Finite-SNR Diversity-Multiplexing Tradeoffs for Half Duplex Protocols in Fading Relay Channels

Erik Stauffer, Özgür Oyman, Ravi Narasimhan, and Arogyaswami Paulraj

***Abstract*— We analyze the diversity-multiplexing tradeoff in a fading relay channel at finite signal-to-noise ratios (SNRs). In this framework, the rate adaptation policy is such that the target system data rate is a multiple of the capacity of an additive white Gaussian noise (AWGN) channel. The proportionality constant determines how aggressively the system scales the data rate and can be interpreted as a finite-SNR multiplexing gain. The diversity gain is given by the negative slope of the outage probability with respect to the SNR. The finite-SNR diversity-multiplexing tradeoff is characterized for three practical decode and forward half-duplex cooperative protocols with different amounts of broadcasting and simultaneous reception. For each configuration, system performance is computed as a function of SNR under a system-wide power constraint on the source and relay transmissions. Our analysis yields the following findings; (i) improved multiplexing performance can be achieved at any SNR by allowing the source to transmit constantly, (ii) both broadcasting and simultaneous reception are desirable in half-duplex relay cooperation for superior diversity-multiplexing performance, and (iii) the diversity-multiplexing tradeoff at finite-SNR is impacted by the power partitioning between the source and the relay terminals. Finally, we verify our analytical results by numerical simulations.**

I. INTRODUCTION

Cooperative relaying has recently gained a lot of interest due to its ability to realize the performance gains of multiple-input multiple-output (MIMO) wireless systems [1]–[4] in networks consisting of single antenna devices. The diversity-multiplexing tradeoff formulation, first proposed by Zheng and Tse in the context of point-to-point [5] and multiuser [6] fading MIMO channels, is a beneficial tool to investigate the role of code design on extracting the available diversity gains and spatial multiplexing gains of cooperative relay systems. While such studies have been carried out in [7], [8], the conclusions were limited to the asymptotically high SNR setting. Recently, asymptotically high SNR results for relay aided multiple access channels have been reported [9] and relay performance has been compared to single user MIMO in [10]. While the diversity-multiplexing tradeoff is derived for asymptotically high SNR, it has been found that it partially informs finite-SNR performance in [11], [12]. The diversity-multiplexing tradeoff over point-to-point MIMO links has been reformulated in [13] to apply for finite-SNRs by finding a new tradeoff between diversity and multiplexing at each SNR. This formulation yields greater insights into system performance at finite-SNR where asymptotic analysis is inaccurate. Additionally, the outage characteristics of relay channels have also been studied in the low SNR regime in [14].

In this paper, we examine the basic building block of cooperative diversity systems, a simple fading relay channel where the source, relay, and destination terminals are each equipped with single antenna transceivers. Considering a Rayleigh-fading channel model and assuming a system-wide power constraint on the source and relay transmissions, we extend the outage probability calculation technique in [2]. We obtain exact closed form expressions for the diversity and multiplexing gains at finite SNR for two different TDMA-based cooperative protocols and estimate the performance of a third using the analytical formulas in [13]. These results are studied in detail, along with a constrained max-flow min-cut bound, in [15].

Considering diversity and multiplexing performance at finite-SNR is important since this represents the practical operating regime. The diversity-multiplexing tradeoff is derived with asymptotically high SNR, and unfortunately neglects constant SNR offsets such as varying the power allocated between the source and the relay terminal. In [9] it was noted that a protocol with superior asymptotic performance was not always superior at finite-SNR. The diversity-multiplexing tradeoff has been shown to partially inform finite-SNR performance to some degree, as it defines different operating regions that correspond to each of the piecewise linear segments of the diversity-multiplexing tradeoff, as discussed in [11], [12]. Since these operating regions are derived using asymptotic SNR arguments, they only approximate the diversity at finite-SNR. Specifically, power partitioning between the source and the relay is neglected and this approximation doesn't accurately account for the transition between the asymptotic piecewise linear operating regions. The finite-SNR diversity-multiplexing tradeoff framework derived in [13], [16], on the other hand, describes the tradeoff between diversity and multiplexing for each SNR.

The superscripts ^{T} and ^{H} stand for transposition and conjugate transposition, respectively. $\mathcal{E}$ denotes the expectation operator, and I_m is the $m \times m$ identity matrix. A random event E_1 occurs with probability $P(E_1)$ and does not occur with probability $1 - P(E_1) = P(E_1^c)$, where E_1^c is the complement event of E_1. Also, $E_1 \cap E_2$ is the intersection of events E_1 and E_2 such that $P(E_1 \cap E_2)$ is the probability of both events occurring simultaneously. Similarly, $E_1 \cup E_2$

This work is supported in part by NSF Contract NSF DMS-0354674 ONR Contract ONR N00014-02-1-0088-P00006.

E. Stauffer, and A. Paulraj are with the Dept. of Electrical Engineering, Stanford University, Stanford, CA 94305, USA. {erstauff,apaulraj}@stanford.edu.

Ö. Oyman is with Intel Research Laboratories, Santa Clara, CA 95054. ozgur.oyman@intel.com

Narasimhan is with the Dept. of Electrical Engineering, University of California, Santa Cruz, CA 95064, USA. ravi@soe.ucsc.edu

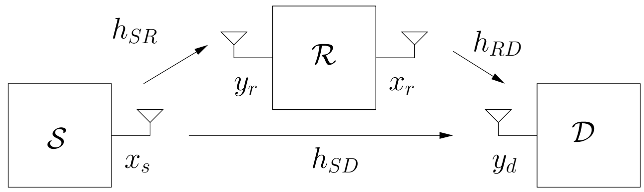

Fig. 1. A fading relay channel. $\mathcal{S}$ is the source node, $\mathcal{R}$ is the relay node, and $\mathcal{D}$ is the destination node.

is the union of the two events such that $P(E_1 \cup E_2)$ is the probability of at least one of the two events occurring.

