Extension of the correspondence principle in relativistic quantum mechanics

K. G. Hernández*
Facultad de Ciencias Naturales y Matemtica, Universidad de El Salvador, Ciudad Universitaria, Final de Av. Mártires y Héroes del 30 julio, San Salvador, El Salvador, América Central.

S. E. Aguilar†
Instituto de Física Teórica, Universidade Estadual Paulista (UNESP), R. Dr. Bento Teobaldo Ferraz 271, Barra Funda São Paulo- SP, 01140-070, Brazil.

J. Bernal‡
División Académica De Ciencias Básicas, Universidad Juárez Autónoma de Tabasco, Carretera Cunduacán-Jalpa Km 1, Cunduacán, Tabasco, México. A.P. 24 C.P. 86690
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In this paper we apply the Bohr's correspondence principle to analize the asymptotic behavior of the Klein-Gordon and Dirac probability densities. It is found that in the non-relativistic limit, the densities reduce to their respective classical single-particle probability distributions plus a series of quantum corrections. The procedure is applied in two basic problems, the relativistic quantum oscillator and the relativistic particle in a box. The particle and antiparticle solutions are found to yield the same classical distribution plus quantum correction terms for the proposed limit. In the quantum oscillator case, a κ parameter modifies the probability distribution. Its origin is briefly discussed in terms of energy.

Keywords: Correspondence principle, Klein-Gordon oscillator, Dirac oscillator, Particle in a box, Non-relativistic limit.

The classical limit of quantum mechanics is a fundamental problem. There are different methodologies that have been proposed to derive classical observables from quantum observables involving constraints. The limit of quantum mechanics as the Planck constant approaches zero, $\hbar \to 0$, is one of the first methods that explore the classical limit of quantum physics [1]. However,the limite $\hbar \to 0$ is misleading since Planck's constant in a nonzero universal constant, and thus, this limit should be understood when $\hbar$ is negligible with respect to the other physical parameters like the Raileigh-Jean's law obtained from the Planck's Law. A second link between quantum and classical mechanics is the Ehrenferest's theorem, which states that the quantum mechanical expectation values of the position and momentum operators satisfy the classical equations of motion [2, 3]. However, the applicability of this theorem is neither necessary nor sufficient, since the classical limit of a quantun system is an ensemble of classical orbits, where its mean position $\langle x \rangle$ not necessarily follow the corresponding classical orbit [4]. Moreover, it implies that where Ehrenfest's theorem is not applicable, a quantum system may have a classic behavior.

Third, the Wigner's distribution function presents congruent results in study quantum corrections to classical statistical mechanics [5, 6]. Neverless, the Wigner's function needs restrictions in order to be interpreted as a probability distribution [1]. Finally, The Bohm's potential formulation of quantum mechanics states that the Hamilton-Jacobi equation can be recovered from the Schrödinger equation, resulting in the WKB method (semiclassical regime), which presents congruent results subject to constraints [7, 8].

It is well known that the partition function of a system of quantum particles can be used to derive the classical partition function of the system and its equations of state [9]. Nonetheless, this correspondence should also be studied from quantum to classical single-particle configurations. Chaotic and regular motions in the Henon-Heiles model of quantum and classical probability distributions has been studied in phase-space, the difference in the centroids of the respective quantum and classical distributions has been calculated and compared with the prediction by the Ehrenfest's theorem [10]. Consistency between classical and quantum models of a localized spin driven by a polarization requires the correspondence of the classical and quantum autocorrelation functions of the spin components [11]. Some authors compare classical and quantum probability densities of single particles [12, 13] but they do not establish a correspondence between their distributions.

Niels Bohr established a correspondence principle where a classical behavior is recovered when the quantum principal number, n, is large. Bohr applied this correspondence for frequencies and orbits of quantum systems [1, 14], and it has been applied successfully in atomic physics [15–17]. However, Ketchum contradicts the notion that classical variables can only be obtained from large principal quantum numbers, suggesting that classical frequencies can be recovered from small quantum numbers [18]. An example of "breakdown" of the Bohr's correspondence principle has been found in the semiclas-

* kevinhernandezbel@hotmail.com
† sergio.ernesto@unesp.br
‡ jorge_bernalarroyo@hotmail.com

sical regime [19] although it has been argued that the conclusions do not hold since the regime of the WKB approximation is limited [20].

