

Orbital evolution for extreme mass-ratio binaries: conservative self forces

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The conservative dephasing effects of gravitational self forces for extreme mass-ratio inspirals are studied. Both secular and non-secular conservative effects may have a significant effect on LISA waveforms that is independent of the mass ratio of the system. Such effects need to be included in generated waveforms to allow for accurate gravitational wave astronomy that requires integration times as long as a year.

A compact object of mass μ (such as a neutron star or a stellar mass black hole) that spirals into a supermassive central object of mass $M \gg \mu$ (such as a black hole in the center of a galaxy) experiences, in addition to dissipative radiation reaction, also conservative effects that are caused by the object's self force [1]. Dissipative effects determine the *adiabatic* waveforms to $O(\mu/M)^{-1}$ [2]. Conservative effects determine corrections to the waveforms at $O(\mu/M)^0$, i.e., they are independent of the mass ratio. The latter include both secular and non-secular effects. Secular conservative effects include an additional precession of the periastron that results in a conservative dephasing of the gravitational waveforms [3, 4]. Indeed, the dephasing of the gravitational waveform per revolution is at $O(\mu/M)$ [3], but as the total number of orbits is at $O(\mu/M)^{-1}$, the cumulative dephasing effect is independent of the mass ratio. Non-secular effects include a change to the frequency of the orbit, that is not accompanied by a change to the object's energy, angular momentum, or Carter's constant [5]. Again, this effect results in dephasing at $O(\mu/M)^0$, which comes about because non-secular conservative effects are smaller than dissipative effects by $O(\mu/M)$ [2, 6].

In this Paper we estimate the magnitude of such effects for EMRI's in the post-Newtonian framework. The conservative piece of the particle's self force is an additional force that acts on the particle. Such a force results in an additional precession of the periastron. At the 1PN level, this additional rate of change of the periastron advance $\langle \dot{\omega} \rangle \sim \mu M^{1/2} r^{-5/2}$. Throughout this Paper we use geometrized units, in which $G = 1 = c$. We denote the semi-major axis with r , and the eccentricity with e . As the radiation reaction time scale is $\tau \sim r/\dot{r} \sim \mu^{-1} M^{-2} r^4$, the conservative dephasing of the gravitational waveform scales with $\langle \dot{\omega} \rangle \tau \sim (r/M)^{3/2} \sim v^{-3}$, where v is the particle's velocity [7]. We show below that this crude approximation captures well the order of magnitude of the phenomenon, specifically for low eccentricities.

Here, we seek a crude order-of-magnitude estimate of the conservative dephasing effect using dissipative terms that are accurate to 3.5PN, and conservative terms to 2PN¹ [8] for an object μ in bound motion around a Schwarzschild black hole of mass M . Note, that some of the technique needed for generalization to a Kerr black hole is already available [9]. Specifically, we take

$$\dot{r} = -\frac{32}{5\pi} \frac{\mu}{M^3} \frac{r}{f} \frac{(2\pi M f)^{11/3}}{(1-e^2)^{9/2}} \times [\alpha(1-e^2) + \beta(2\pi M f)^{2/3}] \quad (1)$$

where $f = (M + \mu)^{1/2}/(2\pi r^{3/2})$ is the orbital frequency, $\alpha = 1 + (73/24)e^2 + (37/96)e^4$ and $\beta = 1273/336 - (2561/224)e^2 - (3885/128)e^4 - (13147/5376)e^6$. The periastron advance is given by

$$\langle \dot{\omega} \rangle = 6\pi \frac{M}{r} f (1-e^2)^{-1} \left[1 + \frac{M}{r} \frac{26-15e^2}{4(1-e^2)} + \frac{\mu}{M} + \frac{\mu}{M} \frac{M}{r} \frac{(26-15e^2)}{2(1-e^2)} \right]. \quad (2)$$

Here, the terms proportional to μ are the effect of the Newtonian and 1PN self force, respectively. The former is, in fact, a consequence of the redefinition of the total mass of the system, $M + \mu$ [10]. The rate of change of the eccentricity e is given by [11]

$$\dot{e} = -\frac{1}{15} \frac{\mu M^2}{r^4} \frac{e}{(1-e^2)^{7/2}} \left[(304 + 121e^2)(1-e^2) - \frac{1}{56} \frac{M}{r} \left(\gamma + \frac{\mu}{M} \delta \right) \right] \quad (3)$$

where $\gamma = 133640 + 108984e^2 - 25211e^4$ and $\delta = 4(9352 + 8421e^2 + 847e^4)$.

