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H_∞ Control of Uncertain Seat Suspension Systems Subject to Input Delay and Actuator Saturation

Yingbo Zhao, Yan Ou, Lixian Zhang, Huijun Gao

***Abstract*— This paper presents a control synthesis method for a class of semi-active seat suspension systems with input time delay, parameter uncertainties and actuator saturation. A vertical vibration model of human body is employed and put together with the plant of the seat suspension system in order to gain a better insight of the control performances. The H_∞ performance is used to measure ride comfort so that more general road disturbances can be considered. By defining a Lyapunov functional and exploring the property of the saturation nonlinearity, the existence conditions of the desired state-feedback controller are derived. An illustrative example is presented to demonstrate the usefulness and effectiveness of the developed theoretical results.**

I. Introduction

Vehicle suspensions serve the basic function of isolating passengers and the chassis from the roughness of the road to provide a more comfortable ride [1], [2]. Time delays [3], [6], [7], [8], [10] and saturation nonlinearities [4], [11] are often encountered in this problem. In this study, the aim is to design a robust H_∞ state-feedback controller for a class of semi-active seat suspension systems with input time delay and actuator saturation. A vertical vibration human-body model is employed and combined with the seat to obtain a better insight of the suspension system performance. By defining a Lyapunov functional and exploring the special property of the saturation nonlinearity, a set of linear matrix inequality (LMI) conditions are obtained. And the desired controller can be designed via solving the LMIs with standard numerical algorithms so that the corresponding closed-loop system is asymptotically stable with a guaranteed disturbance attenuation level. Simulation results are given to show the effectiveness of the proposed controller design method.

The rest of this paper is organized as follows. Section 2 addresses the controller design problem for a semi-active seat suspension system with human body model. Section 3 presents the main results, including stability and performance analysis. An illustrative example demonstrating the effectiveness and advantages of the proposed methodology is given in Section 4 and some concluding remarks are given in Section 5.

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Y. Zhao, Y. Ou, L. Zhang and H. Gao are with the Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, China. Email: {zhaoyingbo1986@gmail.com; ouyanhit@gmail.com; lixianzhang@hit.edu.cn; hjgao@hit.edu.cn;}

II. Problem Formulation

In this study, a three-degree-of-freedom seat suspension model shown in Fig. 1 established by Wei and Griffin in 1998 [9] is considered for controller design. In this figure, m_1 is the mass of seat frame; m_{21} and m_{22} are the masses of human thighs together with buttocks and the seat cushion, respectively, and $m_2 = m_{21} + m_{22}$; m_3 is the mass of the upper body of a seated human. The mass of lower legs and feet is neglected because of their little contribution to the biodynamic response of the seated body. z_0 is the road displacement input, $\omega(t) = \dot{z}_0(t)$ represents the disturbance caused by road roughness; The governing equations of mo-

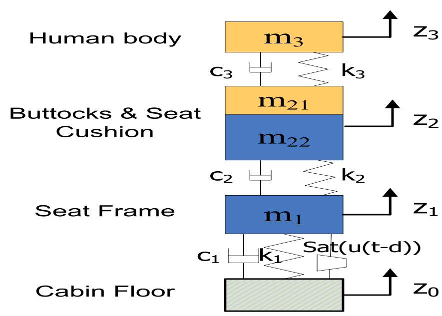

Fig. 1. Vibration model of the seat suspension system

tion for the seat suspension can be expressed as:

$$\begin{aligned} m_1\ddot{z}_1 &= -c_1(\dot{z}_1-\dot{z}_0)-k_1(z_1-z_0)+c_2(\dot{z}_2-\dot{z}_1) \\ &\quad +k_2(z_2-z_1)-\sigma(u), \\ m_2\ddot{z}_2 &= -c_2(\dot{z}_2-\dot{z}_1)-k_2(z_2-z_1)+c_3(\dot{z}_3-\dot{z}_2) \\ &\quad +k_3(z_3-z_2), \\ m_3\ddot{z}_3 &= -c_3(\dot{z}_3-\dot{z}_2)-k_3(z_3-z_2). \end{aligned} \tag{1}$$

By defining the state variable as:

$$x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) & x_5(t) & x_6(t) \end{bmatrix}^T,$$

where

$$\begin{aligned} &x_1(t) = z_1(t)-z_0(t), \quad x_2(t) = \dot{z}_1(t), \quad x_3(t) = z_2(t)-z_1(t), \\ &x_4(t) = \dot{z}_2(t), \quad x_5(t) = z_3(t)-z_2(t), \quad x_6(t) = \dot{z}_3(t), \end{aligned} \tag{2}$$

which are the deflections of the corresponding springs and velocities of the mass segments.