Previous studies have been conducted on the correspondence between classical mechanics and quantum field theory. Earlier studies applied the Planck's limit to quantum field theory with arbitrary interactions, demonstrating that semiclassical limits are always reached [21]. A recent paper by Staunton and Browne suggest that the Poisson brackets in quantum states can be used to derive a classical trajectory [22]. Shin and Rafelski discusse the classical limit of the relativistic quantum transport theory [23]. Brown and goble are exploring the correspondence between quantum and classical electrodynamics and conclude that the bremsstrahlung amplitude has a correspondence with the classical radiation [24]. It has been proven that vacuum classical radiation occurs in the limit of low frequencies and high photon density [25]. The classical limit of the quantum electrodynamics is found when Dente use the path integral formulation in order to eleminate the photon coordinates [26].

In Refs. [1, 27, 28] the authors propose a simple mathematical method to derive the classical probability distribution of a quantum system from its corresponding quantum mechanical analogue. This method has been successfully applied to the harmonic oscillator, the infinite square well and the Kepler's problem. Additionally, the actors show that the Bohr's correspondence principle is applicable to the quantum mechanical probability densities, and its application can be performed for periodic systems (such as atomic orbits).

Here, we propose that the Bohr's correspondence principle shown in Bernal and others [1] can also be applied and extended more general to relativistic quantum systems, and our results show that, in the high enery regime (large principal quantum numbers), the quantum distribution can be written as a power series in $\hbar$, so that the zeroth order corresponds exactly with the classical single-particle probability distribution. In this paper, we focus on particular relativistic quantum systems as described by the Klein-Gordon and the Dirac equations, namely, the infinite square well and the harmonic oscillator

The methodology of our research is based on the one described in Refs. [1, 27, 28], but we include the non-relativistic limit. The formulation of this article is purely in first quantization and we only consider single eigenstates of the 1-dimensional Klein-Gordon and Dirac equations.

The relativistic probability density of the Dirac spinor that corresponds to the n-th energy state is given by $\rho_n^{RQM}(x) = \psi_n^\dagger(x)\psi_n(x)$ where ψ_n is the four component Dirac spinor that corresponds to the n-th energy state and the symbol $\dagger$ stands for conjugate transpose. In the Klein-Gordon case, the probability density is given by $\rho_n^{RQM}(x) = \phi_n^*(x)\phi_n(x)$, where $\phi_n(x)$ is the Klein-Gordon wavefunction that corresponds to the n-th energy state. The procedure can be applied for either the particle or antiparticle solutions. We denote the Fourier transform of the probability density $\rho_n^{RQM}(x)$ from position to momentum space by $f_n^{RQM}(p)$. The extension of the correspondence principle requires the calculation of the Fourier coefficient $f_n^{RQM}(p)$, such that by inverse Fourier transforming this result we obtain, in the zeroth order of approximation, the probability density in coordinate representation. The non-relativistic limit (where the speed of light is larger than the typical velocities of the system) fixes the relativistic quantum energy E_n^{QRM} to a classical value, and also affects the resulting probability distribution, denoted by $\rho_n^{QM}(x)$, so that it becomes into a classical probability density plus a series of quantum correction terms, which appear in a power series in terms of the $\hbar$ constant, as it will be seen in the relativistic quantum oscillator cases.

According to Ref. [29], the wave function of the one-dimensional Klein-Gordon oscillator is exactly the same as that of the one-dimensional Schrödinger oscillator; however, the energy spectrum is modified by the rest energy of the system. For either particle or antiparticle, the probability density for a linear combination of particle and antiparticle stationary states is given by

$$\rho_n^{RQM}(x) = \sqrt{\frac{\alpha}{\pi}}\frac{1}{2^n n!}H_n^2(\sqrt{\alpha}x)e^{-\lambda x^2}, \tag{1}$$

where $H_n(x)$ are the Hermite polynomials, n is an integer, and $\alpha \equiv \frac{m\omega}{\hbar}$, where, m stands for the mass of the particle or antiparticle, ω the frequency of the oscillator, and the energies are given by $E_n^2 = m^2c^4 + 2\big(n+\frac{1}{2}\big)mc^2\hbar\omega$ [29]. The resulting Fourier transform of Eq. (1) can be found in the literature [30]

$$f_n^{RQM}(p) = e^{-\frac{p^2}{4m\omega\hbar}} L_n\Big(\frac{p}{2m\omega\hbar}\Big), \tag{2}$$

where $L_n(x)$ is a Laguerre polynomial of degree n. The assymptotic limit of the Fourier transform can be evaluated in a simmilar way to the non-relativistic case by means of Bessel functions [1]. The inverse Fourier transform in the asymptotic limit is

