Trace formulas and dynamical zeta functions in the Nielsen theory

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Abstract

In this paper we prove the trace formulas for the Reidemeister numbers of group endomorphisms in the following cases:the group is finitely generated and an endomorphism is eventually commutative; the group is finite ; the group is a direct sum of a finite group and a finitely generated free abelian group; the group is finitely generated, nilpotent and torsion free . These results had previously been known only for the finitely generated free abelian group.As a consequence, we obtain under the same conditions on the fundamental group of a compact polyhedron,the trace formulas for the Reidemeister numbers of a continuous map and under suitable conditions the trace formulas for the Nielsen numbers of a continuous map.The trace formula for the Reidemeister numbers implies the rationality of the Reidemeister zeta function. We prove the rationality and functional equation for the Reidemeister zeta function of an endomorphisms of finitely generated torsion free nilpotent group and of a direct sum of a finite group and a finitely generated free abelian group.We give a new proof of the rationality of the Reidemeister zeta function in the case when the group is finitely generated and the endomorphism is eventually commutative and in the case when group is finite.We give also another proof for the positivity of the radius of convergence of the Nielsen zeta function and propose an exast algebraic lower bound estimation for the radius.We connect the Reidemeister zeta function of an endom! orphism of a direct sum of a finite group and a finitely generated free abelian group with the Lefschetz zeta function of the unitary dual map,and as consequence obtain a connection of the Reidemeister zeta function with Reidemeister torsion.

0 Introduction

We assume everywhere X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Taking a dynamical point on view,we consider the iterates of f.In the theory of discrete dynamical systems the following zeta functions are known: the Artin-Mazur zeta function

$$\zeta_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n\right),$$

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where $F(f^n)$ is the number of isolated fixed points of f^n; the Ruelle zeta function [35]

$$\zeta_f{}^g(z) := \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in Fix(f^n)} \prod_{k=0}^{n-1} g(f^k(x))\right),$$

where $g : X \to \mathbb{C}$ is a weight function(if $g = 1$ we recover $\zeta_f(z)$); the Lefschetz zeta function

$$L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \mathrm{Tr}\left[f_{*k}^n : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})\right]$$

are the Lefschetz numbers of the iterates of f; reduced mod 2 Artin-Mazur and Lefschetz zeta functions [19]; twisted Artin-Mazur and Lefschetz zeta functions [20],which have coefficients in the group rings $\mathbb{Z}H$ of an abelian group H.

The above zeta functions are directly analogous to the Hasse-Weil zeta function of an algebraic manifold over finite fields [39].The Lefschetz zeta function is always rational function of z and is given by the formula:

$$L_f(z) = \prod_{k=0}^{\dim X} \det\left(E - f_{*k}.z\right)^{(-1)^{k+1}}.$$

This immediately follows from trace formula for the Lefschetz numbers of the iterates of f.The Artin-Mazur zeta function has a positive radius of convergence for a dence set in the space of smooth maps of a compact smooth manifold to itself [2].Manning proved the rationality of the Artin-Mazur zeta function for diffeomorphisms of a compact smooth manifold satisfying Smale's Axiom A [30].

The Artin-Mazur zeta function and its modification count periodic points of a map geometrically, the Lefschetz's type zeta functions do this algebraically (with weight given by index theory).Another way to count the periodic points is given by Nielsen theory.

Let $p : \tilde{X} \to X$ be the universal covering of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, ie. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$. A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister Number* of f, denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$. The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy type invariants. In the category of compact, connected polyhedra the Nielsen number of a map is equal to the least number of fixed points of maps with the same homotopy type as f. Let G be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be $\phi-$*conjugate* iff there exists $\gamma \in G$ such that $\alpha' = \gamma.\alpha.\phi(\gamma)^{-1}$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of ϕ, denoted by $R(\phi)$.

If we consider the iterates of f and ϕ, we may define several zeta functions connected with Nielsen fixed point theory (see [10, 11, 13, 14, 18]). We assume throughout this article that $R(f^n) < \infty$ and $R(\phi^n) < \infty$ for all $n > 0$. The Reidemeister zeta functions of f and ϕ and the Nielsen zeta function of f are defined as power series:

$$\begin{aligned} R_\phi(z) &:= \exp\left(\sum_{n=1}^\infty \frac{R(\phi^n)}{n} z^n\right), \\ R_f(z) &:= \exp\left(\sum_{n=1}^\infty \frac{R(f^n)}{n} z^n\right), \\ N_f(z) &:= \exp\left(\sum_{n=1}^\infty \frac{N(f^n)}{n} z^n\right). \end{aligned}$$

$R_f(z)$ and $N_f(z)$ are homotopy invariants. The Nielsen zeta function $N_f(z)$ has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map f [18].In section 4 we propose another prove of positivity of the radius and give an exact algebraic lower estimation for radius using the Reidemeister trace formula for the generalised Lefschetz numbers.In section 7 we prove the same result for the radius of convergence of the minimal dynamical zeta function.

