Bayesian Predictions in M/G/1 Queueing System

V.K. Sehgal¹ and Pradeep Kumar Agrawal²

¹Department of Mathematical Sciences & Computer Application
Bundelkhand University, Jhansi (U.P.)
²Department of Mathematics, GLA University, Mathura (U.P.)

Abstract

In this paper we consider M/G/1 queueing system. We assume that there is a single server and the customers arrive according to Poisson process with mean rate $\lambda$ and service time distribution is general. Let the state of the system be defined by a pair of variables the number $X(t)$ in the system at epoch $t$ and the elapsed served time $Y(t)$ of the customer who is under going service. Then we have studied the bivariate Markovian process \{X(t), Y(t)\} in order to obtain the results of Non-Markovian process \{X(t)\}. In M/G/1 queueing system customer’s arrival follow a Poisson process with mean rate $\lambda$ and are served one at a time on FCFS basis. The general aim here is to analyze from Bayesian point of view the available information about the unknown parameter vector $\theta = (\lambda, \mu)$. To obtain sample information about the queue system, we observe $n$ interarrival times say $x_i$ (i=1, 2, ....n) and simultaneously or not, $m$ complete service times $y_j$ (j=1, 2, ....m). This design is not the one most widely used by the classical approach providing complete sample information about the system which consists in observing the arrival and service processes simultaneously and continuously over a fixed period of time.

Keywords: Queue, Bayesian Prediction, M/G/1 Queueing System, Poisson process.

1. Introduction

McGrath et al. [5] worked on a subjective approach to the theory of queues – modeling. Later, McGrath and Singpurwall [6] discussed a subjective Bayesian approach to theory of queues II inference and information in M/M/1 Queues. Armero [7] worked on Bayesian inference in Markovian queues. The main purpose of their work was to make statistical inference for general queues model with Poisson input and exponential services times from a Bayesian point of view. He considered the M/M/1, M/M/C, M/M/, M/M/1/k and M/M/C/k queues. He obtained the joint posterior distribution of the arrival rate and the individual service rate and particular situation of vague prior information. Armero and Bayarri [8] considered prior estimation for prediction in queues. Choi et al. [9] examined an M/G/1 queueing system with multiple types of feedback, gated vacations and FCFS policy. Ausin et al. [10] dealt with Bayesian inference and prediction for M/G/1 queueing systems. They approximated the general service time density with a class of Erlang mixtures which were phase-type distributions. Elteto and Telek [11] proposed a numerical approximation to determine the steady-state distribution of an M/G/1 type Markov process when the forward transition structure has a long tail asymptotic. Mohammadi and Salebi-Rad [12] made use of the Bayesian inference and prediction for an M/G/1 queuing model with optional second re-service. Krishna Kumar et al. [13] analyzed a single server queuing system subject to Bernoulli vacation schedules with server setup and close down periods.

Many reasons have been given in favour and also against Bayesian methodology. We can show here how Bayesian methodology can handle in a natural way some statistical issues of special relevance in the analysis of queueing systems.

2. Bayesian Predictions of M/G/1 Queueing System

In this paper we consider M/G/1 queueing system, we assume that there is a single server, where the customers arrive according to Poisson process with mean and service time distribution is general. Let the state of the system be defined by a pair of variables; the number X(t) in the system at epoch t and the elapsed serviced time Y(t) of the customer who is under going service. Then we may study the bivariate Markovian process \{X(t), Y(t)\} in order to obtain in results of Non-Markovian process \{X(t)\}.

Thus Cox defined

\[ P_x(Y,t) \]

is the joint probability and pdf of n, the number of customers in the system, including the one being served, and if the elapsed service time of the customer in service. The inclusion of one single supplementary variable makes the process Markovian in continuous time. In this case \( (y) \, dy \) is taken as the probability that the service is completed in an interval \( [y, y+dy] \) conditional upon its remaining incomplete upto time y.

In M/G/1 queueing system customers arrive following a Poisson process with mean state and are served one at a time on on FCFS basis. The general aim here is to analyze from Bayesian point of view the available information about the unknown parameter vector \( \lambda = (\lambda, \mu) \). To obtain sample information about the queue system, we observe n interarrival times say \( x_i \) (i=1, 2, ...,n) and simultaneously or not, m complete service times \( y_j \) (j=1, 2, ...,m). This design is not the one most widely used by
Bayesian Predictions in M/G/1 Queueing System

the classical approach providing complete sample information about the system which consists in observing the arrival and service processes simultaneously and continuously over a fixed period of time.

Because both the designs result in likelihood functions that are proportional to each other they should be according to the likelihood principle providing the same information about .

Let the random variable X and Y denote interarrival times and service times. This experiment is expressed in inter-arrival times because in queues with restrictions on the system capacity it may happen that the customers arriving to the queues are not the ones actually entering the system. Since the customers arriving follows Poisson process with mean arrival rate therefore inter-arrival time follow exponential distribution i.e. \( X_i \sim \text{Exp}(\lambda); i = 1, 2, \ldots, n \) and \( f(x_i, y_j) = \lambda e^{-\lambda x_i}; i = 1, 2, \ldots, n \).