Next, we integrate these equations for a quasi-elliptic orbit, and find the periastron advance. Figure 1 shows the additional periastron advance per revolution and the cumulative additional periastron advance as functions of the

¹ Notice, that Eqs. (27)–(31) in Ref. [8] are given to leading order in μ . The conservative self force effect requires an expansion to one order higher in μ in Eq. (29).

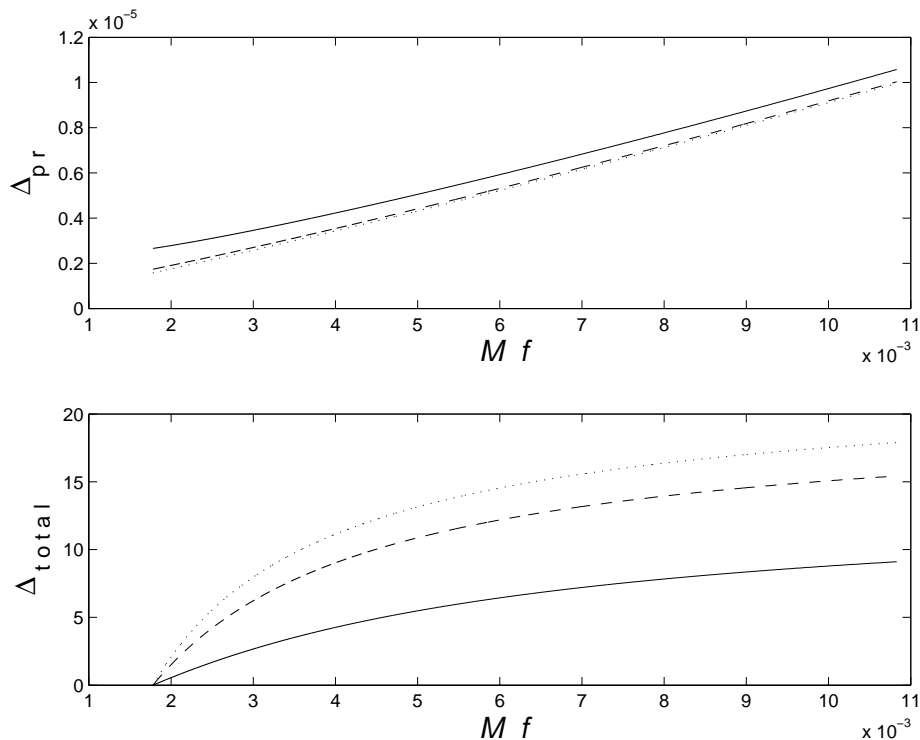


FIG. 1: The conservative dephasing due to periastron precession as a function of the orbital frequency. Upper panel: The dephasing per revolution Δ_{pr} . Lower panel: The total cumulative dephasing Δ_{total} . In all cases the integration starts at semi-major axis of $r/M = 20$, and the initial eccentricities are 0.1 (dotted curves), 0.3 (dashed curves), and 0.6 (solid curves).

orbital frequency, for three values of the initial eccentricity e_i . We do not show in Fig. 1 the periastron advance due to geodesic motion. The dependence of the total conservative dephasing on e_i is shown in Fig. 2. Clearly, in the post-Newtonian framework, the secular conservative dephasing is appreciable, and may be an important effect for LISA.

Notice, that while the method of [3] was correct, the assumed radial self forces that was used there was unrealistic. In fact, Ref. [3] assumed a radial self force at the 3PN order. In fact, the self force enters already at the 1PN level for gravitational self force (unlike scalar field self forces, for which the radial self force indeed is a 3PN effect [12]).

The non-secular conservative effects were studied in [5] for the special case of quasi-circular orbits. Their effect on the orbital frequency is given by

$$\begin{aligned} \frac{\Delta f}{f_0} = & \frac{1}{2} \frac{\mu}{M} \left\{ \left[1 - 2(2\pi M f_0)^{2/3} + \frac{61}{4}(2\pi M f_0)^{4/3} + \dots \right] \right. \\ & \left. - 3 \frac{\mu}{M} (2\pi M f_0)^{2/3} \left[1 - \frac{65}{12}(2\pi M f_0)^{2/3} + \dots \right] + \dots \right\}, \end{aligned} \quad (4)$$

where $f_0 = M^{1/2}/(2\pi r^{3/2})$. This shift in frequency is caused by a conservative radial self force, that can be found using the following method: Consider the correction to Kepler's third law, in a post Newtonian expansion:

