilde{r}_i) = (\tilde{\tilde{r}}_i)/[2\sqrt{2}], \quad i = 1, 2, \ldots, N. \tag{A.29}
\]
The variance \(\sigma_i^2\) of \((\tilde{r}_i)\) will be given by
\[
\sigma_i^2 = \sigma_{\tilde{r}_i}^2/[4(2)], \quad i = 1, 2, \ldots, N. \tag{A.30}
\]
Since the operations (A.9) and (A.29) are linear, the zero mean is maintained. Therefore,
\[
||\tilde{r}||^2 = \sum_{i=1}^{N} (\tilde{r}_i)^2 \\
= \sum_{i=1}^{N} E[(\tilde{r}_i) - E(\tilde{r}_i)]^2 \\
= \sum_{i=1}^{N} \sigma_i^2, \tag{A.31}
\]
where \(E[\cdot]\) denotes the mathematical expectation. By using (A.25) and (A.31), \(||\tilde{r}||^2\) can be estimated. Then, the optimum \(\alpha\) can be obtained from (A.21).

Example A1: In the simulation, white Gaussian noise which yielded an SNR of 30 dB was added to the degraded image of Fig. 5(a). The algorithm was applied and the system of linear equations in the step 4 was solved by using the regularized singular value method described in this section. The noisy degraded image and the restored image are shown in Fig. 9(a) and (b), respectively. The performance factor, defined in the previous section, was 1.33.

Example A2: White Gaussian noise which yielded an SNR of 30 dB was added to the degraded image of Fig. 6(a). The same method as in example A1 was used to restore the noisy image. The noisy degraded image and the restored image are shown in Fig. 10(a) and (b), respectively. The performance factor was 1.14.

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