An endomorphism $\phi : G \to G$ is said to be eventually commutative if there exists a natural number n such that the subgroup $\phi^n(G)$ is commutative. A map $f : X \to X$ is said to be eventually commutative if the induced endomorphism on fundamental group is eventually commutative. We begin the article by proving in section 1 the trace formula for the Reidemeister numbers in the following cases:

(I) G is finitely generated and ϕ is eventually commutative.

(II) G is finite

(III) G is a direct sum of a finite group and a finitely generated free abelian group.

(IV) G is a finitely generated torsion free nilpotent group.

These result had previously been known only for the finitely generated free abelian group [14]. As a consequence, we obtain in section 2, under the same conditions (I) - (IV) on the fundamental group of X ,the trace formula for the Reidemeister numbers of a continuous map and in section 3 under suitable conditions the trace formulas for the Nielsen numbers of a continuous map.

Trace formula for the Reidemeister numbers implies the rationality of the Reidemeister zeta function.In section 5 we prove the rationality and functional equation for the Reidemeister zeta function of an endomorphisms of finitely generated torsion free nilpotent group and of a direct sum of a finite group and a finitely generated free abelian group.We give also a new proof of the rationality of $R_\phi(z)$ in the case when G is finitely generated and ϕ is eventually commutative and in the case when G is finite.

In section 7 we obtain an expression for the Reidemeister torsion of the mapping torus of the dual map of a endomorphism of a direct sum of a finite group and a finitely generated free abelian group, in terms of the Reidemeister zeta function of the endomorphism.The result is obtained by expressing the Reidemeister zeta function in terms of the Lefschetz zeta function of the dual map, and then applying the theorem of D. Fried.What this means is that the Reidemeister torsion counts the fixed point classes of all iterates of map f i.e. periodic point classes of f.These result had previously been known for the finitely generated abelian groups and finite groups[16].

In section 5.5 we describe some conjectures on how the Reidemeister zeta functions should look in general, largely in terms of the growth on the group. In particular we expect that for polynomial growth groups some power of the Reidemeister zeta function is a rational function.

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1 Trace formula for the Reidemeister numbers of a group endomorphism

1.1 Pontryagin Duality

Let G be a locally compact abelian topological group. We write $\hat{G}$ for the set of continuous homomorphisms from G to the circle $U(1) = \{z \in \mathbb{C} : |z| = 1\}$. This is a group with pointwise multiplication. We call $\hat{G}$ the *Pontryagin dual* of G. When we equip $\hat{G}$ with the compact-open topology it becomes a locally compact abelian topological group. The dual of the dual of G is canonically isomorphic to G.

A continuous endomorphism $f : G \to G$ gives rise to a continuous endomorphism $\hat{f} : \hat{G} \to \hat{G}$ defined by

$$\hat{f}(\chi) := \chi \circ f.$$

There is a 1-1 correspondence between the closed subgroups H of G and the quotient groups $\hat{G}/H^*$ of $\hat{G}$ for which H^* is closed in $\hat{G}$. This correspondence is given by the following:

$$H \leftrightarrow \hat{G}/H^*,$$

$$H^* := \{\chi \in \hat{G} \mid H \subset \ker \chi\}.$$

Under this correspondence, $\hat{G}/H^*$ is canonically isomorphic to the Pontryagin dual of H. If we identify G canonically with the dual of $\hat{G}$ then we have $H^{**} = H$. If G is a finitely generated abelian group then a homomorphism $\chi : G \to U(1)$ is completely determined by its values on a basis of G, and these values may be chosen arbitrarily. The dual of G is thus a torus whose dimension is equal to the rank of G.

If $G = \mathbb{Z}/n\mathbb{Z}$ then the elements of $\hat{G}$ are of the form

$$x \to e^{\frac{2\pi i y x}{n}}$$

with $y \in \{1, 2, \dots, n\}$. A cyclic group is therefore (uncanonically) isomorphic to itself.

