Preference relation approach for obtaining OWA operators weights

Byeong Seok Ahn *

College of Business Administration, ChungAng University, 221 Heukseok, Dongjak, Seoul 156-756, South Korea

Received 31 January 2006; received in revised form 30 March 2007; accepted 2 April 2007
Available online 11 April 2007

Abstract

Actual result of aggregation performed by an ordered weighted averaging (OWA) operator heavily depends upon the weighting vector used. A number of approaches for obtaining the associated weights have been suggested in the academic literature. In this paper, we present a method for determining the OWA weights when (1) the preferences of some subset of alternatives over other subset of alternatives are specified in a holistic manner across all the criteria, and (2) the consequences (criteria values) are specified in one of three different formats: precise numerical values, intervals and fuzzy numbers. The OWA weights are to be estimated in the direction of minimizing deviations from the OWA weights implied by the preference relations, thus as consistent as possible with a priori preference relations.

© 2007 Elsevier Inc. All rights reserved.

Keywords: OWA weights; Preference relations; Linear program

1. Introduction

Decision-making involves choosing some preferable course of action among numerous alternatives. In almost all decision-making problems, there are multiple criteria for judging the alternatives. A multiple criteria decision-making (MCDM) method largely consists of two phases: (1) decision problem construction and information specification, and (2) aggregation and exploitation [1,2]. Among others, synthesizing judgments is an important part of MCDM methods. Yager [3] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating multiple inputs that lie between the max and min operators. As the term ‘ordered’ implies, the OWA operator pursues a nonlinear aggregation of objects considered, so it is clearly different from the existent multicriteria aggregation methods such as, for instance, multiattribute utility theory (MAUT) [4,5], the simple weighted sum [6], the analytic hierarchical process (AHP) [7]. The aggregation by the OWA operator is generally composed of the following three steps [8]:

* The part of this work was presented in Optimization: Theories and Applications (OTA) workshop, ICCSA, Singapore, 2005.
* Tel.: +82 2 820 5582; fax: +82 2 821 6385.
E-mail address: bsahn@cau.ac.kr
(1) Reorder the input arguments in descending order.
(2) Determine the weights associated with the OWA operator by using a proper method.
(3) Utilize the OWA weights to aggregate the reordered arguments.

In the short time since its first appearance, the OWA operators have been used in an astonishingly wide range of applications in the fields including neural networks [9,10], database systems [11], fuzzy logic controllers [12,13], group decision-making problems with linguistic assessments [14–16], data mining [17], location based service (LBS) [18] or more generally geographical information system (GIS) [19,20] and so on. The main reason for this is their great flexibility to model a wide variety of aggregators, as their nature is defined by a weighting vector, and not by a single parameter [21]. By appropriately selecting the weighting vector, it is possible to model different kinds of relations among the criteria aggregated. Recently, Xu and Da [22] presented a survey of the main aggregation operators that encompass a broad range of existing operators (more than 20 aggregators). It is clear that actual result of aggregation performed by an OWA operator depends upon the weighting vector, which plays key role in the aggregation process. Filev and Yager [23] presented a way of obtaining weights associated with the OWA aggregation in the situation where we have observed data on the arguments and the aggregated value.

Another appealing point was the introduction of the concept of orness and the definition of an orness measure that could establish how ‘orlike’ a certain operator is, based on the values of its weighting function. Thus the measure can be interpreted as the mode of decision-making circumstances by conferring the semantic meaning to the weights used in aggregation process. If an aggregated value is close to the maximum of the ordered objects, the aggregation pursues the ‘orlike’ aggregation. If an aggregated value is close to the minimum of the ordered objects, on the other hand, the aggregation pursues the ‘andlike’ aggregation. This concept perfectly coincides with the traditional decision-making theory in which max decision principle denotes the optimistic decision context and min decision principle denotes the pessimistic decision context.

On the other hand, Yager [3], based on a measure of entropy, proposed a measure of dispersion which gauges the degree of utilization of information in the sense that each of weighting vectors considered can be different to each other by degree of dispersion though they have the same degree of orness. One of the first approaches, suggested by O’Hagan [24], determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness, algorithmically based on the solution of a constrained optimization problem. The resulting weights are called maximum entropy OWA (MEOWA) weights for a given degree of orness and analytic forms and property for these weights are further investigated by several researchers [25,26]. Instead of maximizing the degree of dispersion, Fuller and Majlender [27] presented a method for deriving the minimal variability weighting vector for any level of orness, using Kuhn–Tucker second-order sufficiency conditions for optimality. Ahn [28] presented analytic forms of OWA operator weighting functions, each of which has properties of rank-based weights and a constant value of orness, irrespective of the number of objectives aggregated. Liu [29,30] proposed a series of weights generating methods in equidifferent forms, which consist of the adjacent weights with a common difference, and their related properties.

Few studies which utilize a priori paired comparison judgments on the set (or subset) of alternatives are devoted to the estimation of unknown OWA weights while maintaining consistency with the paired preference orders as much as possible. The set of a priori ordered pairs could be comprised of

- a set of past decision alternatives,
- a subset of decision alternatives, especially when a set of alternatives is large, or
- a set of fictitious alternatives, consisting of performances of the criteria which can be easily judged by the decision-maker to express his or her global comparisons [31].