Then the dynamic equations in (1) can be written in the following state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w\omega(t), \tag{3}$$

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where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & \frac{c_3}{m_3} & -\frac{k_3}{m_3} & -\frac{c_3}{m_3} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ -\frac{1}{m_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; B_w = \begin{bmatrix} -1 \\ \frac{c_1}{m_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The seat suspension model becomes an uncertain model with input delay when changes in vehicle inertial properties, actuator time delays and saturation nonlinearities are taken into account, which can be expressed as:

$$\dot{x}(t) = A(\lambda)x(t) + B(\lambda)\sigma(u(t-\tau)) + B_w(\lambda)w(t), \quad (4)$$

where τ is the actuator time delay satisfying $0 < \tau \leq \bar{\tau} < \infty$, and $\bar{\tau}$ is the delay bound. Matrices $A(\lambda)$, $B(\lambda)$, and $B_w(\lambda)$ are functions of λ, which is the uncertain parameter vector. It is assumed that matrices $A(\lambda)$, $B(\lambda)$, and $B_w(\lambda)$ are constrained within the polytope Θ given by

$$\Theta \triangleq \left\{ \begin{array}{c} (A,B,B_w)(\lambda): \\ (A,B,B_w)(\lambda) = \sum\limits_{i=1}^{r} \lambda_i (A,B,B_w)_i, \\ \sum\limits_{i=1}^{r} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1,\cdots,r \end{array} \right\}, \quad (5)$$

and the actuator saturation nonlinearity is described by

$$\begin{aligned} \sigma(u(t)) &= \begin{bmatrix} \sigma(u_1(t)) & \sigma(u_2(t)) & \cdots & \sigma(u_q(t)) \end{bmatrix}^T, \\ \sigma(u_i(t)) &\triangleq \begin{cases} u_{im}, & if \quad u_i(t) \geq u_{im}, \\ u_i(t), & if \quad -u_{im} \leq u_i(t) \leq u_{im}, \\ -u_{im}, & if \quad u_i(t) \leq -u_{im}. \end{cases} \end{aligned} \quad (6)$$

Before designing the state-feedback control law for a seat suspension system, we need to consider the following aspects:

(1) Ride comfort: Ride comfort can be generally quantified by the body acceleration in the vertical direction, thus, it is chosen as the first control output, i.e., minimizing the vertical acceleration of human body $\ddot{z}_3(t)$ is one of our most concerned objectives in the controller design, that is,

$$z_{o1}(t) = \ddot{z}_3(t). \quad (7)$$

Moreover, the H_∞ norm is employed to measure the performance, whose value actually gives an upper bound of the root-mean-square gain. Hence, our goal is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $z_{o1}(t)$ in order to improve ride comfort.

(2) Suspension deflection limitation: The controller should be capable to prevent the suspension from hitting its travel limit in order to avoid ride comfort deterioration and mechanical structural damage. The requirement is

$$z_{o2}(t) = |z_1(t) - z_0(t)| \leq z_{\max}, \quad (8)$$

where $z_{\max}$ is the maximum suspension deflection limit, under all road disturbance inputs. The deflection space does not need to be minimized but its peak value needs to be limited. Since the L_∞ norm of a mathematical function in time-domain actually defines the peak value of the function, in order to meet the requirement for the suspension deflection, we can confine the L_∞ norm of the suspension deflection output $\|z\|_\infty$ under the energy-bounded road disturbance input, that is, $\|w\|_2 \triangleq \sqrt{\int_0^\infty w^T(t)w(t)dt} < \infty$.

Therefore, the strategy in the seat suspension system control law designing is to minimize the H_∞ norm of the transfer function from the disturbance $w(t)$ to the control output $z_{o1}(t)$ and guarantee the suspension stroke requirement.

Then, the vehicle seat suspension system can be described by the following state-space equations:

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + B(\lambda)\sigma(u(t-\tau)) + B_w(\lambda)w(t), \\ z_{o1}(t) &= C_1(\lambda)x(t), \\ z_{o2}(t) &= C_2(\lambda)x(t), \end{aligned} \quad (9)$$

where $A(\lambda)$, $B(\lambda)$, $B_w(\lambda)$ are already defined in (5), and

$$\begin{aligned} C_1(\lambda) &= \begin{bmatrix} 0 & 0 & 0 & c_3/m_3 & -k_3/m_3 & -c_3/m_3 \end{bmatrix}, \\ C_2(\lambda) &= \begin{bmatrix} 1/z_{\max} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

In this paper, our goal is to find a state-feedback control law

$$u(t) = Kx(t), \quad (10)$$

such that the following requirements are satisfied:

(1) the closed-loop system is asymptotically stable;

(2) under zero initial condition, the performance $\|T_{z_{o1}w}\|_\infty < \gamma$ is minimized subject to $\|z_{o2}\|_\infty < \gamma_g \|w\|_2$ for all nonzero $w \in L_2[0,\infty)$ and the prescribed constant $\gamma_g > 0$, where $T_{z_{o1}w}$ denotes the closed-loop transfer function from the road disturbance $w(t)$ to the control output $z_{o1}(t)$.

III. Robust H_∞ Controller Design

The sufficient conditions for the closed-loop system robust asymptotically stability and performance requirements can be derived as follows.

To begin with, for feedback gain matrix K, we define

$$L(K) \triangleq \{x \in \mathbb{R}^n : |k_i x| \leq u_{im},\ i = 1,2,\cdots,q\},$$

where k_i is the ith row of K. Then $L(K)$ is the region in the state space where the control input is linear in x.