The dual of $G_1 \oplus G_2$ is canonically isomorphic to $\hat{G}_1 \oplus \hat{G}_2$. From this we see that any finite abelian group is (non-canonically) isomorphic to its own Pontryagin dual group, and that the dual of any finitely generated discrete abelian group is the direct sum of a Torus and a finite group.

Proofs of all these statements may be found, for example in [34]. We shall require the following statement:

Proposition 1 *Let $\phi : G \to G$ be an endomorphism of an abelian group G. Then the kernel* $\ker\left[\hat{\phi} : \hat{G} \to \hat{G}\right]$ *is canonically isomorphic to the Pontryagin dual of* $\mathrm{Coker}\ \phi$.

PROOF We construct the isomorphism explicitly. Let χ be in the dual of Coker $(\phi : G \to G)$. In that case χ is a homomorphism

$$\chi : G/\text{Im } (\phi) \longrightarrow U(1).$$

There is therefore an induced map

$$\overline{\chi} : G \longrightarrow U(1)$$

which is trivial on Im (ϕ). This means that $\overline{\chi} \circ \phi$ is trivial, or in other words $\hat{\phi}(\overline{\chi})$ is the identity element of $\hat{G}$. We therefore have $\overline{\chi} \in \ker(\hat{\phi})$. If on the other hand we begin with $\overline{\chi} \in \ker(\hat{\phi})$, then it follows that χ is trivial on Im ϕ, and so $\overline{\chi}$ induces a homomorphism

$$\chi : G/\text{Im } (\phi) \longrightarrow U(1)$$

and χ is then in the dual of Coker ϕ. The correspondence $\chi \leftrightarrow \overline{\chi}$ is clearly a bijection.

1.2 Eventually commutative endomorphisms

1.2.1 Trace formula for the Reidemeister numbers of eventually commutative endomorphisms.

An endomorphism $\phi : G \to G$ is said to be eventually commutative if there exists a natural number n such that the subgroup $\phi^n(G)$ is commutative. If ϕ is an endomorphism of an abelian group G then x and y are $\phi -$ *conjugate* iff $x - y = \phi(g) - g$ for some $g \in G$.Therefore $R(\phi)$ is the number of cosets of the image of the endomorphism $(\phi - 1) : G \to G$.Then

$$R(\phi) = \#\text{Coker } (1 - \phi).$$

We are now ready to compare the Reidemeister number of an endomorphism ϕ with the Reidemeister number of $H_1(\phi) : H_1(G) \to H_1(G)$, where $H_1 = H_1^{Gp}$ is the first integral homology functor from groups to abelian groups.

Lemma 1 ([26]) *If $\phi : G \to G$ is eventually commutative, then*

$$R(\phi) = R(H_1(\phi) = \#\text{Coker } (1 - H_1(\phi))$$

This means that to find out about the Reidemeister numbers of eventually commutative endomorphisms, it is sufficient to study the Reidemeister numbers of endomorphisms of abelian groups. For the rest of this section G will be a finitely generated abelian group.

Lemma 2 ([14]) *Let $\phi : \mathbb{Z}^k \to \mathbb{Z}^k$ be a group endomorphism. Then we have*

$$R(\phi) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i \text{Tr } (\Lambda^i \phi). \tag{1}$$

where p the number of $\mu \in$ Spec ϕ such that $\mu < -1$, and r the number of real eigenvalues of ϕ whose absolute value is > 1. Λ^i denotes the exterior power.

PROOF Since $\mathbb{Z}^k$ is abelian, we have as before,

$$R(\phi) = \#\text{Coker } (1 - \phi).$$

On the other hand we have

$$\#\text{Coker } (1 - \phi) = \mid \det(1 - \phi) \mid,$$

and hence $R(\phi) = (-1)^{r+p} \det(1 - \phi)$ (complex eigenvalues contribute nothing to the signdet$(1-\phi)$ since they come in conjugate pairs and $(1-\lambda)(1-\bar{\lambda}) = \mid 1 - \lambda \mid^2 > 0$). It is well known from linear algebra that $\det(1-\phi) = \sum_{i=0}^{k}(-1)^i \text{Tr } (\Lambda^i \phi)$. From this we have the trace formula for Reidemeister number.