In this paper, we present a method for determining the OWA weights when preference relations between alternatives are specified. The motivation for including preference relations as input is due to the fact that decision-makers are willing or able to provide such data, and that preference relations between alternatives are revealed from the past decision-making. The OWA weights are to be estimated in the direction of minimizing deviations from the OWA weights implied by the preference relations, thus as consistent as possible with a
priori preference relations. They can be used to prioritize the other alternatives that are not included in a set of a priori ordered pairs of alternatives for reverse decision aiding purpose.

The paper is organized as follows: in Section 2, methods for determining OWA weights are presented considering three cases: the elements in the decision matrix are specified in the form of (1) precise numerical values, (2) intervals, and (3) fuzzy numbers. Concluding remarks follow in Section 3.

2. Identifying the OWA weights consistent with ordered pairs

An OWA operator [3] of dimension \( n \) is a mapping \( f : R^n \rightarrow R \) that has an associated weighting \( n \) vector \( W = (w_1, w_2, \ldots, w_n)^T \) such that \( w_i \in [0,1] \) for \( i \in I = \{1,2,\ldots,n\} \) and \( \sum_{i \in I} w_i = 1 \). Under this type of operator, the function value \( f \) determines the aggregated value of arguments \( a_1, a_2,\ldots,a_n \) in such a manner that \( f(a_1, a_2,\ldots,a_n) = \sum_{i \in I} w_i b_i \), where \( b_i \) is the \( i \)th largest element in the collection, thus satisfying the relation

\[
\min[a_i] \leq f(a_1, a_2,\ldots,a_n) \leq \max[a_i].
\]

The fundamental aspect of the OWA operator is the re-ordering step, in particular, an argument \( a_i \) is not associated with a particular weight \( w_i \), but rather a weight \( w_i \) is associated with a descending ordered position, \( i \) of the arguments \( a_1, a_2,\ldots,a_n \), thus yielding a nonlinear aggregation. Its generality lies in the fact that by selecting appropriate weights, different aggregation can be implemented. Thus, if we place most of the weights near the top of \( W \), we can emphasize the higher scores, while placing the weights near the bottom of \( W \) emphasizes the lower scores in the aggregation [3].

Assume a decision matrix is given such as

\[
A = \begin{bmatrix}
C_1 & C_2 & \cdots & C_n \\
A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_n & a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\]

In the above matrix, the set \( A = \{A_1, A_2,\ldots,A_m\} \) corresponds to a set of alternatives and the set \( C = \{C_1, C_2,\ldots,C_n\} \) corresponds to the set of multiple criteria considered. In the above, \( a_{ij}, i = 1,\ldots,m, j = 1,\ldots,n \) indicates a consequence (or outcome, payoff, value, etc.) for selecting alternative \( A_i \), when the state of nature is \( C_j \) and is assumed to be in the form of precise numerical value or in the form of imprecise value, e.g., bounded descriptions. Let \( \Theta \subseteq A \times A \) denote the set of ordered pairs \( (i,j) \) where \( i \) designates a preferred alternative from a paired comparison involving \( i \) and \( j \). The occurrence of ordered pairs stems from several reasons previously described in Section 1.

We shall now develop a method for determining the OWA weights in three cases in which the consequence \( a_{ij}, i = 1,\ldots,m, j = 1,\ldots,n \), is in the form of precise numerical values, intervals or fuzzy numbers.

2.1. The consequences in the form of precise numerical value

Assume a decision situation in which the consequence \( a_{ij}, i = 1,\ldots,m, j = 1,\ldots,n \) is specified in precise numerical values and a priori ordered pairs on the subset of alternatives are obtained. Let us define an optimal solution \( W^* \) to be a set of the OWA weights \( \{w_k^*\} \) for \( k = 1,\ldots,n \). The solution would be consistent with the decision-maker’s holistic judgments between alternatives if \( f(A_i) > f(A_j) > 0 \) for every a priori ordered pair \( (i,j) \in \Theta \) and for all feasible values of \( W = \{(w_1,\ldots,w_n) : \sum_{k=1}^n w_k = 1, w_k \geq 0, k = 1,\ldots,n\} \). Here \( f(A_i) \) and \( f(A_j) \) denote the aggregated value of input arguments of alternatives \( A_i \) and \( A_j \), respectively. We can state this as, for all \( (i,j) \in \Theta \),

\[
\sum_{k=1}^n (b_{ik} - b_{jk})w_k > 0 \quad \text{for } w_k \in W,
\]

in which \( b_{ik} \) and \( b_{jk} \) are the reordered arguments of the arguments \( a_{i1},\ldots,a_{in} \) and \( a_{j1},\ldots,a_{jn} \) respectively. Thus, the goal of analysis is to determine the solution \( W^* \) for which the conditions such as \( \sum_{k=1}^n (b_{ik} - b_{jk})w_k \geq \varepsilon \) for every a priori ordered pair \( (i,j) \in \Theta \) are violated as minimally as possible in which \( \varepsilon \) is a small arbitrary
positive number to make the problem tractable by linear program. To attain the objective “as minimally as possible”, we use auxiliary variables \( \delta_{ij} \) in \( \sum_{k=1,n} (b_{ik} - b_{jk})w_k + \delta_{ij} \geq \epsilon \) for every ordered pair \((i,j) \in \Theta\) and minimize the sum of auxiliary variables in the objective as shown