Next, as shown in [5], we utilize the technique of auxiliary feedback matrices here to reduce the conservatism of dealing with the actuator saturation. Namely, for two matrices K, $H \in \mathbb{R}^{q\times n}$ and a vector $v \in \mathbb{R}^q$, a matrix set is introduced as

$$W(v,K,H) \triangleq \left\{ W \in \mathbb{R}^{q\times n} : W = \begin{bmatrix} v_1 k_1 + (1-v_1)h_1 \\ \vdots \\ v_q k_q + (1-v_q)h_q \end{bmatrix} \right\},$$

where $v_i = 0$ or 1, define $\psi(v) \triangleq \{v \in \mathbb{R}^q : v_i = 0 \text{ or } 1\}$ and the auxiliary matrix H satisfies $|h_i x| \le u_{im},\ i = 1,2,\cdots,q$. And a subset of the set $L(K)$ will be found and chosen to be an ellipsoid of the form

$$\xi(P,1) \triangleq \left\{x \in \mathbb{R}^n : x^T P x \le 1\right\},$$

where $P > 0$ will be determined. Combine $\xi(P,1)$ with

$$\begin{bmatrix} u_{im} & h_i \\ * & u_{im}P \end{bmatrix} \ge 0,\ i = 1,2,\cdots,q, \tag{11}$$

which means that if $x^T P x \le 1$, we have $2|h_i x| \le u_{im}(1 + x^T P x) \le 2u_{im}$, i.e, $|h_i x| \le u_{im}$. So we can ensure that $\xi(P,1) \subset L(H)$.

Remark 1: There are 2^q elements in $\psi(v)$. v is used to choose from the rows of K and H to form a new matrix $W(v,K,H)$. If $v_i = 0$, then the ith row of $W(v,K,H)$ is h_i, and if $v_i = 1$, then the ith row of $W(v,K,H)$ is k. For example, assume $q = 2$, then

$$\{W(v,K,H) : v \in \psi(v)\} \triangleq \left\{H, \begin{bmatrix} k_1 \\ h_2 \end{bmatrix}, \begin{bmatrix} h_1 \\ k_2 \end{bmatrix}, K\right\}.$$

Based on the above ideas, the following theorem gives the existence conditions of a desired state-feedback controller for system (9).

Theorem 1: Consider system (9) with the input-delayed state-feedback controller in (10), suppose integer $m > 0$ and the gain matrix K of the controller is given. Then the closed-loop system is asymptotically stable and satisfies $\left\|T_{z_{o1}w}\right\|_\infty < \gamma$ for all nonzero $w \in L_2[0,\infty)$ under zero initial condition if there exist matrices $P > 0$, $Q > 0$, $V > 0$, $T > 0$, S, S_m, U, $W(v,K,H)$, $W(s,K,H)$ and T_w satisfying

$$\begin{bmatrix} \Pi_{11} + W_p^T C_1^T(\lambda) C_1(\lambda) W_p & \Pi_{21}^T \\ \Pi_{21} & \hat{\Pi}_{22} \end{bmatrix} < 0, \tag{12}$$

where

$$\begin{aligned} \Pi_{11} &= W_p^T \Phi_{11}(\lambda) W_p + \mathrm{sym}(W_p^T T B(\lambda) W(v,K,H) W_b) \\ &\quad + \mathrm{sym}(\Phi_2 W_\Phi) + W_{v1}^T V W_{v1} - W_{v2}^T V W_{v2} \end{aligned}$$

and

$$\Phi_{11} = A^T(\lambda) T + T A(\lambda),\ \Omega = Q + U + U^T,$$

$$\begin{aligned} \Phi_2^T &= \begin{bmatrix} S^T & S_m^T \end{bmatrix},\ W_p = \begin{bmatrix} I_n & 0_{n,mn} \end{bmatrix}, \\ W_b &= \begin{bmatrix} 0_{n,mn} & I_n \end{bmatrix},\ W_\Phi = \begin{bmatrix} I_n & -I_n & 0_{n,(m-1)n} \end{bmatrix}, \\ W_{v1} &= \begin{bmatrix} I_{mn} & 0_{mn,n} \end{bmatrix},\ W_{v2} = \begin{bmatrix} 0_{mn,n} & I_{mn} \end{bmatrix}, \\ \Pi_{21}(\lambda) &= \begin{bmatrix} P - T + TA(\lambda) & 0_n & 0_{n,(m-2)n} \\ U & -U & 0_{n,(m-2)n} \\ T_w + B_w^T(\lambda) T & -T_w & 0_{1,(m-2)n} \end{bmatrix. \\ &\quad \begin{matrix} TBW(s,K,H) \\ 0_n \\ 0_{1,n} \end{matrix} \Bigg] + \begin{bmatrix} 0_{n,mn} & 0_n \\ -\frac{\tau}{m} S^T & -\frac{\tau}{m} S_m^T \\ 0_{1,mn} & 0_{1,n} \end{bmatrix}, \\ \hat{\Pi}_{22}(\lambda) &= \begin{bmatrix} \frac{\tau}{m} Q - 2T & 0 & TB_w(\lambda) \\ * & -\frac{\tau}{m}\Omega & -\frac{\tau}{m} T_w^T \\ * & * & -\gamma^2 \end{bmatrix}. \end{aligned}$$