Now let ϕ be an endomorphism of finite abelian group G.Let V be the complex vector space of complex valued functions on the group G.The map ϕ induces a linear map $A : V \to V$ defined by

$$A(f) := f \circ \phi.$$

Lemma 3 *Let $\phi : G \to G$ be an endomorphism of a finite abelian group G. Then we have*

$$R(\phi) = TrA \tag{2}$$

We give two proofs of this lemma in this article. The first proof is given here and the second proof is a special case of the proof of theorem 4

PROOF We shall calculate the trace of A in two ways.The characteristic functions of the elements of G form a basis of V,and are mapped to one another by A(the map need not be a bijection).Therefore the trace of A is the number of elements of this basis which are fixed by A.On the other hand, since G is abelian, we have,

$$\begin{aligned} R(\phi) &= \#\text{Coker } (1 - \phi) \\ &= \#G / \#\text{Im } (1 - \phi) \\ &= \#G / \#(G / \ker(1 - \phi)) \\ &= \#G / (\#G / \# \ker(1 - \phi)) \\ &= \# \ker(1 - \phi) \\ &= \#\text{Fix } (\phi) \end{aligned}$$

We therefore have $R(\phi) = \#\text{Fix } (\phi) = TrA$.

For a finitely generared abelian group G we define the finite subgroup $TorsG$ to be the subgroup of torsion elements of G. We denote the quotient $G^\infty := G/TorsG$. The group G^∞ is torsion free. Since the image of any torsion element by a homomorphism must be a torsion element, the endomorphism $\phi : G \to G$ induces endomorphisms

$$\phi^{tor} : TorsG \longrightarrow TorsG, \quad \phi^\infty : G^\infty \longrightarrow G^\infty.$$

As above, the map ϕ^{tor} induces a linear map $A : V \to V$, where V be the complex vector space of complex valued functions on the group $TorsG$.

Theorem 1 *If G is a finitely generated abelian group and ϕ an endomorphism of G .Then we have*

$$R(\phi) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i \mathrm{Tr}\ (\Lambda^i \phi^\infty \otimes A). \tag{3}$$

where k is rgG^∞, p the number of $\mu \in \mathrm{Spec}\ \phi^\infty$ such that $\mu < -1$, and r the number of real eigenvalues of ϕ^∞ whose absolute value is > 1.

Proof By proposition 1, the cokernel of $(1-\phi) : G \to G$ is the Pontrjagin dual of the kernel of the dual map $\widehat{(1-\phi)} : \hat{G} \to \hat{G}$. Since Coker $(1-\phi)$ is finite, we have

$$\#\mathrm{Coker}\ (1-\phi) = \#\ker \widehat{(1-\phi)}.$$

The map $\widehat{1-\phi}$ is equal to $\hat{1} - \hat{\phi}$. Its kernel is thus the set of fixed points of the map $\hat{\phi} : \hat{G} \to \hat{G}$. We therefore have

$$R(\phi) = \#\mathrm{Fix}\ \left(\hat{\phi} : \hat{G} \to \hat{G}\right) \tag{4}$$

The dual group of G^∞ is a torus whose dimension is the rank of G. This is canonically a closed subgroup of $\hat{G}$. We shall denote it $\hat{G}_0$. The quotient $\hat{G}/\hat{G}_0$ is canonically isomorphic to the dual of $TorsG$. It is therefore finite. From this we know that $\hat{G}$ is a union of finitely many disjoint tori. We shall call these tori $\hat{G}_0, \ldots, \hat{G}_t$.

We shall call a torus $\hat{G}_i$ periodic if there is an iteration $\hat{\phi}^s$ such that $\hat{\phi}^s(\hat{G}_i) \subset \hat{G}_i$. If this is the case, then the map $\hat{\phi}^s : \hat{G}_i \to \hat{G}_i$ is a translation of the map $\hat{\phi}^s : \hat{G}_0 \to \hat{G}_0$ and has the same number of fixed points as this map. If $\hat{\phi}^s(\hat{G}_i) \not\subset \hat{G}_i$ then $\hat{\phi}^s$ has no fixed points in $\hat{G}_i$. From this we see

$$\#\mathrm{Fix}\ \left(\hat{\phi} : \hat{G} \to \hat{G}\right) = \#\mathrm{Fix}\ \left(\hat{\phi} : \hat{G}_0 \to \hat{G}_0\right) \times \#\{\hat{G}_i \mid \hat{\phi}(\hat{G}_i) \subset \hat{G}_i\}.$$