II. SYSTEM MODELS

The relay system model will be described as follows. In Subsection II-A, we describe the relay channel and we present three different cooperative relaying schemes that differ in the degree of broadcasting and simultaneous reception. In Subsection II-B, we provide the physical-layer channel and signal models assumed in establishing the cooperative links.

A. General Setup and Protocol Descriptions

Consider a fading relay channel shown in Fig. 1. Data is to be transmitted from the source terminal $\mathcal{S}$ to the destination terminal $\mathcal{D}$ with the assistance of the relay terminal $\mathcal{R}$. All terminals are equipped with single antenna transmitters and receivers. The relay terminal $\mathcal{R}$ assists the source-destination communication by decoding the source transmission and forwarding the message in order to increase the reliability of decoding at the destination terminal.

We now describe three different cooperative protocols adapted from [3], which implement varying degrees of broadcasting and simultaneous reception in the network. Each of the protocols discussed is half duplex decode and forward, since the relay node is either listening or transmitting during a time slot, but never both simultaneously, and decodes the message from the source before forwarding it on to the destination. Each time slot consists of an equal amount of time. The degree of broadcasting is given by the number of nodes simultaneously (i.e., in the same time slot) listening to the source node (i.e., 2 if both $\mathcal{R}$ and $\mathcal{D}$ listen, 1 if only $\mathcal{R}$ or only $\mathcal{D}$ listens). Furthermore, the degree of reception is said to be maximum if the destination node receives information simultaneously from both $\mathcal{S}$ and $\mathcal{R}$.

Protocol I: In this protocol, the source terminal communicates with the relay and destination terminals over the first time slot. In the second time slot, only the relay terminal communicates with the destination terminal. This protocol realizes a maximum degree of broadcasting and exhibits no simultaneous reception. Also, this protocol allows for data reception by the source terminal from another terminal during the second time slot.

Protocol II: In this protocol, the source terminal communicates with only the relay terminal over the first time slot. In the second time slot, the source and relay terminals communicate with the destination terminal. This protocol does not implement broadcasting but realizes simultaneous reception. Also, this protocol allows for data transmission by the destination terminal to another terminal during the first time slot.

Protocol III: The source terminal communicates the entire message with the relay and destination terminals during the first time slot. In the second time slot, both source and relay terminals communicate the entire message with the destination terminal. This protocol realizes maximum degrees of broadcasting and simultaneous reception.

B. Channel and Signal Models

Throughout this paper, we assume frequency-flat fading, no channel knowledge at the transmitters, perfect channel state information (CSI) at the receivers, and perfect synchronization. Perfect channel state information at the receivers implies that the $\mathcal{S} \rightarrow \mathcal{R}$ channel is known to the relay terminal, while the individual $\mathcal{S} \rightarrow \mathcal{D}$ and $\mathcal{R} \rightarrow \mathcal{D}$ channels are known to the destination terminal. The signals transmitted by terminal $\mathcal{T} \in \{\mathcal{S}, \mathcal{R}\}$ over the first and second time slots shall be denoted as $x_{\mathcal{T},1}$ and $x_{\mathcal{T},2}$, respectively. Similarly, the signals received by terminal $\mathcal{T} \in \{\mathcal{R}, \mathcal{D}\}$ over the first and second time slots shall be denoted as $y_{\mathcal{T},1}$ and $y_{\mathcal{T},2}$, respectively. The following statistical properties are assumed on the transmit signals

$$\mathcal{E}\{x_{\mathcal{T},i}\} = 0 \tag{1}$$

and

$$\begin{gathered} \mathcal{E}\left\{\frac{1}{2}\sum_{i=1}^{2}|x_{\mathcal{R},i}|^2\right\} = \alpha P \\ \mathcal{E}\left\{\frac{1}{2}\sum_{i=1}^{2}|x_{\mathcal{S},i}|^2\right\} = \beta P \\ \alpha + \beta = 1 \text{ and } \alpha, \beta \geq 0 \end{gathered} \tag{2}$$

where P is the average total network transmit power over the two time slots. Additionally, α and β describe the power split between the relay and the source terminals, respectively. The additive noise at the relay and destination terminals $\mathcal{T} \in \{\mathcal{R}, \mathcal{D}\}$ over time slot $i \in \{1, 2\}$ shall be denoted as $n_{\mathcal{T},i}$ and is assumed to be drawn from a complex-valued white Gaussian noise process with zero mean and variance σ_n^2. Based on the system-wide power constraint in (2) across source and relay transmissions, we can define the network SNR as

$$\rho = \frac{P}{\sigma_n^2}. \tag{3}$$

The channel gains for the source-relay, source-destination, and relay-destination links are denoted by h_{SR}, h_{SD}, and h_{RD}, respectively, which are independent complex-valued Gaussian random variables with zero mean and variances σ_{SR}^2, σ_{SD}^2, and σ_{RD}^2, respectively. The channel gains are assumed to follow the Rayleigh fading distribution and the differences in the variances account for path loss and shadowing. Although the calculations that follow account for arbitrary positive variances, we take $\sigma_{SR}^2 = \sigma_{SD}^2 = \sigma_{RD}^2 = 1$ in the numerical examples. Furthermore, the channel coefficients $\{h_{SR}, h_{SD}, h_{RD}\}$ are assumed to be quasi-static, i.e., the channel remains constant for a fixed block duration but channels over different blocks fade independently.