Proof. In the first place, we define a Lyapunov-Krasovskii functional candidate and use the delay partitioning approach for system (4) as:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{13}$$

and

$$\begin{aligned} V_1(t) &= x^T(t) P x(t), \\ V_2(t) &= \int_{-\frac{\tau}{m}}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) Q \dot{x}(\alpha) d\alpha d\beta, \\ V_3(t) &= \int_{t-\frac{\tau}{m}}^{t} \Upsilon^T(\alpha) V \Upsilon(\alpha) d\alpha, \end{aligned}$$

where

$$\Upsilon(\alpha) = \begin{bmatrix} x(\alpha) \\ x(\alpha - \frac{1}{m}\tau) \\ \vdots \\ x(\alpha - \frac{m-1}{m}\tau) \end{bmatrix}$$

and $P > 0$, $Q > 0$, $V > 0$ are matrices to be determined.

Then, the derivative of $V(t)$ along the solution of system (4) satisfies

$$\dot{V}(t) \le \frac{m}{\tau} \int_{t-\frac{\tau}{m}}^{t} \Phi(t,\alpha) d\alpha, \tag{14}$$

where

$$\begin{aligned} \Phi(t,\alpha) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) + \frac{\tau}{m} \dot{x}^T(t) Q \dot{x}(t) \\ &\quad + \Upsilon^T(t) V \Upsilon(t) - \Upsilon^T(t - \frac{\tau}{m}) V \Upsilon(t - \frac{\tau}{m}) \\ &\quad - \frac{\tau}{m} \dot{x}^T(\alpha) Q \dot{x}(\alpha). \end{aligned}$$

Besides, for any appropriately dimensioned matrices S_m, S, U, T_w and $T > 0$ we have

$$\frac{2m}{\tau} \int_{t-\frac{\tau}{m}}^{t} \Omega_1 \left[x(t) - x(t - \frac{\tau}{m}) - \frac{\tau}{m} \dot{x}(\alpha)\right] d\alpha = 0, \tag{15}$$

and

$$2\Omega_2 \left[-\dot{x}(t) + Ax(t) + B\sigma(Kx(t-\tau)) + B_w w\right] = 0, \tag{16}$$

where

$$\begin{aligned} \Omega_1 &= \Upsilon^T(t) S + x^T(t-\tau) S_m + \dot{x}^T(\alpha) U + w^T T_w, \\ \Omega_2 &= x^T(t) T + \dot{x}^T(t) T. \end{aligned}$$

Noticing that the following equations hold

$$\begin{aligned} 2x^T T B(\lambda) \sigma(Kx(t-\tau)) &= 2\sum_{i=1}^{q} x^T T b_i \sigma(k_i x(t-\tau)), \\ 2\dot{x}^T T B(\lambda) \sigma(Kx(t-\tau)) &= 2\sum_{i=1}^{q} \dot{x}^T T b_i \sigma(k_i x(t-\tau)). \end{aligned}$$

Then, according to (6), we have

$$2x^T T B(\lambda) \sigma(Kx(t-\tau)) \le 2x^T T B(\lambda) W(v,K,H) x(t-\tau),$$

and

$$2\dot{x}^T T B(\lambda) \sigma(Kx(t-\tau)) \le 2\dot{x}^T T B(\lambda) W(s,K,H) x(t-\tau),$$

where $v(x) \in \psi(v)$, $s(x) \in \psi(s)$.

Hence, we can see from (16) that for every $x \in \xi(P,1)$ it holds that

$$2\Omega_2\left[-\dot{x}(t)+A(\lambda)x(t)+B(\lambda)\sigma(Kx(t-\tau))+B_w(\lambda)w\right] \leq \Sigma, \quad (17)$$

where

$$\Sigma = 2x^T T\varphi_1 + 2\dot{x}^T T\varphi_2,$$

and

$$\begin{aligned}\varphi_1 &= -\dot{x}(t)+A(\lambda)x(t)+B(\lambda)W_v x(t-\tau)+B_w(\lambda)w,\\ \varphi_2 &= -\dot{x}(t)+A(\lambda)x(t)+B(\lambda)W_s x(t-\tau)+B_w(\lambda)w.\end{aligned}$$

Adding Eq. (15) and (17) to Eq. (14) yields

$$\dot{V}(t) \leq \frac{m}{\tau}\int_{t-\frac{\tau}{m}}^{t} \zeta^T(t,\alpha)\Pi(\lambda)\zeta(t,\alpha)d\alpha, \quad (18)$$

where $\zeta^T(t,\alpha) = \left[\Upsilon^T(t)\ x^T(t-\tau)\ \dot{x}^T\ \dot{x}^T(\alpha)\ w^T\right]$ and

$$\Pi(\lambda) = \begin{bmatrix} \Pi_{11} & \Pi_{21}^T \\ \Pi_{21} & \Pi_{22} \end{bmatrix}, \quad (19)$$

where

$$\Pi_{22} = \begin{bmatrix} \frac{\tau}{m}Q-2T & 0 & TB_w \\ * & -\frac{\tau}{m}\Omega & -\frac{\tau}{m}T_w^T \\ * & * & 0 \end{bmatrix}.$$