We now rephrase this

$$\begin{aligned} &\#\mathrm{Fix}\ \left(\hat{\phi} : \hat{G} \to \hat{G}\right) \\ &= \ \#\mathrm{Fix}\ \left(\widehat{\phi^\infty} : \hat{G}_0 \to \hat{G}_0\right) \times \#\mathrm{Fix}\ \left(\widehat{\phi^{tor}} : \hat{G}/(\hat{G}_0) \to \hat{G}/(\hat{G}_0)\right). \end{aligned}$$

From this we have product formula for Reidemeister numbers

$$R(\phi) = R(\phi^\infty).R(\phi^{tors}).$$

The trace formula for $R(\phi)$ follow from the previous two lemmas and formula

$$Tr(\Lambda^i \phi^\infty).Tr(A) = Tr(\Lambda^i \phi^\infty \otimes A).$$

In the paper [16] we have connected Reidemeister number of endomorphism ϕ with Lefschetz number of the dual map.From this we have following trace formula

Theorem 2 ([16]) *Let $\phi : G \to G$ be an endomorphism of a finitely generated abelian group. Then*

$$R(\phi) = \mid L(\hat{\phi}) \mid = (-1)^{r+p} \sum_{k=0}^{\dim \hat{G}} (-1)^k \mathrm{Tr}\ \left[\hat{\phi}_{*k} : H_k(\hat{G}; \mathbb{Q}) \to H_k(\hat{G}; \mathbb{Q})\right] \tag{5}$$

where $\hat{\phi}$ is the continuous endomorphism of $\hat{G}$ defined in §2.2 and $L(\hat{\phi})$ is the Lefschetz number of $\hat{\phi}$ thought of as a self-map of the topological space $\hat{G}$ and r and p are the constants described in theorem 1. If G is finite then this reduces to

$$R(\phi) = L(\hat{\phi}) = \mathrm{Tr}\ \left[\hat{\phi}_{*0} : H_0(\hat{G}; \mathbb{Q}) \to H_0(\hat{G}; \mathbb{Q})\right].$$

1.3 Endomorphisms of finite groups

In this section we consider finite non-abelian groups. We shall write the group law multiplicatively. We generalize our results on endomorphisms of finite abelian groups to endomorphisms of finite non-abelian groups. We shall write $\{g\}$ for the ϕ-conjugacy class of an element $g \in G$. We shall write $< g >$ for the ordinary conjugacy class of g in G. We first note that if ϕ is an endomorphism of a group G then ϕ maps conjugate elements to conjugate elements. It therefore induces an endomorphism of the set of conjugacy classes of elements of G. If G is abelian then a conjugacy class of element consists of a single element. The following is thus an extension of lemma 3:

Theorem 3 ([15]) *Let G be a finite group and let $\phi : G \to G$ be an endomorphism. Then $R(\phi)$ is the number of ordinary conjugacy classes $< x >$ in G such that*

$$< \phi(x) >=< x > .$$

PROOF From the definition of the Reidemeister number we have,

$$R(\phi) \;=\; \sum_{\{g\}} 1$$

where $\{g\}$ runs through the set of ϕ-conjugacy classes in G. This gives us immediately

$$\begin{aligned} R(\phi) &= \sum_{\{g\}} \sum_{x\in\{g\}} \frac{1}{\#\{g\}} \\ &= \sum_{\{g\}} \sum_{x\in\{g\}} \frac{1}{\#\{x\}} \\ &= \sum_{x\in G} \frac{1}{\#\{x\}}. \end{aligned}$$

We now calculate for any $x \in G$ the order of $\{x\}$. The class $\{x\}$ is the orbit of x under the G-action

$$(g, x) \longmapsto gx\phi(g)^{-1}.$$

We verifty that this is actually a G-action:

$$\begin{aligned} (id, x) &\longmapsto id.x.\phi(id)^{-1} \\ &= x, \\ (g_1g_2, x) &\longmapsto g_1g_2.x.\phi(g_1g_2)^{-1} \\ &= g_1g_2.x.(\phi(g_1)\phi(g_2))^{-1} \\ &= g_1g_2.x.\phi(g_2)^{-1}\phi(g_1)^{-1} \\ &= g_1(g_2.x.\phi(g_2)^{-1})\phi(g_1)^{-1}. \end{aligned}$$