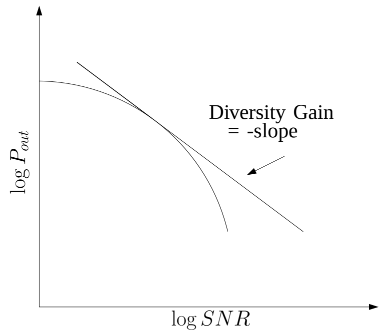

Fig. 2. Illustration of diversity gain at finite SNR.

III. FINITE-SNR DIVERSITY-MULTIPLEXING TRADEOFF

In this section, we compute the finite-SNR diversity-multiplexing tradeoff of a specific class of channels whose instantaneous SNR is the weighted sum of two exponential random variables. The results of this analysis will then be used to compute the finite-SNR diversity-multiplexing tradeoff of the half-duplex protocols described in Section II.

While the conventional definitions of diversity and multiplexing gains of a system refer to asymptotic quantities as the SNR approaches infinity [5], here we consider the multiplexing and diversity definitions of [13], [16] that extend these tradeoffs to finite SNRs. In the finite-SNR case, the multiplexing gain r is defined as the ratio of the system data rate R (in bps/Hz) to the capacity of an AWGN channel at SNR ρ and indicates how aggressively the system increases the throughput with SNR:

$$r = \frac{R}{\log_2(1+\rho)}. \tag{4}$$

Additionally, diversity was defined in [16] as the negative slope of the log-log plot of outage probability versus SNR, as illustrated in Fig. 2, for fixed multiplexing gain r and SNR ρ:

$$d(r,\rho) = -\frac{\rho}{P_{out}(r,\rho)}\frac{\partial P_{out}(r,\rho)}{\partial \rho}. \tag{5}$$

The significance of this definition is that the diversity gain at a particular SNR can be used to estimate the additional SNR required to decrease the outage probability by a specified amount for a given multiplexing gain.

Next, we consider the finite-SNR diversity-multiplexing tradeoff of channels whose mutual information can be expressed as

$$I = \log_2(1+\sigma_1^2 X_1 + \sigma_2^2 X_2), \tag{6}$$

where we assume that X_1 and X_2 are exponential random variables with mean one and that the source and relay terminals employ Gaussian codebooks for transmission. This class of channels includes fading 1×2 single-input multiple-output (SIMO), 2×1 multiple-input single-output (MISO), and two parallel channels using repetition coding and maximum ratio combining (MRC) at the receiver. These channel models [1] will be revisited when analyzing the performance of the relay channel.

To derive the finite-SNR diversity-multiplexing tradeoff for the channel described in (6), we need to compute the probability of outage and its derivative. The probability of outage for a given target data rate R is given as

$$\begin{aligned} P_{out} &= P(I<R) \\ &= P(\log_2(1+\sigma_1^2X_1+\sigma_2^2X_2)<R) \\ &= P(\sigma_1^2X_1+\sigma_2^2X_2<2^R-1) \\ &= P(z<2^R-1) \end{aligned} \tag{7}$$

where $z=\sigma_1^2X_1+\sigma_2^2X_2$. The pdf of z depends on whether $\sigma_1^2 \neq \sigma_2^2$ with $\sigma_1^2,\sigma_2^2>0$, $\sigma_1^2=\sigma_2^2$, $\sigma_1^2=0$, or $\sigma_2^2=0$. In the first case, the pdf of z is derived in [17] and is shown below for $\sigma_1^2 \neq \sigma_2^2$ with $\sigma_1^2,\sigma_2^2>0$.

$$p_z(z) = \tfrac{1}{\sigma_2^2-\sigma_1^2}\left[\exp\left(\tfrac{-z}{\sigma_2^2}\right) - \exp\left(\tfrac{-z}{\sigma_1^2}\right)\right] \quad \text{for } z \geq 0 \tag{8}$$

From the above pdf, the outage probability is given as follows:

$$\begin{aligned} &P_{out} = P(z<2^R-1) = \textstyle\int_0^{2^R-1} p_z(z)dz \\ &= 1-\tfrac{1}{\sigma_2^2-\sigma_1^2}\left[\sigma_2^2\exp\left(\tfrac{-(2^R-1)}{\sigma_2^2}\right)-\sigma_1^2\exp\left(\tfrac{-(2^R-1)}{\sigma_1^2}\right)\right] \end{aligned} \tag{9}$$

Note that R depends on ρ via (4):

$$2^R = (1+\rho)^r. \tag{10}$$

Substitution yields the following:

$$\begin{aligned} P_{out} = 1 - \tfrac{1}{\sigma_2^2-\sigma_1^2}\Big[&\sigma_2^2\exp\left(\tfrac{-((1+\rho)^r-1)}{\sigma_2^2}\right) \\ &-\sigma_1^2\exp\left(\tfrac{-((1+\rho)^r-1)}{\sigma_1^2}\right)\Big] \end{aligned} \tag{11}$$

Next, substitute in for $\sigma_1^2=\gamma_1\rho$ and $\sigma_2^2=\gamma_2\rho$.

$$\begin{aligned} P_{out} = 1 - \tfrac{1}{\gamma_2\rho-\gamma_1\rho}\Big[&\gamma_2\rho\exp\left(\tfrac{-((1+\rho)^r-1)}{\gamma_2\rho}\right) \\ &-\gamma_1\rho\exp\left(\tfrac{-((1+\rho)^r-1)}{\gamma_1\rho}\right)\Big] \end{aligned} \tag{12}$$

Finally, we can now compute $\frac{\partial P_{out}(r,\rho)}{\partial\rho}$, which is given in (13). The diversity $d(r,\rho)$ for condition (6) can be computed by substituting (12) and (13) into (5).