\[
\text{Minimize } \sum_{(i,j) \in \Theta} \delta_{ij} \tag{1a}
\]

\[
\text{s.t. } \sum_{k=1}^n (b_{ik} - b_{jk})w_k + \delta_{ij} \geq \epsilon \quad \text{for all } (i,j) \in \Theta \tag{1b}
\]

\[
w_k \in W \quad \text{for } k = 1, \ldots, n, \delta_{ij} \geq 0 \quad \text{for all } (i,j) \in \Theta, \quad \epsilon > 0. \tag{1c}
\]

The preference relations other than the strictly ordinal paired orders can be included in the model. Weak ordinal relations between alternatives (e.g., \( A_i \) is at least as preferred as alternative \( A_j \)) or preferences with ratio comparisons of some paired alternatives (e.g., \( A_i \) is \( z_{ij} \) times more important than alternative \( A_j \)) are some examples that can be included in the incomplete holistic judgments. These holistic judgments are then used to determine the OWA weights as the system of constraints restricting the feasible region of the weights.

**Remark.** If the goal of analysis is to determine the maximum entropy OWA weights \( W^\epsilon \) for which the conditions such as \( \sum_{k=1,n} (b_{ik} - b_{jk})w_k \geq \epsilon \) for every a priori ordered pair \((i,j) \in \Theta\) are satisfied, this consideration leads to following mathematical program as shown below:

\[
\text{Maximize } \sum_{k=1}^n -w_k \ln w_k \tag{2a}
\]

\[
\text{s.t. } \sum_{k=1}^n (b_{ik} - b_{jk})w_k \geq \epsilon \quad \text{for all } (i,j) \in \Theta \tag{2b}
\]

\[
w_k \in W \quad \text{for } k = 1, \ldots, n, \quad \text{for all } (i,j) \in \Theta, \quad \epsilon > 0 \tag{2c}
\]

Instead of the maximal entropy objective function in (2a), other objective functions can be considered as well, depending on the goal to attain and they are listed below:

- minimize: \((1/n) \sum_k (w_k - 1/n)^2\) for a minimum variance approach [27],
- minimize: \(\max \sum_k |w_i - w_{i+1}|\) for a minimax disparity approach [32],
- minimize: \(\sum_k (w_k - 1/n)^2\) for a least square OWA approach [33],
- minimize: Max\(_k\)\(_w_k\) for another expression of a measure of entropy [34].

In addition to the alternation of the objective function, experts’ knowledge can be included as constraints in the proposed model to influence the relative importance of some criteria over others. The possible formats might include, but are not limited to:

\[
w_1 \geq w_2 \geq \cdots \geq w_n \quad \text{(in some cases, a strict preference } w_1 > w_2 > \cdots > w_n \text{ is required)}
\]

\[
\sum_{k=1}^n w_k \geq \sum_{k=i+1}^n w_k, \quad \text{(the value } i<n \text{ is chosen by decision-maker)}
\]

\[
w_i \geq k \cdot w_j, \quad k \geq 0, \quad i \neq j, \quad (k \text{ represents twice or three times, etc.}).
\]

**Example.** Suppose that there is a decision matrix with five alternatives characterized by three criteria as shown in Table 1. Further, we assume that a decision-maker indicates paired judgments on the alternatives such as \( \Theta = \{(2,3),(2,4),(3,4),(4,1),(1,5)\} \), i.e., alternative \( A_2 \) is preferred to alternative \( A_3 \), alternative \( A_2 \) to alternative \( A_4 \), and so on.

The holistic judgment, for example, “alternative \( A_2 \) is more preferred to alternative \( A_3 \)” implies that an aggregated value of alternative \( A_2 \) by the OWA weights is greater than that of \( A_3 \) resulted by applying the same OWA weights. Accordingly, this can be denoted as a constraint such as \( 0.9w_1 + 0.7w_2 + 0.6w_3 > 0.8w_1 + 0.8w_2 + 0.5w_3 \), which can be further rearranged as \( 0.1w_1 - 0.1w_2 + 0.1w_3 > 0 \). Similarly, the other constraints corresponding to the ordered pairs can be constructed and the entire system of constraints becomes:
The OWA weights are to be calculated in the direction of satisfying this system of constraints, thus as consistent as the ordered pairs. This consideration leads to the following linear program (an arbitrary small number $e$ is replaced by 0.0001):

\[
\begin{align*}
\text{Minimize} & \quad \delta_{23} + \delta_{24} + \delta_{34} + \delta_{41} + \delta_{15} \\
\text{s.t.} & \quad 0.1w_1 - 0.1w_2 + 0.1w_3 + \delta_{23} \geq 0.0001 \\
& \quad -0.1w_1 + 0.3w_2 + 0.2w_3 + \delta_{24} \geq 0.0001 \\
& \quad -0.2w_1 + 0.4w_2 + 0.1w_3 + \delta_{34} \geq 0.0001 \\
& \quad 0.2w_1 - 0.1w_2 + \delta_{41} \geq 0.0001 \\
& \quad 0.1w_2 + \delta_{15} \geq 0.0001 \\
& \quad \delta_{23}, \delta_{24}, \delta_{34}, \delta_{41}, \delta_{15} \geq 0, w_1 + w_2 + w_3 = 1, w_1, w_2, w_3 \geq 0 
\end{align*}
\]

Commercial softwares associated with linear program can be used to solve the program, which results in the OWA weights $W^* = (0.5998, 0.4002, 0)$ with the objective function value equal to zero. This implies that the OWA weights $W^*$ is perfectly consistent with the ordered pairs assessed by the decision-maker.