$$\rho_n^{RQM}(x) \sim \frac{1}{\pi}\frac{1}{\sqrt{\kappa_n^2 - x^2}} + \frac{1}{2\pi\kappa_n}\sum_{j=1}^{\infty}\left(\frac{-\hbar^2}{S_n^2}\right)^j i_j(x,\kappa_n), \tag{3}$$

where

$$\kappa_n \equiv \sqrt{\frac{2\hbar(n+\frac{1}{2})}{m\omega}} = \sqrt{\frac{E_n^2 - m^2c^4}{m^2\omega^2c^2}}, \tag{4}$$

is a parameter found in the relativistic quantum mechanical case; $S_n = 4\sqrt{2\pi}m\omega\kappa_n^2$ and $i_j(x,\kappa_n)$ is the j-th dimensionless integral, defined in Ref. [1].

According to Ref. [1], the principal quantum number is fixed by the classical energy of the system, and this is achived by equating the expressions for the classical and quantum energies. This allows us to express the value of the principal quantum number in terms of the classical amplitude of the oscillator. In the present case, this procedure yields to $|E_n| \to mc^2 + \frac{1}{2}m\omega^2 x_0^2$, where x_0 is the amplitude of the oscillator.

It is easy to verify that in the non-relativistic case $\kappa_n \to x_0$, and therefore

$$\rho_n^{QM}(x) \sim \frac{1}{\pi}\frac{1}{\sqrt{x_0^2 - x^2}} + \frac{1}{2\pi x_0}\sum_{j=1}^{\infty}\left(\frac{-\hbar^2}{S^2}\right)^j i_j(x, x_0), \tag{5}$$

where $S = 4\sqrt{2\pi} m\omega x_0^2$ is the classical action of the particle up to a constant factor. As we can see in Eq. (5), the zeroth order result, which is $\hbar$-independent, corresponds to the classical probability distribution of a single harmonic oscillator, while the higher order terms can be interpreted as quantum corrections. This is the same result derived in non-relativistic quantum mechanics [1].

Note that we have not specified whether the wavefunction is for particles or antiparticles, it follows that for stationary states of particles or antiparticles or a linear combination of them yield to the classical probability distribution for the limit that have been considered.

Consider a Klein-Gordon stationary state for either particle or antiparticle trapped in an infinite square well of length $0 \le x \le L$. The probability density is the same as the one that would be found from the Schrödinger equation [31]

$$\rho_n^{RQM}(x) = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right), \tag{6}$$

only difference with respect to the non-relativistic case is the energy spectrum, which is given by

$$E_n^2 = mc^4 + c^2\hbar^2\frac{n^2\pi^2}{L^2}. \tag{7}$$

According to Ref. [1], by writing the probability density as a Fourier expansion, the asymptotic behavior of the corresponding Fourier coefficients is

$$f_n^{RQM}(p) \sim \frac{i\hbar}{pL}\left(e^{-i\frac{Lp}{\hbar}} - 1\right). \tag{8}$$

such that by inverse Fourier transforming we find

$$\rho_n^{RQM}(x) \sim \frac{1}{L}[H(L-x) - H(-x)], \tag{9}$$

where $H(x)$ is the Heaviside step function.

We observe that the Klein-Gordon solution yields the same classical limit as that obtained by using the non-relativistic Schrödinger equation. the reason is simply, independently of the particle's velocity, the classical-relativistic probability density for a particle trapped in a box will always be $1/L$, without apply the non-relativistic limit and, of course, taking into account that in the asymptotic limit the energies still have a discrete spectrum. Comparing this result with the Klein-Gordon oscillator, it can be interpreted that this happens because the wavefunctions and the asymptotic condition are the same as the non-relativist case of the particle in a box [1], where there is no need of explicitly fixing the energies to the classical value. The leading term of the probability density becomes independent of relativistic quantum corrections (which is later contrasted with the Dirac version of this problem). It should also be remarked that the resulting relativistic probability density is independent of the particle's velocity. An observer, at rest with respect to the well of length L, would predict the same probability of finding the particle at a an arbitrary x coordinate not matter how fast the particle is moving, as long it moves freely around the well.