Next, we establish the asymptotic stability of the system in (9) with $w(t)=0$, that is,

$$\dot{x}(t) = A(\lambda)x(t)+B(\lambda)\sigma(u(t-\tau)).$$

For the above system, $\dot{V}(t)$ in (18) reduces to

$$\dot{V}(t) \leq \frac{m}{\tau}\int_{t-\frac{\tau}{m}}^{t} \bar{\zeta}^T(t,\alpha)\tilde{\Pi}\bar{\zeta}(t,\alpha)d\alpha,$$

where $\bar{\zeta}^T(t,\alpha) = \left[\Upsilon^T(t)\ x^T(t-\tau)\ \dot{x}^T\ \dot{x}^T(\alpha)\right]$ and

$$\tilde{\Pi} = \begin{bmatrix} \Pi_{11} & \tilde{\Pi}_{21}^T \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} \end{bmatrix},$$

with

$$\begin{aligned}\tilde{\Pi}_{21} &= \begin{bmatrix} P-T+TA(\lambda) & 0_n & 0_{n,(m-2)n} \\ U & -U & 0_{n,(m-2)n} \end{bmatrix.}\\ &\quad \begin{matrix} TBW(s,K,H) \\ 0_n \end{matrix}\Bigg] + \begin{bmatrix} 0_{n,mn} & 0_n \\ -\frac{\tau}{m}S^T & -\frac{\tau}{m}S_m^T \end{bmatrix},\\ \tilde{\Pi}_{22} &= \begin{bmatrix} \frac{\tau}{m}Q-2T & 0 \\ * & -\frac{\tau}{m}\Omega \end{bmatrix}.\end{aligned}$$

It is obvious that (12) guarantees $\tilde{\Pi}<0$, which further leads to $\dot{V}(t)<0$ for any $\bar{\zeta}(t,\alpha)\neq 0$. Therefore, we conclude that system (9) with $w(t)=0$, parameter uncertainty (5), actuator saturation and time delay τ satisfying $0<\tau\leq\bar{\tau}$ is robust asymptotically stable.

Finally, we shall establish the H_∞ performance of the uncertain time-delay system under zero initial condition. Consider the following index:

$$J \triangleq \int_0^\infty \left[z_{o1}^T(t)z_{o1}(t)-\gamma^2 w^T(t)w(t)\right]dt. \quad (20)$$

Then we have

$$J \leq \int_0^\infty \left[z_{o1}^T(t)z_{o1}(t)-\gamma^2 w^T(t)w(t)+\dot{V}(t)\right]dt,$$

for any non-zero $w\in L_2[0,\infty)$.

Via some algebraic manipulations and Schur complement, it is not difficult to obtain

$$\begin{aligned}&z_{o1}^T(t)z_{o1}(t)-\gamma^2 w^T(t)w(t)+\dot{V}(t)\\ &\leq \frac{m}{\tau}\int_{t-\frac{\tau}{m}}^{t}\zeta^T(t,\alpha)\hat{\Pi}\zeta(t,\alpha)d\alpha,\end{aligned}$$

where

$$\hat{\Pi} = \begin{bmatrix} \Pi_{11}+W_p^T C_1^T(\lambda)C_1(\lambda)W_p & \Pi_{21}^T \\ \Pi_{21} & \hat{\Pi}_{22} \end{bmatrix},$$

and

$$\hat{\Pi}_{22} = \begin{bmatrix} \frac{\tau}{m}Q-2T & 0 & TB_w(\lambda) \\ * & -\frac{\tau}{m}\Omega & -\frac{\tau}{m}T_w^T \\ * & * & -\gamma^2 \end{bmatrix},$$

which is the same as (12) in Theorem 1.

Therefore, if (12) holds, i.e. $\hat{\Pi}<0$, we have $z_{o1}^T(t)z_{o1}(t)-\gamma^2 w^T(t)w(t)+\dot{V}(t)<0$, which indicates $J<0$. Hence $\|z_{o1}\|_2<\gamma\|w\|_2$ is guaranteed for any non-zero $w\in L_2[0,\infty)$, and the proof is completed. □

Theorem 2: Suppose γ, γ_g, τ are prescribed positive scalars, and $m>0$ is a known integer. Consider the semi-active suspension system in (9), if there exist matrices $\bar{P}>0$, $\bar{Q}>0$, $\bar{V}>0$, $\bar{T}>0$, $\bar{S}$, $\bar{S}_m$, $\bar{U}$, $\bar{W}(v,K,H)$, $\bar{W}(s,K,H)$ and $\bar{T}_w$ satisfying

$$\begin{bmatrix} \bar{\Pi}_{11i} & \bar{\Pi}_{21i}^T & \bar{\Pi}_{31i}^T \\ \bar{\Pi}_{21i} & \bar{\Pi}_{22i} & 0 \\ \bar{\Pi}_{31i} & 0 & -I \end{bmatrix} < 0, \quad (21)$$