We therefore have from the orbit-stabilizer theorem,

$$\#\{x\} = \frac{\#G}{\#\{g \in G \mid gx\phi(g)^{-1} = x\}}.$$

The condition $gx\phi(g)^{-1} = x$ is equivalent to

$$x^{-1}gx\phi(g)^{-1} = 1 \quad \Leftrightarrow \quad x^{-1}gx = \phi(g)$$

We therefore have

$$R(\phi) = \frac{1}{\#G} \sum_{x \in G} \#\{g \in G \mid x^{-1}gx = \phi(g)\}.$$

Changing the summation over x to summation over g, we have:

$$R(\phi) = \frac{1}{\#G} \sum_{g \in G} \#\{x \in G \mid x^{-1}gx = \phi(g)\}.$$

If $< \phi(g) > \neq < g >$ then there are no elements x such that $x^{-1}gx = \phi(g)$. We therefore have:

$$R(\phi) = \frac{1}{\#G} \sum_{\substack{g \in G \text{ such that} \\ <\phi(g)>=<g>}} \#\{x \in G \mid x^{-1}gx = \phi(g)\}.$$

The elements x such that $x^{-1}gx = \phi(g)$ form a coset of the subgroup satisfying $x^{-1}gx = g$. This subgroup is the centralizer of g in G which we write $C(g)$. With this notation we have,

$$\begin{aligned} R(\phi) &= \frac{1}{\#G} \sum_{\substack{g \in G \text{ such that} \\ <\phi(g)>=<g>}} \#C(g) \\ &= \frac{1}{\#G} \sum_{\substack{<g> \subset G \text{ such that} \\ <\phi(g)>=<g>}} \# < g > .\#C(g). \end{aligned}$$

The last identity follows because $C(h^{-1}gh) = h^{-1}C(g)h$. From the orbit stabilizer theorem, we know that $\# < g > .\#C(g) = \#G$. We therefore have

$$R(\phi) = \#\{< g > \subset G \mid < \phi(g) > = < g >\}.$$

Let W be the complex vector space of complex valued class functions on the group G.A class function is a function which takes the same value on every element of a usual congruency class.The map ϕ induces a linear map $B : W \to W$ defined by

$$B(f) := f \circ \phi.$$

Theorem 4 *Let $\phi : G \to G$ be an endomorphism of a finite group G. Then we have*

$$R(\phi) = TrB \tag{6}$$

PROOF We shall calculate the trace of B in two ways.The characteristic functions of the congruency classes in G form a basis of W,and are mapped to one another by B(the map need not be a bijection).Therefore the trace of B is the number of elements of this basis which are fixed by B.By theorem 3,this is equal to the Reidemeister number of ϕ.

1.4 Endomorphisms of the direct sum of a free abelian and a finite group

In this section let F be a finite group and r a natural number.The group G will be $G = \mathbb{Z}^k \oplus F$.The torsion elements of G are exastly the elements of the finite, normal subgroup F.For this reason we have $\phi(F) \subset F$.Let $\phi^{tors} : F \to F$ be the restriction of ϕ to F, and let $\phi^\infty : G/F \to G/F$ be the induced map on the quotient group.

Lemma 4 *[25] In the notation described above*

$$R(\phi) = R(\phi^\infty).R(\phi^{tors})$$

Let W be the complex vector space of complex valued class functions on the group F.The map ϕ induces a linear map $B : W \to W$ defined as above.

Theorem 5 *If G is the direct sum of a free abelian and a finite group and ϕ an endomorphism of G .Then we have*

$$R(\phi) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i \mathrm{Tr}\ (\Lambda^i \phi^\infty \otimes B). \tag{7}$$

where k is $rg(G/F)$, p the number of $\mu \in \mathrm{Spec}\ \phi^\infty$ such that $\mu < -1$, and r the number of real eigenvalues of ϕ^∞ whose absolute value is > 1.

PROOF Theorem follows from lemmas 2 and 4 , theorem 4 and formula

$$Tr(\Lambda^i \phi^\infty).Tr(B) = Tr(\Lambda^i \phi^\infty \otimes B).$$

1.5 Endomorphisms of nilpotent groups

In this section we consider finitely generated torsion free nilpotent group Γ.It is well known [29] that such group Γ is a uniform discrete subgroup of a simply connected nilpotent Lie group G (uniform means that the coset space G/Γ is compact).The coset space $M = G/\Gamma$ is called a nilmanifold.Since $\Gamma = \pi_1(M)$ and M is a $K(\Gamma, 1)$, every endomorphism $\phi : \Gamma \to \Gamma$ can be realized by a selfmap $f : M \to M$ such that $f_* = \phi$ and thus $R(f) = R(\phi)$.Any endomorphism $\phi : \Gamma \to \Gamma$ can be uniquely extended to an endomorphism $F : G \to G$.Let $\tilde{F} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra endomorphism induced from F.