The Dirac oscillator was originally proposed by Moshinsky [32], and its eigenfunctions in the 1-dimensional case can be found in Ref. [33]. For either particles or antiparticles, the probability density for single eigenstates (either spin up or down), has the form

$$\rho_n^{RQM}(x) = e^{-\alpha x^2}\left[|a_n|^2 H_n^2(\sqrt{\alpha}x) + |a_n'|^2 H_{n-1}^2(\sqrt{\alpha}x)\right], \tag{10}$$

where $\alpha \equiv \frac{m\omega}{\hbar}$, $|a_n|^2 = \frac{\sqrt{\alpha}(E_n+mc^2)}{2^{n+1}n!E_n\sqrt{\pi}}$, $|a_n'|^2 = \frac{\sqrt{\alpha}(E_n-mc^2)}{2^n(n-1)!E_n\sqrt{\pi}}$ and $E_n^2 = m^2c^4 + 2n\hbar\omega mc^2$ are the energy values for this model, denoting $E_n > 0$ for the particle solution, $E_n < 0$ for the antiparticle solution.

The Fourier transforms for the relevant terms of the probability density can be found analogously to the Klein-Gordon case, which leads to the expression

$$\begin{aligned} f_n^{RQM}(p) \sim & e^{-\frac{p^2}{4m\omega\hbar}}\left|\frac{a_n}{A_n}\right|^2 L_n\Big(\frac{p}{2m\omega\hbar}\Big) \\ & + e^{-\frac{p^2}{4m\omega\hbar}}\left|\frac{a_n'}{A_{n-1}}\right|^2 L_{n-1}\Big(\frac{p}{2m\omega\hbar}\Big), \end{aligned} \tag{11}$$

where $|A_n|^2 = \sqrt{\frac{\alpha}{\pi}}\frac{1}{2^n n!}$.

As in the Klein-Gordon case, now we compute the Fourier transform of the probability density Eq. (11) and next we calculate its asymptotic behavior. The result is

$$\begin{aligned} \rho_n^{RQM}(x) \sim & \frac{1}{\pi}\left[\left|\frac{a_n}{A_n}\right|^2 \frac{1}{\sqrt{\kappa_{1,n}^2 - x^2}} + \left|\frac{a_n'}{A_{n-1}}\right|^2 \frac{1}{\sqrt{\kappa_{2,n}^2 - x^2}}\right] \\ & + \left|\frac{a_n}{A_n}\right|^2 \frac{1}{2\pi\kappa_{1,n}}\sum_{j=1}^{\infty}\left(\frac{-\hbar^2}{S_{1,n}^2}\right)^j i_j(x, \kappa_{1,n}) \\ & + \left|\frac{a_n'}{A_{n-1}}\right|^2 \frac{1}{2\pi\kappa_{2,n}}\sum_{j=1}^{\infty}\left(\frac{-\hbar^2}{S_{2,n}^2}\right)^j i_j(x, \kappa_{2,n}), \end{aligned} \tag{12}$$

where

$$\kappa_{1,n} = \sqrt{\frac{2\hbar(n+\frac{1}{2})}{m\omega}}, \quad \kappa_{2,n} = \sqrt{\frac{2\hbar(n-\frac{1}{2})}{m\omega}},$$

and $S_{1,n} = 4\sqrt{2\pi} m\omega\kappa_{1,n}^2$, $S_{2,n} = 4\sqrt{2\pi} m\omega\kappa_{2,n}^2$. The non-relativistic case with quantum corrections, as in the Klein Gordon case, are found by fixing $E_N \to \pm\left(mc^2 + \frac{1}{2}m\omega^2 x_0^2\right)$ and choosing $N = n \pm \frac{1}{2}$, where the plus and minus sign stands for particles and antiparticles respectively. In the particle case we find that $|a_n|^2 \to |A_n|^2$, $a'_n \to 0$ and $\kappa_{1,n} \to x_0$. In the antiparticle case $a_n \to 0$, $|a'_n|^2 \to |A_{n-1}|^2$ and $\kappa_{2,n} \to x_0$. Thus for either case Eq. (12) becomes

$$\rho_n^{QM}(x) \sim \frac{1}{\pi}\frac{1}{\sqrt{x_0^2 - x^2}} + \frac{1}{2\pi x_0}\sum_{j=1}^{\infty}\left(\frac{-\hbar^2}{S^2}\right)^j i_j(x, x_0), \tag{13}$$

where $S = 4\sqrt{2\pi} m\omega x_0^2$. It should be emphasized that the system can be in an up or down spin state, or a linear combination of both, but in either case the probability density adopts the same form shown in Eq. (10), which corresponds to the classical expressions dictated by the correspondence principle. We also observe that in the fermionic case there are two $\kappa_{1,n}$ parameters appear, and both reduce in the classical limit to the same amplitude x_0, although only one of them might contribute to the probability density depending on whether the state we consider is for particles or antiparticles, or both in the case a superposition of states.