where

$$\begin{aligned}\bar{\Pi}_{11i} &= W_p^T\bar{\Phi}_{11i}W_p+\mathrm{sym}(W_p^T B_i\bar{W}(v,K,H)W_b)\\ &\quad +\mathrm{sym}(\bar{\Phi}_2 W_\Phi)+W_{v1}^T\bar{V}W_{v1}-W_{v2}^T\bar{V}W_{v2},\\ \bar{\Phi}_{11i} &= \bar{T}A_i^T+A_i\bar{T},\quad \bar{\Phi}_2^T = \begin{bmatrix}\bar{S}^T & \bar{S}_m^T\end{bmatrix},\\ \bar{\Pi}_{21i} &= \begin{bmatrix} \bar{P}-\bar{T}+A_i\bar{T} & 0_n & 0_{n,(m-2)n} \\ \bar{U} & -\bar{U} & 0_{n,(m-2)n} \\ \bar{T}_w+B_{wi}^T & -\bar{T}_w & 0_{1,(m-2)n} \end{bmatrix.}\\ &\quad \begin{matrix} B_i\bar{W}(s,K,H) \\ 0_n \\ 0_{1,n} \end{matrix}\Bigg] + \begin{bmatrix} 0_{n,mn} & 0_n \\ -\frac{\tau}{m}\bar{S}^T & -\frac{\tau}{m}\bar{S}_m^T \\ 0_{1,mn} & 0_{1,n} \end{bmatrix},\\ \bar{\Pi}_{22i} &= \begin{bmatrix} \frac{\tau}{m}\bar{Q}-2\bar{T} & 0 & B_{wi} \\ * & -\frac{\tau}{m}\bar{\Omega} & -\frac{\tau}{m}\bar{T}_w^T \\ * & * & 0 \end{bmatrix},\\ \bar{\Pi}_{31i} &= \begin{bmatrix} C_{1i}\bar{T} & 0_{1,mn} \end{bmatrix},\quad \bar{\Omega} = \bar{Q}+\bar{U}+\bar{U}^T. \quad (22)\end{aligned}$$

then a stabilizing controller in the form of (10) exists, such that the corresponding closed-loop system in (4) satisfies:

1) asymptotically stable;

2) the H_∞ performance $\left\|T_{z_{o1}w}\right\|_\infty<\gamma$ is guaranteed for all nonzero $w\in L_2[0,\infty)$ under zero initial condition.

Moreover, if inequalities (21) have a feasible solution, the control gain K in (10) is given by $K=\bar{K}\bar{T}^{-1}$.

Proof. First, from the matrices $A(\lambda)$, $B(\lambda)$, $B_w(\lambda)$, $C_1(\lambda)$ and $C_2(\lambda)$ defined in (5), it follows that the feasibility of (21) ensures that

$$\begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{21}^T & \bar{\Pi}_{31}^T \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} & 0 \\ \bar{\Pi}_{31} & 0 & -I \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned} \bar{\Pi}_{11} &= W_p^T \bar{\Phi}_{11}(\lambda) W_p + \mathrm{sym}(W_p^T B(\lambda) \bar{W}(v,K,H) W_b) \\ &\quad + \mathrm{sym}(\bar{\Phi}_2 W_\Phi) + W_{v1}^T \bar{V} W_{v1} - W_{v2}^T \bar{V} W_{v2}, \\ \bar{\Phi}_{11} &= \bar{T} A^T(\lambda) + A(\lambda)\bar{T},\ \bar{\Phi}_2^T = \begin{bmatrix} \bar{S}^T & \bar{S}_m^T \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \bar{\Pi}_{21} &= \begin{bmatrix} \bar{P} - \bar{T} + A(\lambda)\bar{T} & 0_n & 0_{n,(m-2)n} \\ \bar{U} & -\bar{U} & 0_{n,(m-2)n} \\ \bar{T}_w + B_w^T & -\bar{T}_w & 0_{1,(m-2)n} \end{bmatrix} \\ &\quad \begin{bmatrix} B\bar{W}(s,K,H) \\ 0_n \\ 0_{1,n} \end{bmatrix} + \begin{bmatrix} 0_{n,mn} & 0_n \\ -\frac{\tau}{m}\bar{S}^T & -\frac{\tau}{m}\bar{S}_m^T \\ 0_{1,mn} & 0_{1,n} \end{bmatrix}, \\ \bar{\Pi}_{22} &= \begin{bmatrix} \frac{\tau}{m}\bar{Q} - 2\bar{T} & 0 & B_w \\ * & -\frac{\tau}{m}\bar{\Omega} & -\frac{\tau}{m}\bar{T}_w^T \\ * & * & 0 \end{bmatrix}, \\ \bar{\Pi}_{31} &= \begin{bmatrix} C_1(\lambda)\bar{T} & 0_{1,mn} \end{bmatrix},\ \bar{\Omega} = \bar{Q} + \bar{U} + \bar{U}^T. \end{aligned}$$

Then, according to Schur complement, (23) is equivalent to

$$\begin{bmatrix} \bar{\Pi}_{11} + W_p^T \bar{T} C_1^T(\lambda) C_1(\lambda) \bar{T} W_p & \bar{\Pi}_{21}^T \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} \end{bmatrix} < 0. \tag{24}$$