### 2.2. The consequences in the form of interval

Assume a decision situation in which the consequence $a_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$ is specified in the form of intervals and a priori ordered pairs on the subset of alternatives are obtained. The possible reasons regarding the interval values are attributed to time pressure, lack of data, information difficult to quantify by a precise value due to its nature, decision-maker’s limited attention and information processing capabilities, and/or high cost of its computation [1,2]. Thus, many attempts have been made with a view to relaxing the burden of preference specifications imposed on the decision-maker and thus taking into account the inexactness or vagueness of human judgment.

Similarly, the optimal OWA weights $\{w_k^*\}$ for $k = 1, \ldots, n$ would be consistent with the decision-maker’s preferences if $f(A_i) - f(A_j) > 0$ subject to the given input arguments for every a priori ordered pair $(i,j) \in \Theta$ and for all feasible values of $W$.

One way to derive strict weights that satisfy the relation $f(A_i) - f(A_j) > 0$ is that, for $(i,j) \in \Theta$,

\[
\sum_{k=1}^{n} y_k(i,j) \cdot w_k > 0 \quad \text{for} \quad w_k \in W, \\
y_k(i,j) = \min(b_{ik} - b_{jk}) \\
\text{s.t.} \quad b_{ik} \in [b_{ik}^l, b_{ik}^u], b_{jk} \in [b_{jk}^l, b_{jk}^u]
\]

### Table 1

An example with precise numerical values

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_1$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>0.8</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.5</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.4</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
0.1w_1 & + 0.1w_3 > 0 \quad \text{for} \quad (2, 3) \in \Theta \\
-0.1w_1 + 0.3w_2 + 0.2w_3 > 0 \quad \text{for} \quad (2, 4) \in \Theta \\
-0.2w_1 + 0.3w_2 + 0.1w_3 > 0 \quad \text{for} \quad (3, 4) \in \Theta \\
0.2w_1 - 0.1w_2 & > 0 \quad \text{for} \quad (4, 1) \in \Theta \\
0.1w_2 & > 0 \quad \text{for} \quad (1, 5) \in \Theta
\end{align*}
\]
in which $b_{jk}$ and $b_{jk}$ are the reordered arguments of the interval arguments $a_{i1}, \ldots, a_{in}$ and $a_{i1}, \ldots, a_{jn}$ respectively, and the weights set $W = \{ (w_1, \ldots, w_n); w_1 + \cdots + w_n = 1, w_k \geq 0, k = 1, \ldots, n \}$. To proceed a further analysis, we have to reorder the input arguments that are described in intervals. Let us denote $a_{ij} = [a_{ij}, a_{ij}']$, $i = 1, \ldots, m, j = 1, \ldots, n$ in which $a_{ij}$ is a lower bound and $a_{ij}'$ is an upper bound. The problem of prioritizing intervals, or fuzzy numbers more in general, is well known and it has been deeply investigated since 1970s. A first attempt to solve this problem is generally attributed to Baas and Kwakernaak [35]. Since then, not a few proposals have been proposed to deal with the problem (see some reviews of the proposed approaches [36–40]. Recently, Xu and Da [41] presented formulas for comparing input arguments when each of them is specified in the form of interval number, thus yielding $b_i$, the $i$th largest element of the collection of $n$ interval objects, $a_1, a_2, \ldots, a_n$. (see Appendix A). Considering that the interval numbers within the interval sometimes do not have the same meaning for decision-maker as is implied by the use of interval ranges, Ahn [42] presented a way of prioritizing the interval numbers, taking into account the strength of preference based on the probabilistic measure (see Appendix B). The similarity between two interval numbers can be gauged by two measures characterizing the intervals: the ratio of overlapping portion of two interval numbers and the level of closeness of midpoints between two interval numbers [43]. These considerations lead a plausible formula which is described in Appendix C.

If the input arguments that are represented in interval numbers are prioritized by one of three methods, the goal of analysis is to determine the solution $W^*$ for which the conditions such as $\sum_{k=1,n}y_k(i,j) \cdot w_k \geq \varepsilon$ is a small arbitrary positive number) for every a priori ordered pair $(i,j) \in \Theta$ are violated as minimally as possible. The constraints $\sum_{k=1,n}y_k(i,j) \cdot w_k \geq \varepsilon$ can be written simply as $\sum_{k=1,n}(b_{ik}^l - b_{jk}^u)w_k \geq \varepsilon$. Thus the OWA weights satisfying a priori preference relations between alternatives can be obtained by solving the following linear program:

Minimize $\sum_{(i,j) \in \Theta} \delta_{ij}$

s.t. $\sum_{k=1,n}(b_{ik}^l - b_{jk}^u)w_k + \delta_{ij} \geq \varepsilon$ for all $(i,j) \in \Theta$

$w_k \in W$ for $k = 1, \ldots, n, \delta_{ij} \geq 0$ for all $(i,j) \in \Theta$, $\varepsilon > 0$.