Another remark is that the energy was fixed to E_N, with $N = n \pm \frac{1}{2}$ for particles or antiparticles, because of a particular feature of the Moshinsky model for the Dirac oscillator. The Moshinsky model does not reproduce the non-relativistic energy values that would be found in quantum harmonic oscillator from the Schrödinger equation, because of how the harmonic term is added in the Hamiltonian [32], thus the non-relativistic limit of the energy spectrum creates a factor $n\hbar\omega$ instead of $(n + \frac{1}{2})\hbar\omega$ [33].

Let's consider the Dirac particle wavefunction, $\psi_k^{(+)}(x)$, for a 1-dimensional infinite square potential, which is is found explicitly in Ref. [34]. The corresponding antiparticle solution, $\psi_k^{(-)}(x)$, can be calculated as $\psi_k^{(-)}(x) = \gamma^5 \psi_{-k}^{(+)}(-x)$, where we follow the gamma matrix representation of Ref. [31] where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Either particle or antiparticle, having spin up or down, produce a probability density of the form

$$\begin{aligned}\rho_k^{RQM}(x) =& |B_k|^2 \cos^2\left(kx - \frac{\delta_k}{2}\right) \\ &+ |B_k|^2 \Phi_k^2 \sin^2\left(kx - \frac{\delta_k}{2}\right),\end{aligned} \tag{14}$$

for the interval $0 \le x \le L$, and the probability density vanishes outside this interval. We denote $|B_k|^2 \equiv \left[\frac{\Phi_k^2 - 1}{4k}(2kL - \sin(kL + \delta_k) - \sin\delta_k) + L\right]^{-1}$, $k \equiv \frac{1}{\hbar}\sqrt{\frac{E_k^2}{c^2} - m^2c^2}$, $\Phi_k \equiv \frac{\hbar kc}{E_k + mc^2}$ and $\delta_k \equiv \tan^{-1}\left(\frac{2\Phi_k}{\Phi_k^2 - 1}\right)$. The energy of the particle E_k can be found from the condition $\tan(kL) = -\frac{\hbar k}{mc^2}$ [34]. As in Eq. (8) for infinite well from Klein-Gordon, the asymptotic limit the Fourier transform of the probability density reduces to

$$f_k^{RQM}(p) \sim (1 + \Phi_k^2)|B_k|^2 \frac{i\hbar}{2p}\left(e^{-i\frac{Lp}{\hbar}} - 1\right). \tag{15}$$

The inverse Fourier transform results into

$$\rho_k^{RQM}(x) \sim \frac{1 + \Phi_k^2}{2}|B_k|^2 [H(L - x) - H(-x)]. \tag{16}$$

It should be noticed that the terms Φ_k^2 and $|B_k|^2$ of the probability density appear as fermionic quantum parameters, given that Eq. (9) for the Klein Gordon system does not contain similar factors.

The non-relativistic limit can be stated as the condition such that $\hbar k \ll mc$, which implies that $\Phi_k \to 0$ and $|B_k|^2 \to 2/L$. The probability density becomes

$$\rho_k^{QM}(x) \simeq \frac{1}{L}[H(L - x) - H(-x)]. \tag{17}$$

The resulting probability distribution shows invariance under speed boost of the particle, as it was already discussed for the Klein-Gordon particle in a box. It should be observed that the non-relativistic asymptotic probability density of the Klein-Gordon and Dirac particles do not yield a series of quantum corrections because the asymptotic term has no dependence on $\hbar$, which is also found in the non-relativistic case of Ref. [1].

The proposed extension of the Bohr's correspondence principle allowed us to derive particular classical distributions from relativistic quantum mechanical ones. The procedure applied to the Klein-Gordon equation (a relativistic theory) was very similar to Ref. [1] where the classical probability distribution is recovered through the Schrödinger equation with infinite well and the quantum harmonic oscillator potentials, with the difference lying on the energy spectrum, which in turn modifies the probability distribution when it has dependence on the particle's velocity. Our results show that the mathematical procedure proposed in this letter is applicable to the Dirac equation, and that the big component of the Dirac spinor is the only contribution that leads to the non-relativistic probability density after implementing the respective limit.

Whether we considered particle or antiparticle solutions, the same classical single particle picture plus a series of quantum corrections is found after applying the non-relativistic limit, with results that coincide with those in [1]. However if this limit is not considered, the relativistic asymptotic probability density shows manifest difference between the particle and antiparticle probability distributions, and the dependence on the c constant leads to relativistic corrections, which should be experimentally verified in a future research.

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