Theorem 6 *If G is a finitely generated torsion free nilpotent group and ϕ an endomorphism of G .Then*

$$R(\phi) = (-1)^{r+p} \sum_{i=0}^{m} (-1)^i \mathrm{Tr}\ \Lambda^i \tilde{F}, \tag{8}$$

where m is $rg\Gamma = \dim M$, p the number of $\mu \in \mathrm{Spec}\ \tilde{F}$ such that $\mu < -1$, and r the number of real eigenvalues of $\tilde{F}$ whose absolute value is > 1.

PROOF: Let $f : M \to M$ be a map realizing ϕ on a compact nilmanifold M of dimension m.We suppose in this article that the Reidemeister number $R(f) = R(\phi)$ is finite.The finiteness of $R(f)$ implies the nonvanishing of the Lefschetz number $L(f)$ [17].A strengthened version of Anosov's theorem [1] is proven in [31] which states, in particular, that if $L(f) \neq 0$ than $N(f) = |L(f)| = R(f)$.It is well known that $L(f) = \det(\tilde{F} - 1)$ [1].From this we have

$$R(\phi) = R(f) = |L(f)| = |\det(1-\tilde{F})| = (-1)^{r+p}\det(1-\tilde{F}) = (-1)^{r+p}\sum_{i=0}^{m}(-1)^i \mathrm{Tr}\ \Lambda^i\tilde{F}.$$

1.6 The trace formulas and group extensions.

Suppose we are given a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G \\ \downarrow p & & \downarrow p \\ \overline{G} & \xrightarrow{\overline{\phi}} & \overline{G} \end{array} \qquad (9)$$

of groups and homomorphisms. In addition let the sequence

$$0 \to H \to G \xrightarrow{p} \overline{G} \to 0 \qquad (10)$$

be exact. Then ϕ restricts to an endomorphism $\phi\mid_H : H \to H$.

Definition 1 *The short exact sequence (16) of groups is said to have a normal splitting if there is a section $\sigma : \overline{G} \to G$ of p such that $\mathrm{Im}\ \sigma = \sigma(\overline{G})$ is a normal subgroup of G. An endomorphism $\phi : G \to G$ is said to preserve this normal splitting if ϕ induces a morphism of (16) with $\phi(\sigma(\overline{G})) \subset \sigma(\overline{G})$.*

In this section we study the relation between the traces formulas of type (7) for Reidemeister numbers $R(\phi)$, $R(\overline{\phi})$ and $R(\phi\mid_H)$.

Theorem 7 *Let the sequence (14) have a normal splitting which is preserved by $\phi : G \to G$. If we have a trace formulas of the type (7) for $R(\overline{\phi})$ and $R(\phi\mid_H)$ then we have a trace formula of the same type for $R(\phi)$.*

PROOF From the assumptions of the theorem it follows that

$$R(\phi) = R(\overline{\phi}).R(\phi\mid_H) \qquad \text{(see [24])}.$$

Trace formula for $R(\phi)$ now follows from linear algebra formula:

$$TrA.TrB = TrA \otimes B.$$

1.6.1 Direct Sums

If $G = G_1 \oplus G_2$ is a direct sum and if $\phi(G_i) \subset G_i$ for $i = 1, 2$ then it has been shown (see [24]) that $R(\phi) = R(\phi_1) \cdot R(\phi_2)$ where ϕ_i is the restriction of ϕ to G_i.So if we have the trace formula of type (7) for $R(\phi_1)$ and $R(\phi_2)$ then we have the trace formula for $R(\phi)$.

2 Trace formulas for Reidemeisters numbers of a continuous map.

Let $f : X \to X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as reference. Let Γ be the group of covering translations of $\tilde{X}$ over X. Then every lifting of f can be written uniquely as $\gamma \circ \tilde{f}$, with $\gamma \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\gamma \in \Gamma$ the composition $\tilde{f} \circ \gamma$ is a lifting of f so there is a unique $\gamma' \in \Gamma$ such that $\gamma' \circ \tilde{f} = \tilde{f} \circ \gamma$. This correspondence $\gamma \to \gamma'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism.