Remark. The minimization of potential violations is the simplest and most natural of several possible objectives. However, positive error terms for some ordered pairs in the optimal solution might produce contrary results to the original paired preference judgments even though the model is trying to attaining as minimum a value as possible. When these cases occur, further interactive modifications with the decision-maker can be performed for reducing the estimation errors: (1) checking and possible modification of consequences and (2) checking and possible rearrangement of preference orders between alternatives specified by the decision-maker. In doing so, we may identify the ordered pairs which contribute to the violating terms of the objective function. The ordered pairs most significantly violating the preference orders are presented to the decision-maker to check if the preference order of the ordered pairs can be reversed or the ordered pairs can be removed. The interactive modifications with the decision-maker might result in the OWA weights that progressively fit the preference orders.

Remark. Instead of the objective of minimization of potential violations, we can consider the objective which minimizes the number of violations since the number of violations might increase although the sum of the $\delta_{ij}$ is attained at a minimum. This consideration can be set forth by the following formulation:

Minimize $\sum_{(i,j) \in \Theta} \delta_{ij}$

s.t. $\sum_{k=1,n}(b_{ik}^l - b_{jk}^u)w_k + M\delta_{ij} \geq \varepsilon$ for all $(i,j) \in \Theta$

$w_k \in W$, for $k = 1, \ldots, n, \delta_{ij} = 0, 1$ for all $(i,j) \in \Theta$, $\varepsilon > 0$. 
\(M\) denotes some very large number. Therefore the minimum value of \(\sum_{(i,j) \in \Theta} \delta_{ij}\) must occur when the number of violations of ordered pairs is minimized. The above model can be solved by a mixed integer programming code. Of course, with regard to the objectives there is no guarantee which objective is better for predicting the criteria weights. One interesting point to be noted is that two objectives were tested in a real world application and the objective of minimizing the number of violation appeared to have an edge over the objective of minimizing the amount of violation [44].

**Example.** Suppose that there is a decision matrix with five alternatives characterized by three criteria. As can be seen in Table 2, \(a_{ij} (i = 1, \ldots, 5, j = 1, 2, 3)\) indicating the consequence for selecting alternative \(A_i\) when the state of nature is \(C_j\), is assumed to be specified in the form of interval numbers.

The input arguments of each alternative have to be reordered to make the aggregated values by the OWA code. Of course, with regard to the objectives there is no guarantee which objective is better for predicting the criteria weights. One interesting point to be noted is that two objectives were tested in a real world application and the objective of minimizing the number of violation appeared to have an edge over the objective of minimizing the amount of violation [44].

Suppose that a decision-maker indicates paired judgments on the alternatives such as \(\Theta = \{(2,3), (2,4), (3,4), (1,4), (1,5)\}\), i.e., alternative \(A_2\) is preferred to alternative \(A_3\), alternative \(A_2\) to alternative \(A_4\), and so on. The holistic judgment, for example, "alternative \(A_2\) is preferred to alternative \(A_3\)" constitutes a constraint such as

\[f(A_2) = a_{23}w_1 + a_{22}w_2 + a_{21}w_3 > a_{33}w_1 + a_{32}w_2 + a_{31}w_3 = f(A_3)\]

This inequality can be further rearranged as \(-0.2w_1 - 0.1w_2 - 0.2w_3 > 0\) since

\[(d_{23}^{l} - d_{33}^{u})w_1 + (d_{22}^{l} - d_{32}^{u})w_2 + (d_{21}^{l} - d_{31}^{u})w_3 > 0\]

Similarly, the other constraints can be constructed for the holistic judgments between alternatives. The OWA weights are to be calculated in the direction of satisfying this system of constraints, thus as consistent as the ordered pairs. This consideration leads to the following linear program:

\[
\begin{align*}
\text{Minimize} & \quad \delta_{23} + \delta_{24} + \delta_{34} + \delta_{14} + \delta_{15} \\
\text{s.t.} & \quad 0.1w_1 + 0.05w_3 + \delta_{23} \geq 0.0001 \\
& \quad 0.1w_1 + 0.1w_2 + 0.05w_3 + \delta_{24} \geq 0.0001 \\
& \quad -0.05w_1 - 0.1w_2 - 0.05w_3 + \delta_{34} \geq 0.0001 \\
& \quad -0.05w_1 - 0.05w_2 + 0.05w_3 + \delta_{14} \geq 0.0001 \\
& \quad -0.05w_3 + \delta_{15} \geq 0.0001 \\
& \quad \delta_{23}, \delta_{24}, \delta_{34}, \delta_{14}, \delta_{15} \geq 0, w_1 + w_2 + w_3 = 1, w_1, w_2, w_3 \geq 0
\end{align*}
\]