In the second case, when $\sigma_1^2=\sigma_2^2=\gamma\rho$, the outage probability is given as

$$P_{out} = 1-\exp\left(\frac{-((1+\rho)^r-1)}{\gamma\rho}\right)\left(1+\frac{(1+\rho)^r-1}{\gamma\rho}\right). \tag{14}$$

Similarly, in the third case, when $\sigma_1^2=0$ and $\sigma_2^2=\gamma\rho$, or $\sigma_1^2=\gamma\rho$ and $\sigma_2^2=0$, the outage probability is given as

$$P_{out} = 1-\exp\left(\frac{-((1+\rho)^r-1)}{\gamma\rho}\right). \tag{15}$$

IV. FINITE-SNR DIVERSITY-MULTIPLEXING TRADEOFF OF HALF DUPLEX PROTOCOLS

In this section, we consider the finite-SNR diversity-multiplexing tradeoff of half-duplex protocols discussed in Section II-A. For Protocols I and II, we allow the destination node the opportunity to decode the message without the

$$\begin{array}{rl}\frac{\partial P_{out}(r,\rho)}{\partial \rho} = & \frac{\gamma_2-\gamma_1}{(\gamma_2\rho-\gamma_1\rho)^2}\left[\gamma_2\rho\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_2\rho}\right)-\gamma_1\rho\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_1\rho}\right)\right] \\ & -\frac{1}{\gamma_2\rho-\gamma_1\rho}\left[\gamma_2\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_2\rho}\right)+\gamma_2\rho\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_2\rho}\right)\left(\frac{-r(1+\rho)^{r-1}}{\gamma_2\rho}+\frac{\gamma_2((1+\rho)^r-1)}{(\gamma_2\rho)^2}\right)\right] \\ & +\frac{1}{\gamma_2\rho-\gamma_2\rho}\left[\gamma_1\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_1\rho}\right)+\gamma_1\rho\exp\left(\frac{-((1+\rho)^r-1)}{\gamma_1\rho}\right)\left(\frac{-r(1+\rho)^{r-1}}{\gamma_1\rho}+\frac{\gamma_1((1+\rho)^r-1)}{(\gamma_1\rho)^2}\right)\right]\end{array} \quad (13)$$

help of the relay node when the relay node fails to decode the message. Finally, we compare the outage probability expressions and bounds with Monte Carlo simulations.

A. Protocol I

We begin by considering Protocol I. Note that in this protocol, the source is transmitting in only one time slot. Because of this, this protocol allows for 3dB more power during the time slot when the source is on compared to the case when the source is on during both time slots.

$$\begin{array}{c}\mathcal{E}\left\{|x_{\mathcal{S},1}|^2\right\}=2\beta P \\ \mathcal{E}\left\{|x_{\mathcal{S},2}|^2\right\}=0\end{array} \quad (16)$$

Similarly, the relay node is only transmitting during the second time slot.

$$\begin{array}{c}\mathcal{E}\left\{|x_{\mathcal{R},1}|^2\right\}=0 \\ \mathcal{E}\left\{|x_{\mathcal{R},2}|^2\right\}=2\alpha P\end{array} \quad (17)$$

In this protocol, we focus on the special case of a shared codebook between the relay and the source. Specifically, after receiving the message transmitted by the source in the first time slot, the relay then retransmits the same codeword in the second time slot. This constraint allows for the receiver to perform MRC to combine the received signals in time slots 1 and 2 before decoding.

To analyze Protocol I, we will first define three events: E_1 when the relay fails to decode the message, E_2 when the destination fails to decode the message with the relay helping, and E_3 when the destination fails to decode the message after the relay has failed. These events occur when the following inequalities are satisfied, which are all of the form discussed in Section III.

$$\begin{array}{l}E_1:\log_2(1+|h_{SR}|^2 2\beta\rho)<2R \\ E_2:\log_2(1+|h_{SD}|^2 2\beta\rho+|h_{RD}|^2 2\alpha\rho)<2R \\ E_3:\log_2(1+|h_{SD}|^2 2\beta\rho)<2R\end{array} \quad (18)$$

Since this protocol has only one of two time slots to transmit from the source to the relay, the data rate of the link in E_1 is $2R$, double that of the system data rate R. Similarly, the link to the destination is effectively one time slot after MRC combining, whether or not the relay is helping, and so the data rate of the links in E_2 and E_3 are double that of the system as well. An outage occurs if both the link through the relay and the direct link fail simultaneously [2]. When the relay has decoded the message, an outage occurs when the link from both the source and the relay to the destination fails. If the relay fails to decode the message, an outage occurs when the link between the source and the destination fails. Thus, an outage event is defined by the following:

$$E_{out}=(E_1\cup E_2)\cap E_3 \quad (19)$$

The probability of outage is equivalently given by

$$P_{out}=P(E_1\cup E_2|E_3)P(E_3). \quad (20)$$

Next, note that E_1 and E_2 are independent. Thus,

$$P_{out}=[1-(1-P(E_1|E_3))(1-P(E_2|E_3))]P(E_3). \quad (21)$$

Note that E_1 and E_3 are independent, but E_2 and E_3 are not. Therefore,

$$P_{out}=P(E_1)P(E_3)+P(E_2\cap E_3)-P(E_1)P(E_2\cap E_3). \quad (22)$$

Next, the probabilities $P(E_1)$, $P(E_3)$, and $P(E_2,E_3)$ need to be computed. First, $P(E_1)$ and $P(E_3)$ are SISO channels, so Eq. (15) applies. Secondly, for $P(E_2\cap E_3)$ we realize that E_3 occurs whenever E_2 does; hence, $P(E_2\cap E_3)=P(E_2)$. So, we can simply compute $P(E_2)$ using results derived in Section III. The probability of outage and the derivative with respect to the SNR ρ for Protocol I is given as follows.