$$\begin{aligned} \Omega^2 = & \frac{M + \mu}{R^3} \left[1 + (-3 + \nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2 \right) \gamma^2 \right. \\ & \left. + \left(-10 + \rho\nu + \frac{19}{2}\nu^2 + \nu^3 \right) \gamma^3 + \dots \right], \end{aligned} \quad (5)$$

where R is the harmonic radial coordinate, $\nu := M\mu/(M + \mu)^2$, $\gamma := (M + \mu)/R$, and ρ is a certain (known) expression [13]. Note, that while Ω^2 is a gauge invariant quantity, it is here expressed in terms of gauge dependent quantities, specifically the harmonic coordinate R . We first recognize that the ν -independent terms inside the square brackets in Eq. (5) are the leading terms in the expansion of $(1 + \gamma)^{-3}$ for $\gamma \ll 1$. Using this, we notice that

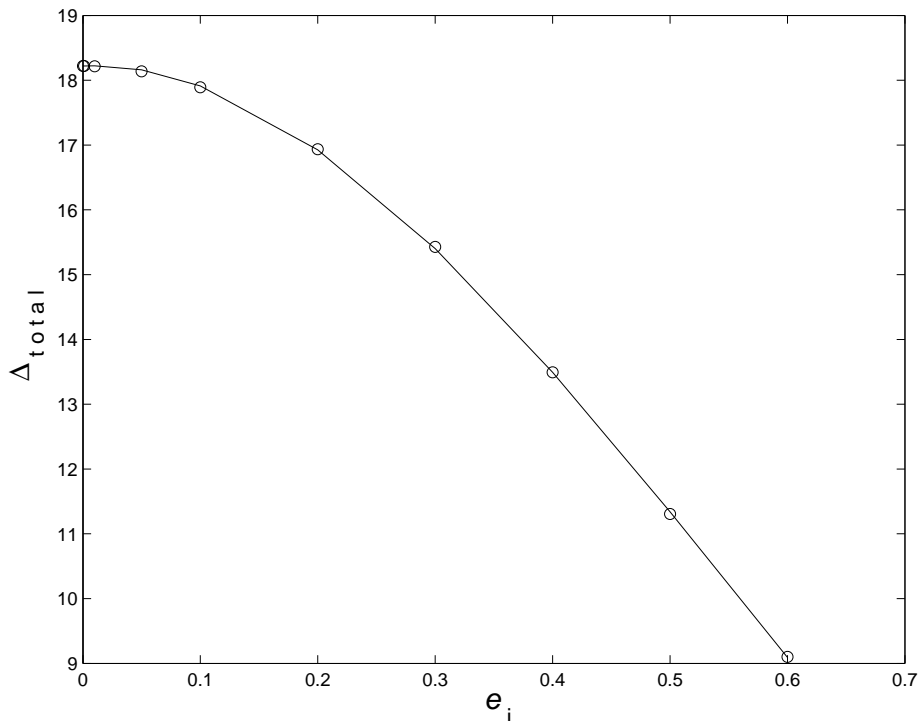


FIG. 2: The cumulative conservative dephasing Δ_{total} as a function of the initial eccentricity e_i . In all cases integration starts at semi-major axis of $r/M = 20$. The circles depict the numerical data, and the solid curve is the fitted curve of $18.219 + 0.833e_i - 41.4e_i^2 + 24.4e_i^3$.

$R^{-3}(1 + \gamma)^{-3} = r^{-3}[1 - 3\mu/r + O(\mu^2)]$. With this substitution, we express the angular velocity as

$$\begin{aligned} \Omega^2 = & \frac{M}{r^3} + \frac{\mu}{r^3} - 2\frac{M\mu}{r^4} + \frac{61}{4}\frac{M^2\mu}{r^5} + \dots \\ & - 3\frac{\mu^2}{r^4} + \frac{65}{4}\frac{M\mu^2}{r^5} + \dots, \end{aligned} \quad (6)$$

where following the Keplerian term on the rhs we present in the first line of (6) the Newtonian self force correction [10], followed by the 1PN and 2PN first-order self-force corrections, and where in the second line we present the Newtonian and 1PN second-order self-force corrections.

From Eq. (6) we can extract the form of the radial self force, in the post Newtonian gauge [5]:

$$\begin{aligned} f_r = & -\frac{\mu^2}{r^2} - \frac{\mu^2 M}{r^3} - \frac{73}{4}\frac{\mu^2 M^2}{r^4} + O(\mu^2 M^3/r^5) \\ & + 2\frac{\mu^3}{r^3} - \frac{37}{4}\frac{\mu^3 M}{r^4} + O(\mu^3 M^2/r^5). \end{aligned} \quad (7)$$

In the first line of Eq. (7) we present the first order self force, and in the next line the second order self force.