Table 2
An example with interval values

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>Criteria</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>[0.7, 0.8]</td>
<td>[0.65, 0.7]</td>
<td>[0.6, 0.7]</td>
<td></td>
</tr>
<tr>
<td>(A_2)</td>
<td>[0.6, 0.75]</td>
<td>[0.7, 0.8]</td>
<td>[0.85, 0.9]</td>
<td></td>
</tr>
<tr>
<td>(A_3)</td>
<td>[0.5, 0.55]</td>
<td>[0.5, 0.7]</td>
<td>[0.7, 0.75]</td>
<td></td>
</tr>
<tr>
<td>(A_4)</td>
<td>[0.5, 0.6]</td>
<td>[0.7, 0.75]</td>
<td>[0.5, 0.55]</td>
<td></td>
</tr>
<tr>
<td>(A_5)</td>
<td>[0.5, 0.65]</td>
<td>[0.65, 0.7]</td>
<td>[0.6, 0.65]</td>
<td></td>
</tr>
</tbody>
</table>
The solution of the linear program shows the OWA weights $W^* = (0.499, 0.501, 0)$ with the objective function value equal to 0.07525 ($\delta_{34} = 0.07515, \delta_{15} = 0.0001$). The positive values of the auxiliary variables imply that the constraints concerning the preference orders $\{(3, 4), (1, 5)\} \subset \Theta$ are not satisfied, thus the affirmations “the alternative $A_3$ is preferred to $A_4$” and “the alternative $A_1$ is preferred to $A_5$” are not valid. This problem arises since the two reordered arguments are so similar that it is difficult to strictly satisfy the specified preference order. For example, the determination of the OWA weights that satisfy the preference order $f(A_3) > f(A_4)$ seems difficult for the case $(3, 4) \in \Theta$ (the coefficients of the functional equation, $f(A_3) - f(A_4) > 0$ are all negative) since the reordered arguments are close each other such as

$$b_{31} = [0.7, 0.75], \quad b_{32} = [0.5, 0.7], \quad b_{33} = [0.5, 0.55]$$

$$b_{41} = [0.7, 0.75], \quad b_{42} = [0.5, 0.6], \quad b_{43} = [0.5, 0.55]$$

If the decision-maker retracts the preference order $(3, 4)$ and changes the imprecise consequences of alternative $A_5$ to $[0.50, 0.65]$ for $C_1$, $[0.60, 0.66]$ for $C_2$, and $[0.50, 0.67]$ for $C_3$, we can obtain optimal OWA weights such as $(w_1, w_2, w_3) = (0.499, 0.173, 0.328)$.

If both precise and inexact numerical values are mixed in the input arguments and they have to be reordered, the precise numerical values can be transformed to the interval values, taking into account the estimation error $\delta$ which is a small number. Thus a precise numerical value $N$ can be converted into $[N - \delta, N + \delta]$.

Further analysis can be performed to derive the OWA weights as shown below.

**Example.** A chief information officer (CIO) at a medium-sized company is reviewing the past decision made to select a project among four competing projects denoted by PROJ-1, PROJ-2, PROJ-3, and PROJ-4, which were evaluated by three criteria:

- whether the company has experience related to the project (Experience),
- how much it acquires prominence after it finishes the project (Prominence),
- the possibility of fulfillment by the due date (Fulfillment).

The CIO’s preference relations between projects were such that $\Theta = \{(1,4), (2,3)\}$ and the elements in decision matrix are composed of deterministic and interval numbers as shown in Table 3.

Set $\delta = 0.05$ to make the consequences of the projects with respect to the criterion “Experience” into interval expression. The OWA weights generated by applying the program (3a)–(3c) are $w_E = 0.6004$, $w_P = 0.3996$, and $w_F = 0$ with the sum of errors equal to zero.

### 2.3. The consequences in the form of fuzzy number

Assume a decision situation in which the consequence $a_{ij} = [a_{lij}, a_{mij}, a_{uij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ is specified in the form of fuzzy numbers and a priori ordered pairs on the subset of alternatives are obtained. Rather than performing fuzzy OWA aggregation for obtaining preference-based OWA weights, the fuzzy decision matrix that is composed of the fuzzy numbers is transformed into an expected decision matrix for simplicity of calculation [45,46]. Thus an element in the expected decision matrix becomes

$$\bar{a}_{ij} = \frac{1}{2}[(1 - \eta)a_{lij} + a_{mij} + \eta a_{uij}], \quad \eta \in [0, 1], \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

### Table 3
An example with deterministic and interval values

<table>
<thead>
<tr>
<th>Projects</th>
<th>Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Experience ($w_E$)</td>
</tr>
<tr>
<td>PROJ-1</td>
<td>0.7</td>
</tr>
<tr>
<td>PROJ-2</td>
<td>0.6</td>
</tr>
<tr>
<td>PROJ-3</td>
<td>0.4</td>
</tr>
<tr>
<td>PROJ-4</td>
<td>0.2</td>
</tr>
</tbody>
</table>
where $\eta$ is an index that reflects a decision-maker’s attitude toward risk. If $\eta > 0.5$, $\eta < 0.5$, or $\eta = 0.5$, then it is said that the decision-maker shows a risk-taking, risk-averse, or risk-neutral tendency respectively. The elements in the expected decision matrix are further normalized in column-wise to obtain comparable scales. If the criteria are benefit criteria, that is, more is better, then

$$
\tilde{r}_{ij} = \frac{\sum_{i=1}^{m} a_{ij}}{\sum_{i=1}^{m} a_{ij}}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.
$$