$$\begin{array}{rcl}P_{out} & = & P(E_1)P(E_3)+P(E_2)-P(E_1)P(E_2) \\ \frac{\partial P_{out}}{\partial\rho} & = & \frac{\partial P(E_1)}{\partial\rho}\left[P(E_3)-P(E_2)\right] \\ & & +P(E_1)\left[\frac{\partial P(E_3)}{\partial\rho}-\frac{\partial P(E_2)}{\partial\rho}\right]+\frac{\partial P(E_2)}{\partial\rho}\end{array} \quad (23)$$

B. Protocol II

In the case of Protocol II, the source is active during both time slots. Because of this, the transmit power of the source is shared across the two time slots.

$$\begin{array}{c}\mathcal{E}\left\{|x_{\mathcal{S},1}|^2\right\}=\beta P \\ \mathcal{E}\left\{|x_{\mathcal{S},2}|^2\right\}=\beta P\end{array} \quad (24)$$

As in the case of Protocol I, the relay is only active during the second time slot, and so (17) applies in this case as well. Additionally, since the relay and the source know the message during the second time slot, it is possible for them to use any transmit covariance matrix $\Sigma=\mathcal{E}\{[x_{\mathcal{S},2};x_{\mathcal{R},2}][x_{\mathcal{S},2};x_{\mathcal{R},2}]^H\}$. Since neither the source or the relay possesses CSI, we consider the case where $\Sigma=\mathbf{diag}([\beta P,2\alpha P])$. Cooperation between the source and the relay can be accomplished using the Alamouti space time code [18], where the source and the relay act as a transmit antenna in the space time coding scheme.

To analyze Protocol II, we will first define three events: E_1 when the relay fails to decode the message, E_2 when the destination fails to decode the message with the relay helping, and E_3 when the destination fails to decode the message after the relay has failed. These events occur when the following inequalities are satisfied.

$$\begin{array}{l}E_1:\log_2(1+|h_{SR}|^2\beta\rho)<2R \\ E_2:\log_2(1+|h_{SD}|^2\beta\rho+|h_{RD}|^2 2\alpha\rho)<2R \\ E_3:\log_2(1+|h_{SD}|^2\beta\rho)<2R\end{array} \quad (25)$$

Since this protocol uses the first time slot to transmit from the source to the relay and the second time slot to communicate with the destination, the data rate of the links in E_1, E_2, and E_3 are $2R$, double that of the system data rate R. Note, that the inequalities above are of the same form as those in (18) (and Section III), except the power is scaled here relative to Protocol I due to the amount of time that the source transmitter is on. Additionally, (23) can be used to compute the system outage probability and diversity for this protocol.

C. Protocol III

Protocol III, unlike protocols I and II, allows the source to transmit to the destination during both time slots. Specifically, we allow the source to transmit the same data in both time slots, but using different codewords. While the first two protocols were limited to multiplexing gain 1/2, this will allow Protocol III to reach a multiplexing gain of 1. Additionally, we consider the case when the destination may be able to decode the message even if the relay fails. Unfortunately, due to the effective parallel channels exact analysis is not tractable when the relay helps to forward the message. The two parallel channels come from the two time slot transmissions describing the same message but with different SNRs. Because of this, we turn to outage probability bounding derived in [13].

First, we define the following events, where E_1 and E_3 are of the form discussed in Section III, but E_2 is not due to the parallel channels.

$$\begin{array}{l} E_1 : \log_2(1+|h_{SR}|^2\beta\rho) < 2R \\ E_2 : \log_2((1+|h_{SD}|^2\beta\rho+|h_{RD}|^2 2\alpha\rho) \\ \qquad \times(1+|h_{SD}|^2\beta\rho)) < 2R \\ E_3 : \log_2(1+|h_{SD}|^2\beta\rho) < R \end{array} \tag{26}$$

Since this protocol has only one of two time slots to transmit from the source to the relay, the data rate of the link in E_1 is $2R$, double that of the system data rate R. In this case, though, the link to the destination spans both time slots, whether the relay is helping or not, and so the data rate of the link in E_3 is equal to the system data rate. Each of the parallel channels in E_2 span one time slot, so the data rate of the link is double that of the system data rate. An outage occurs if both the link through the relay and the direct link fail simultaneously. Thus, an outage event is defined by the following.

$$E_{out} = (E_1 \cup E_2) \cap E_3 \tag{27}$$

Note that the above equation is of the same form as (19). Following the steps (19) to (22), the probability of outage $P(E_{out})$ is given as follows, since E_1 and E_2 are independent and E_1 and E_3 are independent.

$$P_{out} = P(E_1)P(E_3) + P(E_2 \cap E_3) - P(E_1)P(E_2 \cap E_3) \tag{28}$$

Additionally, the derivative of the outage probability is given by (23). Finally, what remains is to compute these probabilities. First, $P(E_1)$ and $P(E_3)$ are SISO channels, so Eq. (15) applies. Secondly, we realize that E_3 occurs whenever E_2 does; hence, $P(E_2) = P(E_2 \cap E_3)$.