Note that (24) is equivalent to (12) in Theorem 1 by defining

$$\begin{aligned} \bar{T} &= T^{-1},\ \bar{R} = T^{-1} R T^{-1},\ \bar{S} = J_2 S T^{-1}, \\ \bar{S}_m &= T^{-1} S_m T^{-1},\ \bar{U} = T^{-1} U T^{-1},\ \bar{P} = T^{-1} P T^{-1}, \\ \bar{T}_w &= T_w T^{-1},\ \ \bar{Q} = T^{-1} Q T^{-1}, \\ J_2 &= \mathrm{diag}\left\{ \overbrace{T^{-1}, \cdots T^{-1}}^{m} \right\},\ \ \bar{V} = J_2 V J_2, \end{aligned}$$

$$\bar{W}(v,K,H) = W(v,K,H)T^{-1},\ \bar{W}(s,K,H) = W(s,K,H)T^{-1}$$

and performing a congruence transformation to (12) via $J_1 = \mathrm{diag}\left\{ \overbrace{T^{-1}, \cdots T^{-1}}^{(m+3)}, I \right\}$. Hence, the closed-loop system is asymptotically stable with an H_∞ disturbance attenuation level of γ if (21) holds and the proof is completed. □

Furthermore, using Schur complement, the feasibility of the following inequality

$$\begin{bmatrix} P & C_2^T(\lambda) \\ * & \gamma_g^2/\gamma^2 \end{bmatrix} > 0 \tag{25}$$

guarantees that $C_2^T(\lambda) C_2(\lambda) < \frac{\gamma_g^2}{\gamma^2} P$, where γ_g is the prescribed GH_2 performance bound. Meanwhile it is not difficult to see that $x^T P x < \gamma^2 \int_0^t w^T(s) w(s) ds$ if (12) holds. Then, it is obvious that

$$\begin{aligned} z_{o2}^T(t) z_{o2}(t) &= x^T C_2^T(\lambda) C_2(\lambda) x < \frac{\gamma_g^2}{\gamma^2} x^T(t) P x(t) \\ &< \gamma_g^2 \int_0^t w^T(s) w(s) ds \le \gamma_g^2 \int_0^\infty w^T(s) w(s) ds. \end{aligned}$$

And taking the supremum over $t \ge 0$ yields $\|z_{o2}\|_\infty < \gamma_g \|w\|_2$ for all $w \in L_2[0,\infty)$, which means the GH_2 performance is established.

Similar with the transformation procedure above, (25) is equivalent to

$$\begin{bmatrix} \bar{P} & \bar{P} C_2^T(\lambda) \\ * & \gamma_g^2/\gamma^2 \end{bmatrix} > 0, \tag{26}$$

which can be ensured by

$$\begin{bmatrix} \bar{P} & \bar{P} C_{2i}^T \\ * & \gamma_g^2/\gamma^2 \end{bmatrix} > 0,\ i = 1, \cdots, r. \tag{27}$$

Therefore, the feasibility of (27) guarantees that the GH_2 performance bound requirement of the suspension stroke is satisfied.

From Theorem 2 we can see that the conditions are LMIs not only over the matrix variables, but also over the objective scalar γ when γ_g is given, which implies that γ can be included as an optimization variable to obtain a lower bound of the guaranteed H_∞ performance. That is, the controller design problem has been transformed into a set of LMI conditions. Based on these conditions, the robust multi-objective state-feedback controller design can be accomplished by solving the following convex optimization problem:

$$\min \gamma \quad \text{s.t. (21), (27).} \tag{28}$$

IV. A Design Example

In order to evaluate the effectiveness and usefulness of the controller design method proposed in the above section, an example is introduced in this section. The schematic and biodynamical parameters are listed in Table 1 and the maximum suspension deflection is defined as $z_{\max} = 0.06$m. Furthermore, assume that the time delay $\tau = 10$ ms, and the human body mass m_{21} and m_3 contain uncertainties, which are expressed as

$$m_{21} = 7.8 \times (1+\lambda)\text{kg},\ \ m_3 = 43.4 \times (1+\lambda)\text{kg},$$

where λ satisfies $|\lambda| \le 0.1$.

Table 1 System parameters of the proposed seat suspension

Mass		Damping coefficient		Spring constant	
m_1	15	c_1	830	k_1	31000
m_2	1+7.8	c_2	200	k_2	18000
m_3	43.4	c_3	1485	k_3	44130

In the context of seat suspension performance, road disturbances can be generally assumed as bumps. In this work, this case of road profile is considered first to reveal the transient response characteristic, which is given by

$$z_0(t) = \begin{cases} \frac{a}{2}(1 - \cos(\frac{2\pi V_0}{l} t)), & 0 \le t \le \frac{l}{V_0}, \\ 0, & t > \frac{l}{V_0}, \end{cases} \tag{29}$$

, where a is the height of the bump, and l is the length of the bump. Here we choose $a = 0.1$ m, $l = 2$ m and the vehicle forward velocity $V_0 = 30$ (km/h).