If $f(A_i)$ and $f(A_j)$ denote the aggregated value of input arguments of alternatives $A_i$ and $A_j$ respectively, then, for all $(i,j) \in \Theta$,

$$
\sum_{k=1}^{n} (\tilde{r}_{ik} - \tilde{r}_{jk})w_k > 0, \quad \text{for } w_k \in W
$$
in which $\tilde{r}_{ik}$ and $\tilde{r}_{jk}$ are the reordered arguments of the arguments $\tilde{r}_{i1}, \ldots, \tilde{r}_{in}$ and $\tilde{r}_{j1}, \ldots, \tilde{r}_{jn}$ respectively. Further analysis follows the procedure in Section 2.1.

3. Concluding remarks

The paper deals with a method for determining the OWA weights when a priori preference relations between alternatives are obtained. Basically the paper assumes that a decision-maker specifies a whole set of preference relations sequentially without checking the feasibility of the preference relations added. An interactive approach, on the other hand, verifies whether added preference relation violates the feasibility which is maintained until additional preference relation is included, thus the parameters can be modified if necessary at the point in which a violation occurs.

The mathematical programs are used to optimally locate the OWA weights which are as consistent as the ordered pairs between alternatives. There exist two options for optimally locating the OWA weights: minimizing the amount of violations and minimizing the number of violations.

The sensitivity analysis is necessary to check how viable the OWA weights are when some of the consequences in the decision matrix change due to the problem caused by the difficulty of evaluation of precise numerical values. The method that determines the OWA weights under interval values in Section 2.2 presents how to deal with the sensitivity analysis when the consequences on some criteria in Section 2.1 are perturbed and thus represented by interval numbers.

The methods presented in the paper can be even further expanded to the consequence table including fuzzy numbers since challenges will be in the ranking of the fuzzy numbers and in the aggregation with unknown weights. This case will be considered as a future research topic.

Acknowledgements

The author thanks the anonymous referees for their thorough reviews and helpful suggestions, which leads us to a significant improvement of the presentation and quality. The author is also deeply indebted to Young Soon Kwak for her helpful and constructive comments. This research was supported by the Chung Ang University Research Grants in 2001.

Appendix A

Xu and Da [41] presented formulas for comparing input arguments when each of them is specified in the form of interval number, thus yielding $b_i$, the $i$th largest element of the collection of $n$ interval objects, $a_1, a_2, \ldots, a_n$. Let $a = [a_l, a_u]$ and $b = [b_l, b_u]$, and let $l_a = a^u - a^l$ and $l_b = b^u - b^l$. Then the degree of possibility of $a \geq b$ is defined as

$$
p(a \geq b) = \max \left\{ 1 - \max \left( \frac{b^u - a^l}{l_a + l_b}, \ 0 \right), \ 0 \right\}. \quad (A1)
$$
Similarly, the degree of possibility of \( b \geq a \) is defined as

\[
p(b \geq a) = \max \left\{ 1 - \max \left( \frac{a^u - b^l}{l_u + l_b}, 0 \right), \ 0 \right\}.
\]  (A2)

**Appendix B**

To obtain a rank order of interval numbers by probabilistic approach, Ahn [42] defines a function \( d : A \times A \to [0,1] \), representing the strength of preference, where \( A \) is a finite set of arguments for comparisons. Let \( d(a, b) = P(a \geq b) \) where \( P(\cdot) \) states a probability of an interval number \( a \) being greater than or equal to an interval number \( b \). Let \( f_a(x) \) and \( f_b(y) \) be the probability density functions of interval numbers \( a \) and \( b \) which lie in \([a^l, a^u]\) and \([b^l, b^u]\), respectively. If one is to make a comparison between two interval numbers, three possible cases have to be considered, depending upon the end points of the interval numbers, which are summarized in Table 4.

**Appendix C**

The relation assumed to occur between two interval numbers can be included in one of six cases as shown in Fig. 1 (actually this figure is reproduced to take into account the reverse cases of three cases dealt in Appendix B). Let us denote \( a_i \) as an interval for comparison with other intervals from Case 1 to Case 6. There is no overlapping range for Case 1 and Case 2. The range in Case 3 is totally included in the range of \( a_i \), which is to the contrary totally included in the range in Case 6. Some portion of the ranges overlaps in Case 4 and Case 5.