Next, we focus on the computation of $P(E_2)$, which unfortunately cannot be computed in closed form. Instead, we turn to the bounding technique described in [13].

$$\begin{aligned} P(E_2) &= P[(1+|h_{SD}|^2\beta\rho+|h_{RD}|^2 2\alpha\rho) \\ &\quad (1+|h_{SD}|^2\beta\rho) < 2^{2R}] \\ &= P[(1+|h_{SD}|^2\beta\rho+|h_{RD}|^2 2\alpha\rho) \\ &\quad (1+|h_{SD}|^2\beta\rho) < (1+\rho)^{2r}] \\ &\geq P[(1+|h_{SD}|^2\beta\rho) < (1+\rho)^{a_1} \cap \\ &\quad (1+|h_{SD}|^2\beta\rho+|h_{RD}|^2 2\alpha\rho) < (1+\rho)^{a_2}] \\ &= P[(1+\sigma_1^2 X_1) < (1+\rho)^{a_1} \cap \\ &\quad (1+\sigma_1^2 X_1+\sigma_2^2 X_2) < (1+\rho)^{a_2}] \end{aligned} \tag{29}$$

Where we have defined $(1+\rho)^{2r} = \prod_{l=1}^{2}(1+\rho)^{a_l}$. Also, X_1 and X_2 are exponential random variables with mean one.Next, note that since $1+\sigma_1^2 X_1 < (1+\rho)^{a_1}$, the following inequality holds.

$$1+\sigma_1^2 X_1+\sigma_2^2 X_2 < (1+\rho)^{a_1}+\sigma_2^2 X_2 \tag{30}$$

Rearranging (29) and using (30) yields.

$$\begin{aligned} P(E_2) > \; & P[(X_1 < \tfrac{1}{\sigma_1^2}((1+\rho)^{a_1}-1))] \times \\ & P[(X_2 < \tfrac{1}{\sigma_2^2}((1+\rho)^{a_2}-(1+\rho)^{a_1}))] \end{aligned} \tag{31}$$

Where, $\eta_{(\rho,1)} = \frac{1}{\sigma_1^2}((1+\rho)^{a_1}-1)$.Following [13], the lower bound in (31) is maximized over the exponents a_1 and a_2 for each ρ to obtain accurate diversity gains at finite SNRs. A feasible point for this optimization is determined by the fact that each $\eta_{(\rho,i)} \geq 0$ for $1 \leq i \leq 2$. This yields the following conditions.

$$\begin{array}{c} 0 \leq a_1 \leq a_2 \leq r \\ a_1 + a_2 = 2r \end{array} \tag{32}$$

Once the optimal a_i's are found, we can compute the outage probability and its derivative. Taking the derivative of the bound on $P(E_2)$ in (31) with respect to ρ yields the following.

$$\begin{aligned} \frac{\partial P(E_2)}{\partial \rho} &= \exp(-\eta_{(\rho,1)})\frac{\partial \eta_{(\rho,1)}}{\partial \rho}(1-\exp(-\eta_{(\rho,2)})) \\ &\quad +(1-\exp(-\eta_{(\rho,1)}))\exp(-\eta_{(\rho,2)})\frac{\partial \eta_{(\rho,2)}}{\partial \rho} \end{aligned} \tag{33}$$

The derivatives of $\eta_{(\rho,1)}$ and $\eta_{(\rho,2)}$ with respect to ρ are given below.

$$\begin{aligned} \frac{\partial \eta_{(\rho,1)}}{\partial \rho} &= \frac{-\gamma_1}{\sigma_1^4}((1+\rho)^{a_1}-1)+\frac{1}{\sigma_1^2}(a_1(1+\rho)^{(a_1-1)}) \\ \frac{\partial \eta_{(\rho,2)}}{\partial \rho} &= \frac{-\gamma_2}{\sigma_2^4}((1+\rho)^{a_2}-(1+\rho)^{a_1}) \\ &\quad +\frac{1}{\sigma_2^2}(a_2(1+\rho)^{(a_2-1)}-a_1(1+\rho)^{(a_1-1)}) \end{aligned} \tag{34}$$

D. Protocol Comparisons and Numerical Results

In this section, we compare the finite-SNR performance of the protocols. First, we consider the finite-SNR diversity-multiplexing performance in Fig. 3. Here, we compare the finite-SNR diversity-multiplexing tradeoff performance of a SISO system, Protocol I, Protocol II, and Protocol III at SNR values of 0 dB (low SNR) and 50 dB (high SNR), and $\alpha =$

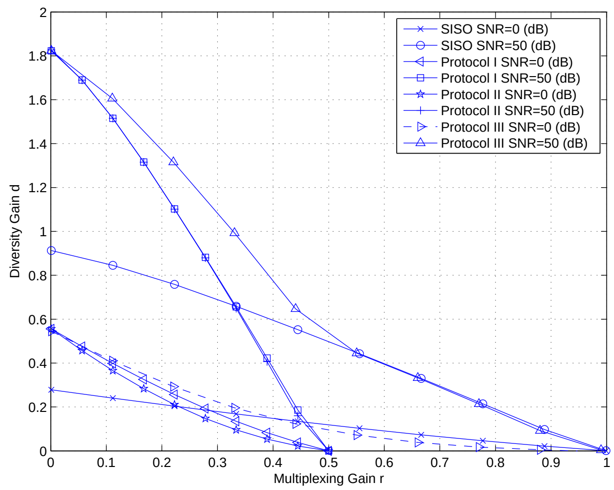

Fig. 3. Finite-SNR diversity-multiplexing comparison of a SISO system, Protocol I, Protocol II, and Protocol III. $\alpha = \beta = 1/2$