From Tables 2, it can be seen that even the delay is not partitioned $(m = 1)$, the closed-loop system is asymptotically stable with a guaranteed H_∞ performance, which is further improved after the delay is partitioned $(m = 2)$.

Table 2 Comparison of the nominal system peak values for bump responses

$\lambda = 0$	$u_{\max}(N)$	$\ddot{z}_{3\max}(m/s^2)$	$z_{\max}(mm)$
$m = 1$	500	12.3520	45.7
$m = 1$	1500	11.1930	52.6
$m = 2$	500	12.3432	45.6
$m = 2$	1500	11.1709	52.6

The bump responses of the passive suspension and the active suspension are compared in Fig. 3, where the nominal case and the two vertex cases are plotted for clarity. It demonstrates that the closed-loop system is asymptotically stable and has a much better performance. Hence, a significant improvement in ride comfort has been made through the designed state-feedback controller in spite of the parameter uncertainty, actuator time delay and saturation.

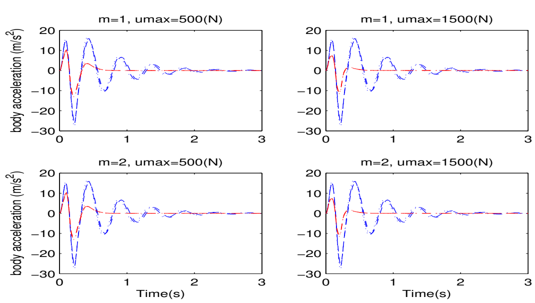

Fig. 2. Vertical accelerations of open- and closed-loop systems under bump condition

Fig. 4 depicts the active control forces in different conditions, which are confined within a reasonable range which can be generated by hydraulic or electrorheological actuators in practice. It is confirmed that the designed robust active seat suspension system is capable to guarantee a better performance under a pronounced bump disturbance.

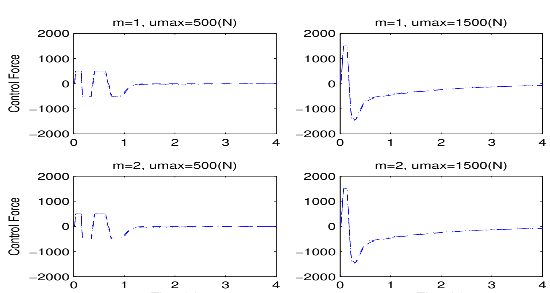

Fig. 3. Control forces of closed-loop systems

V. Concluding Remarks

The problem of multi-objective H_∞ control for a class of uncertain seat suspension systems with input time delay and actuator saturation has been dealt with in this paper by proposing a robust state-feedback controller. The delay partitioning method and an auxiliary feedback matrix have been introduced to reduce conservatism so that a better result can be obtained. The controller design has been cast into a convex optimization problem with LMI constraints. Finally, an illustrative example has been given to demonstrate the effectiveness and advantages of the proposed controller design approach.

References

[1] H. Du, J. Lam, and K. Sze. Non-fragile output feedback H_∞ vehicle suspension control using genetic algorithm. *Engineering Applications of Artificial Intelligence*, 16(7-8):667–680, 2003.

[2] H. Gao, J. Lam, and C. Wang. Multi-objective control of vehicle active suspension systems via load-dependent controllers. *Journal of Sound and Vibration*, 290(3-5):654–675, 2006.

[3] H. Gao and C. Wang. Delay-dependent robust H_∞ and L_2-L_∞ filtering for a class of uncertain nonlinear time-delay systems. *Automatic Control, IEEE Transactions on*, 48(9):1661–1666, 2003.

[4] T. Hu and Z. Lin. Exact characterization of invariant ellipsoids for single inputlinear systems subject to actuator saturation. *Automatic Control, IEEE Transactions on*, 47(1):164–169, 2002.

[5] T. Hu, Z. Lin, and B. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38(2):351–359, 2002.

[6] P. Shi, E. Boukas, and R. Agarwal. Control of Markovian jump discrete-time systems with norm boundeduncertainty and unknown delay. *IEEE Transactions on Automatic Control*, 44(11):2139–2144, 1999.

[7] Z. Wang, Y. Liu, and X. Liu. On global asymptotic stability of neural networks with discrete and distributed delays. *Physics Letters A*, 345(4-6):299–308, 2005.

[8] Z. Wang, F. Yang, D. Ho, and X. Liu. Robust H_∞ filtering for stochastic time-delay systems with missing measurements. *IEEE Transactions on Signal Processing*, 54(7):2579–2587, 2006.

[9] L. Wei and J. Griffin. The Prediction of seat transmissibility from measures of seat impedance. *Journal of Sound and Vibration*, 214(1):121–137, 1998.

[10] L. Wu, P. Shi, C. Wang, and H. Gao. Delay-dependent robust H_∞ and L_2-L_∞ filtering for LPV systems with both discrete and distributed delays. *IEE Proceedings Control theory and applications*, 153(4):483–492, 2006.

[11] L. Zhang, E. Boukas, and A. Haidar. Delay-range-dependent control synthesis for time-delay systems with actuator saturation. *Automatica*, 44(10):2691–2695, 2008.