The ratio of overlapping range between \( a_i \) and \( a_j \) can be calculated by the following formula [43]:

\[
OR = \frac{2\{\min(a_i^u, a_j^u) - \max(a_i^l, a_j^l)\}}{(a_i^u - a_i^l) + (a_j^u - a_j^l)} \text{ for all } j
\]

### Table 4

The calculation of strength of preference based on probabilistic method

<table>
<thead>
<tr>
<th>Case</th>
<th>Pictorial representation</th>
<th>The strength of preference of ( a ) over ( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b^u \leq a^l )</td>
<td>( b^l \quad b^u \quad a^d \quad a^u )</td>
<td>( d(a, b) = \int_{a^l}^{a^u} f_a(x) , dx = \int_{b^l}^{b^u} f_b(y) , dy = 1 )</td>
</tr>
<tr>
<td>( b^l &lt; a^l &lt; b^u \leq a^u )</td>
<td>( b^l \quad a^l \quad b^u \quad a^u )</td>
<td>( d(a, b) = \int_{b^l}^{b^u} f_b(y) , dy + \int_{a^l}^{a^u} f_a(x) , dx + \int_{b^l}^{a^l} f_a(x) , dx + \int_{b^u}^{a^u} f_a(x) , dx )</td>
</tr>
<tr>
<td>( d^l \leq b^l &lt; b^u \leq a^u )</td>
<td>( d^l \quad b^l \quad b^u \quad a^u )</td>
<td>( d(a, b) = \int_{d^l}^{a^l} f_a(x) , dx + \int_{b^l}^{b^u} f_b(y) , dy \int_{d^l}^{a^l} f_a(x) , dx )</td>
</tr>
</tbody>
</table>

**Fig. 1.** The possible relations between two interval numbers.
Here, the overlapping range (OR) is the ratio of the overlapping range to the sum of ranges of two interval numbers and falls in the range between 0 and 1. Higher value means better the similarity. The level of closeness to midpoint (CM) is used as a second criterion for similarity measure and is calculated as follows [43]:

\[
CM = 1 - \frac{a^l_i + a^u_i - a^l_j - a^u_j}{2} \quad \text{for all } j
\]

Here the value CM falls in the range between 0 and 1, higher value means better the similarity. Finally the level of similarity (SIM) between the two numbers can be calculated as follows:

\[
SIM = \alpha \cdot \text{OR} + (1 - \alpha) \cdot \text{CM}
\]

(C1)

A value between 0 and 1 can be assigned to the value \(\alpha\) to take into account decision-maker’s preference between the amount of overlapping range and the midpoint of two interval numbers. If there is no special preference, a value \(\alpha = \frac{1}{2}\) is usually employed to calculate the measure.

**Example.** Suppose four arguments of an alternative \(A_i\) are specified in interval numbers such as

\[a_{i1} = [0.3, 0.5], \quad a_{i2} = [0.4, 0.6], \quad a_{i3} = [0.4, 0.7], \quad a_{i4} = [0.3, 0.6]\]

First, one can obtain a pairwise comparison matrix when applying Xu and Da’s [41] degree of possibility between the input arguments:

\[
\begin{bmatrix}
0.5 & 0.25 & 0.2 & 0.4 \\
0.75 & 0.5 & 0.4 & 0.6 \\
0.8 & 0.6 & 0.5 & 0.8 \\
0.6 & 0.4 & 0.3 & 0.5 \\
\end{bmatrix}
\]

for Appendix A

Second, a pairwise comparison matrix below can be obtained, assuming that the interval numbers are uniformly distributed over the end points of interval numbers (refer Ahn [42] for a method for obtaining the reordered arguments with a pairwise comparison matrix):

\[
\begin{bmatrix}
0.5 & 0.125 & 0.083 & 0.333 \\
0.875 & 0.5 & 0.333 & 0.667 \\
0.917 & 0.667 & 0.5 & 0.778 \\
0.667 & 0.333 & 0.222 & 0.5 \\
\end{bmatrix}
\]

for Appendix B

Finally, two measures by similarity can be calculated as follows:

\[
\text{OR} = \begin{bmatrix} 1 & 0.5 & 0.4 & 0.8 \\
0.5 & 1 & 0.8 & 0.67 \\
0.4 & 0.8 & 1 & 0.67 \\
0.8 & 0.67 & 0.67 & 1 \end{bmatrix}, \quad \text{CM} = \begin{bmatrix} 1 & 0.9 & 0.85 & 0.95 \\
0.9 & 1 & 0.95 & 0.95 \\
0.85 & 0.95 & 1 & 0.9 \\
0.95 & 0.95 & 0.9 & 1 \end{bmatrix}
\]

Applying the formula in (C1) with \(\alpha = 1/2\) results in the following similarity matrix:

\[
\begin{bmatrix}
1 & 0.7 & 0.625 & 0.875 \\
0.7 & 1 & 0.875 & 0.81 \\
0.625 & 0.875 & 1 & 0.785 \\
0.875 & 0.81 & 0.785 & 1 \\
\end{bmatrix}
\]

for Appendix C

From the similarity matrix, we can infer that an interval number \(a_{i1}\) is most similar to \(a_{i4}\) and least similar to \(a_{i2}\), that is, \(a_{i1} \approx a_{i4} \approx a_{i2} \approx a_{i3}\), in which \(\approx\) denotes ‘similar to’. Therefore the reordered arguments become \(b_{i1} = a_{i3}, \ b_{i2} = a_{i2}, \ b_{i3} = a_{i4}, \ b_{i4} = a_{i1}\).
References