The likelihood function of \( \lambda \) for the sample data is

\[
L(\lambda, \mu) = \prod_{i=1}^{n} f(x_i, \lambda) \prod_{j=1}^{m} b(y_j) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \prod_{j=1}^{m} \eta(y_j) e^{-\int_{\eta}^{u} \eta(u) du}
\]

\[
= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{j=1}^{m} \eta(y_j) e^{-\int_{\eta}^{u} \eta(u) du}
\]

A reasonable choice for a conjugate family of prior distribution directly based on the form of the likelihood function is given in (1).

Let us consider \( (y_j) = \mu y_j, \forall j = 1, 2, \ldots, m \)

Then the likelihood function reduced to

\[
L(\lambda, \mu) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{j=1}^{m} (\mu y_j) e^{-\mu \int_{\eta}^{u} \eta(u) du} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \mu^m \prod_{j=1}^{m} \left( y_j e^{-\mu y_j / 2} \right)
\]

\[
= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \mu^m \prod_{j=1}^{m} y_j e^{-\mu \sum_{j=1}^{m} y_j / 2} = \lambda^n e^{-\lambda T} \mu^m \prod_{j=1}^{m} y_j e^{-\mu \sum_{j=1}^{m} y_j / 2}
\]

where \( T = \sum_{i=1}^{n} x_i \)

Then we assume that a prior for \( \lambda \) and \( \mu \) are independent random variable with joint prior distribution.

\[
p(\lambda, \mu) = \gamma(\lambda | \alpha, \beta) \gamma(\mu | \alpha', \beta')
\]

where \( \gamma(\lambda | \alpha, \beta) = \frac{\alpha^\beta e^{-\alpha \lambda} \lambda^{\beta-1}}{\Gamma(\beta)} ; 0 < \lambda < \infty \)

\[
\gamma(\mu | \alpha', \beta') = \frac{\alpha'^\beta e^{-\alpha' \mu} \mu^{\beta'-1}}{\Gamma(\beta')} ; 0 < \mu < \infty
\]
Let \( z = \left( T, \sum_{j=1}^{m} y_j^2 / 2 \right) \)

The posterior distribution is given by using (2) and (3)

\[
p(\lambda, \mu | z) = \lambda^m e^{-\lambda T} \mu \left( \prod_{j=1}^{m} y_j \right) e^{-\mu \sum_{j=1}^{m} y_j^2 / 2} \frac{\alpha^2 e^{-\alpha \lambda^2 \beta - 1}}{\Gamma(\beta)} \frac{(\alpha)^{\beta' \mu - \alpha' \mu} \beta'^{-1}}{\Gamma(\beta')}
\]

\[
= \frac{\alpha^2 \lambda^2 \beta + n - 1 - \lambda (\alpha + T)}{\mu} \frac{(\alpha)^{\beta' \mu - \alpha' \mu} \beta'^{-1}}{\Gamma(\beta')} \frac{\beta^2 \Gamma(\beta - 1) \Gamma(\beta')}{\Gamma(\beta) \Gamma(\beta')}
\]

In Bayesian inference the probabilistic description of a situation of vogue prior knowledge about the parametric vector is a controversial subject. The corresponding non-informative posterior distribution of \( \pi(\lambda, \mu | z) \) is given by:

\[
\pi(\lambda, \mu | z) = \frac{p(\lambda, \mu | z)}{\int \int p(\lambda, \mu | z) d\lambda d\mu}
\]

Now

\[
= \int \int \lambda^n e^{-\lambda T} \mu \left( \prod_{j=1}^{m} y_j \right) e^{-\mu \sum_{j=1}^{m} y_j^2 / 2} \frac{\alpha^2 e^{-\alpha \lambda^2 \beta - 1}}{\Gamma(\beta)} \frac{(\alpha)^{\beta' \mu - \alpha' \mu} \beta'^{-1}}{\Gamma(\beta')} d\lambda d\mu
\]

\[
= \int \int \lambda^n e^{-\lambda T} \mu \left( \prod_{j=1}^{m} y_j \right) e^{-\mu \sum_{j=1}^{m} y_j^2 / 2} \frac{\alpha^2 e^{-\alpha \lambda^2 \beta - 1}}{\Gamma(\beta)} \frac{(\alpha)^{\beta' \mu - \alpha' \mu} \beta'^{-1}}{\Gamma(\beta')} d\lambda d\mu
\]

Substituting from equation (6) and equation (8) in equation (7), we get
Furthermore, after \( n > 1 \) arrivals and \( m > 1 \) service times are observed, (9) can be regarded as a limiting case of prior distribution (3) when all the parameters approach to zero, so that \((\lambda, \mu)\) can be regarded as a limiting case of the prior distribution (3) when all parameters approach to zero.

As a consequence this non-informative prior distribution provides an intuitive and straightforward interpretations of the relative contribution of the prior and sample information to the posterior knowledge of \( \lambda = (\lambda, \mu) \).

The decision problem usually requires a statistical analysis of other measure of performance of the system of the queue such as waiting times, idle and busy times, number of customers in the system whose statistical behaviour depends on the stability of the system.

References