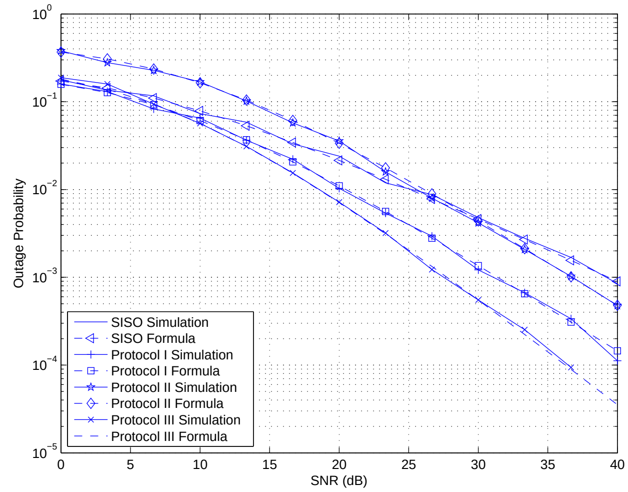

Fig. 4. Comparison of analytical forms and bounds verses simulated performance. $r = 0.25$ and $\alpha = \beta = 1/2$.

$\beta = 1/2$. The SISO system can represented by the power allocation

$$\begin{aligned} &\mathcal{E}\left\{\tfrac{1}{2}\sum_{i=1}^{2}|x_{\mathcal{R},i}|^2\right\} = 0 \\ &\mathcal{E}\left\{\tfrac{1}{2}\sum_{i=1}^{2}|x_{\mathcal{S},i}|^2\right\} = P, \end{aligned} \tag{35}$$

and outage event

$$E : \log_2(1 + |h_{SD}|^2\rho) < R. \tag{36}$$

While the SISO system and Protocol III achieve a multiplexing gain of one, the SISO system is diversity suboptimal for low multiplexing gains. On the other hand, Protocols I and II are multiplexing suboptimal compared to the SISO system and Protocol III. Diversity-multiplexing curves for the SISO system and Protocol I and II cross at multiplexing gains near $0.3 \leq r \leq 0.4$ for the considered SNR range. Finally, the lower bound on Protocol III achieves superior performance to that of Protocol I and II. While Protocol III can achieve larger diversity gains than a SISO system at low multiplexing gain, a SISO system can achieve larger diversity gain than Protocol III for high multiplexing gain due to the power sharing of Protocol III between the source and the relay (the source is allocated all the available power in the SISO case).

Next, we consider the outage probability of a SISO system, Protocol I, Protocol II, and Protocol III in Fig. 4. The analytical results are verified through Monte Carlo simulations at a multiplexing gain of $r = 0.25$ and $\alpha = \beta = 1/2$. First, note the superior diversity performance of all three protocols over that of the SISO system. Secondly, note the superior performance of Protocol I over that of Protocol II, as the source in Protocol II must share power over two time slots. In addition, Protocol III is superior to Protocols I and II at high SNR due to the utilization of both time slots for the source to communicate with the destination. Finally, note that the lower bound for Protocol III is very close to the simulation performance.

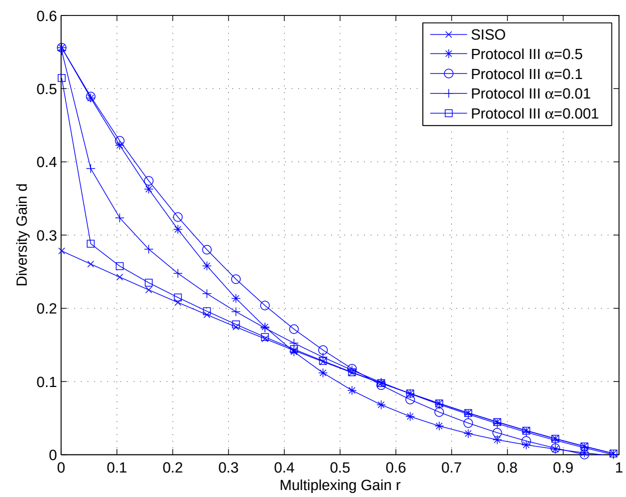

Fig. 5. Finite-SNR diversity-multiplexing performance comparison between a SISO link and Protocol III for different power partitions at SNR=0 (dB).

Finally, we consider the finite-SNR diversity-multiplexing tradeoff dependence of Protocol III on the power partition α and $\beta = 1 - \alpha$ shown in Fig. 5 at an SNR of 0 dB. Varying the power partition of Protocol III changes the performance. For low multiplexing gains $r < 0.5$, Protocol III has the best diversity performance with $\alpha \approx 0.1$. For higher multiplexing gains $r > 0.5$, Protocol III has the best diversity performance for small α, as Protocol III approaches the performance of a SISO system. Intuitively, if α is set to zero, then Protocol III becomes a SISO system. This behavior can be seen in Fig. 5 as α is decreased. Performance will also change with varying time allocation between the time slots, but is not considered here. As the system SNR increases, the finite-SNR diversity-multiplexing performance of Protocol III converges to the