Figure 3 shows the waveforms for an orbit that starts at $r = 10M$, and decays to the inner-most stable circular orbit (ISCO) at $r = 6M$ for self-force correction at the Newtonian, 1PN and 2PN orders. Not surprisingly, close to the ISCO the post Newtonian approximation breaks down.

Figure 3 suggests that this non-secular dephasing effect may be large enough to affect LISA observations.

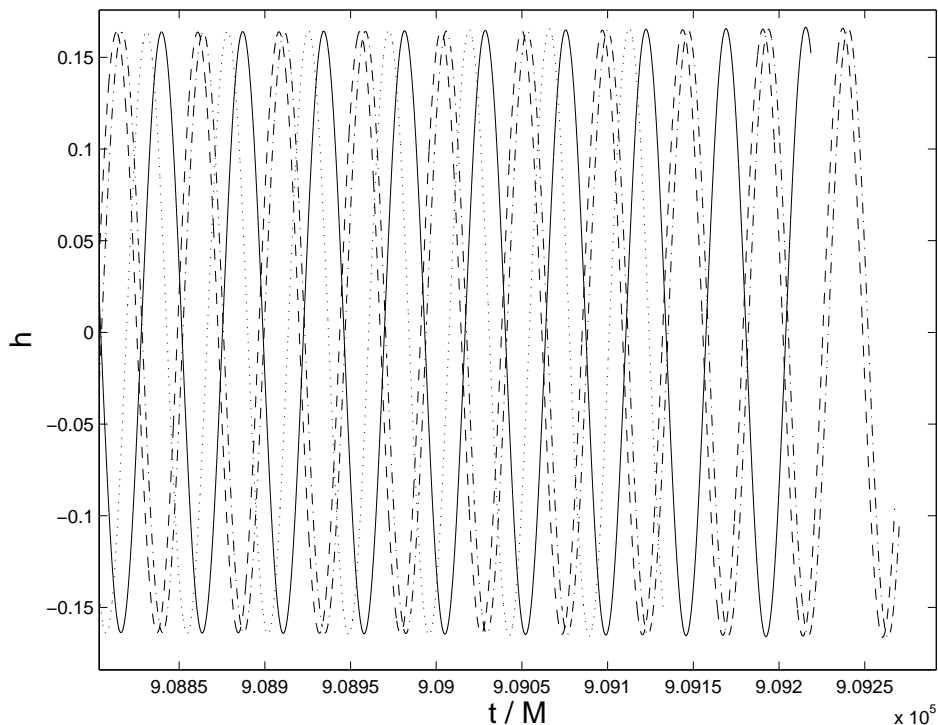


FIG. 3: The waveform for an orbit starting at $r = 10M$ at $t = 0$, and decaying to the ISCO at $6M$ for $\mu = 10^{-4}M$. Dotted line: no self force. Dash-dotted line: Newtonian self force. Dashed line: 1PN self force. Solid line: 2PN self force.

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- [1] E. Poisson, *Living Rev. Rel.* **7**, 6 (2004).
 - [2] L. M. Burko, *Phys. Rev. D* **67**, 084001 (2003).
 - [3] L. M. Burko, *Int. J. Mod. Phys. A* **16**, 1471-1479 (2001).
 - [4] A. Pound, E. Poisson, and B. G. Nickel, *Phys. Rev. D* **72**, 124001 (2005).
 - [5] L. M. Burko *Class. Quantum Grav.* **23**, 4281-4288 (2006).
 - [6] S. Drasco, É. É. Flanagan, and S. A. Hughes, *Class. Quantum Grav.* **22**, S801-846 (2005).
 - [7] E. Poisson, Talk given at the *Ninth Capra Meeting on Radiation Reaction*, (2006) (unpublished).
 - [8] L. Barack and C. Cutler, *Phys. Rev. D* **69**, 082005 (2004).
 - [9] L. M. Burko, *Class. Quantum Grav.* **22**, S847-857 (2005).
 - [10] S. Detweiler and E. Poisson, *Phys. Rev. D* **69**, 084019 (2004).
 - [11] W. Junker and G. Schäfer, *Mon. Not. R. Astr. Soc.* **254**, 146-164 (1992).
 - [12] L. M. Burko, *Phys. Rev. Lett.* **84**, 4529-4532 (2000).
 - [13] L. Blanchet, G. Faye, and B. Ponsot, *Phys. Rev. D* **58**, 124002 (1998).