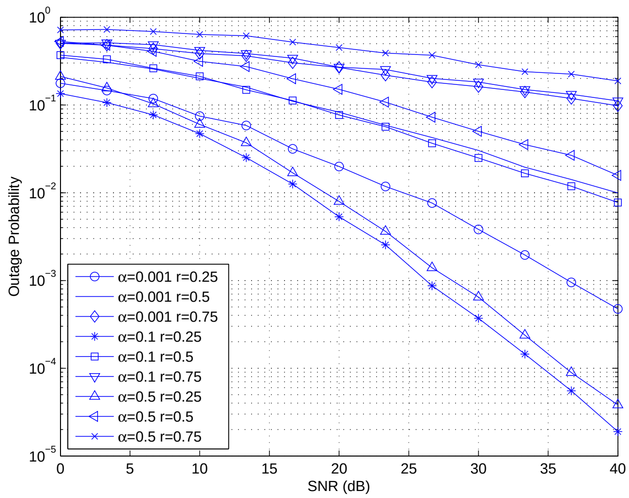

Fig. 6. Simulated outage probability comparison for Protocol III with different power partitions.

asymptotic performance, regardless of the power partition. We can now compare the finite-SNR diversity-multiplexing tradeoff with the outage probability shown in Fig. 6. Here, we can confirm that different power partitions are optimal for different multiplexing gains. Specifically, $\alpha \approx 0.1$ is optimal for low multiplexing gains and small α is optimal for large multiplexing gains, as indicated in Fig. 5.

V. CONCLUSION

In this work, we analyzed the diversity-multiplexing tradeoff for a relay channel in the finite-SNR regime. We have derived closed form expressions on the diversity and multiplexing gains under a system-wide power constraint on the source and relay transmissions for two half-duplex cooperative relaying protocols that differ in the degree of broadcasting and simultaneous reception. An outage probability bound was then used to estimate the performance of a third protocol that combined broadcasting and simultaneous reception. Using these analytical results and Monte Carlo simulations, we have shown performance enhancements through relay cooperation at finite-SNR and quantified gains in terms of diversity and multiplexing over direct (SISO) transmissions. These performance gains yield additional insight over that of asymptotic diversity-multiplexing results that are only approximate at finite-SNR, including the diversity-multiplexing tradeoff dependence on the power partition.

REFERENCES

[1] A. Paulraj, R. Nabar, and D. Gore, *Introduction to Space-Time Wireless Communications*. Cambridge University Press, 2003.

[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior." *IEEE Transactions on Information Theory*, vol. 50, no. 12, pp. 3062 – 80, December 2004.

[3] R. U. Nabar, H. Bolcskei, and F. W. Kneubuhler, "Fading relay channels: performance limits and space-time signal design." *IEEE Journal on Selected Areas in Communications*, vol. 22, no. 6, pp. 1099 – 109, August 2004.

[4] A. Sendonaris, E. Erkip, and B. Aazhang, "User cooperation diversity. part i. system description." *IEEE Transactions on Communications*, vol. 51, no. 11, pp. 1927 – 38, November 2003.

[5] L. Zheng and D. Tse, "Diversity and multiplexing: A fundemental tradeoff in multiple antenna channels," *IEEE Transactions on Information Theory*, vol. 49, May 2003.

[6] D. N. C. Tse, P. Viswanath, and L. H. Zheng, "Diversity-multiplexing tradeoff in multiple-access channels." *IEEE Transactions on Information Theory*, vol. 50, no. 9, pp. 1859 – 74, September 2004.

[7] K. Azarian, H. E. Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels." *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4152 – 72, December 2005.

[8] N. Prasad and M. K. Varanasi, "Diversity and multiplexing tradeoff bounds for cooperative diversity protocols." *2004 IEEE International Symposium on Information Theory, 27 June-2 July 2004, Chicago, IL, USA*, p. 268, 2004.

[9] D. Chen and J. N. Laneman, "The diversity-multiplexing tradeoff for the multiaccess relay channel," *CISS, Princeton, NJ*, March 2006.

[10] M. Yuksel and E. Erkip, "Diversity-multiplexing tradeoff in cooperative wireless systems," *CISS, Princeton, NJ*, March 2006.

[11] H. Yao, "Efficient signal, code, and receiver designs for MIMO communication systems," *MIT Ph.D Thesis*, June 2003.

[12] K. Azarian and H. E. Gamal, "The throughput reliability tradeoff in MIMO channels," *Submitted to IEEE Transactions on Information Theory*.

[13] R. Narasimhan, "Finite-SNR diversity performance of rate-adaptive MIMO systems," *Globecom 2005, St. Louis, Mo*, 2005.

[14] S. Avestimehr and D. Tse, "Outage optimal relaying in the low SNR regime," *Proc. IEEE Int. Symp. Inform. Theory, Adelaide, ISIT 2005*.

[15] E. Stauffer, O. Oyman, R. Narasimhan, and A. Paulraj, "Finite-SNR diversity-multiplexing tradeoffs in fading relay channels," *Submitted to IEEE Transactions*.

[16] R. Narasimhan, A. Ekbal, and J. M. Cioffi, "Finite-SNR diversity-multiplexing tradeoff of space-time codes," *IEEE International Conference on Communications*, vol. 1, pp. 458 – 462, 2005.

[17] M. K. Simon, *Probability Distributions Involving Gaussian Random Variables: A Handbook for Engineers and Scientists*. Springer, 2002.

[18] S. Alamouti, "A simple transmit diversity teachnique for wireless communications," *IEEE J. Sel. Areas Comm*, no. 16(8), pp. 1451 – 1458, October 1998.