Definition 2 *The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of f is defined by*

$$\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$

It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the following way. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$ once and for all. Now points of $\tilde{X}$ are in 1-1 correspondence with homotopy classes of paths in X which start at x_0: for $\tilde{x} \in \tilde{X}$ take any path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$ and project it onto X; conversely for a path c starting at x_0, lift it to a path in $\tilde{X}$ which starts at $\tilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\tilde{X}$ with a path class $< c >$ in X starting from x_0. Under this identification, $\tilde{x}_0 =< e >$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha =< a >\in \pi_1(X, x_0)$ on $\tilde{X}$ is then given by

$$\alpha =< a >:< c >\to \alpha.c =< a.c > .$$

Now we have the following relationship between $\tilde{f}_* : \pi \to \pi$ and

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0)).$$

Lemma 5 *Suppose $\tilde{f}(\tilde{x}_0) =< w >$. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\ & \tilde{f}_* \searrow & \downarrow w_* \\ & & \pi_1(X, x_0) \end{array}$$

where w_* is isomorphism induced by the path w.

In other words, for every $\alpha =< a >\in \pi_1(X, x_0)$, we have

$$\tilde{f}_*(< a >) =< w(f \circ a)w^{-1} >$$

Remark 1 *In particular, if $x_0 \in p(Fix(\tilde{f})$ and $\tilde{x}_0 \in Fix(\tilde{f})$,then $\tilde{f}_* = f_*$.*

Lemma 6 *Lifting classes of f are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in π, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $\tilde{f}_*$-conjugacy class of γ. We therefore have $R(f) = R(\tilde{f}_*)$.*

We shall say that the fixed point class $p(Fix(\gamma \circ \tilde{f}))$, which is labeled with the lifting class $[\gamma \circ \tilde{f}]$, *corresponds* to the $\tilde{f}_*$-conjugacy class of γ. Thus $\tilde{f}_*$-conjugacy classes in π serve as coordinates for fixed point classes of f, once a reference lifting $\tilde{f}$ is chosen.

Let us consider a homomorphisms from π sending an $\tilde{f}_*$-conjugacy class to one element:

Lemma 7 ([26]) *The composition $\eta \circ \theta$,*

$$\pi = \pi_1(X, x_0) \xrightarrow{\theta} H_1(X) \xrightarrow{\eta} \text{Coker } \left[H_1(X) \stackrel{1-f_{1*}}{\longrightarrow} H_1(X) \right],$$

where θ is abelianization and η is the natural projection, sends every $\tilde{f}_*$-conjugacy class to a single element. Moreover, any group homomorphism $\zeta : \pi \to G$ which sends every $\tilde{f}_*$-conjugacy class to a single element, factors through $\eta \circ \theta$.

Definition 3 *A map $f : X \to X$ is said to be eventually commutative if there exists an natural number n such that $f_*^n(\pi_1(X, x_0))$ $(\subset \pi_1(X, f^n(x_0)))$ is commutative.*

By means of Lemma 5, it is easily seen that f is eventually commutative iff $\tilde{f}_*$ is eventually commutative (see [26]).

Now using lemma 6 we may apply all theorems of §1 to the Reidemeister numbers of continuous maps.

2.1 Trace formulas and Serre bundles.

For example, let us consider topological counterpart of theorem 7.

Let $p : E \to B$ be a Serre bundle in which E, B and every fibre are connected, compact polyhedra and $F_b = p^{-1}(b)$ is a fibre over $b \in B$. A Serre bundle $p : E \to B$ is said to be *(homotopically) orientable* if for any two paths w, w' in B with the same endpoints $w(0) = w'(0)$ and $w(1) = w'(1)$, the fibre translations $\tau_w, \tau_{w'} : F_{w(0)} \to F_{w(1)}$ are homotopic. A map $f : E \to E$ is called a *fibre map* if there is an induced map $\bar{f} : B \to B$ such that $p \circ f = \bar{f} \circ p$. Let $p : E \to B$ be an orientable Serre bundle and let $f : E \to E$ be a fibre map. Then for any two fixed points b, b' of $\bar{f} : B \to B$ the maps $f_b = f\mid_{F_b}$ and $f_{b'} = f\mid_{F_{b'}}$ have the same homotopy type; hence they have the same Reidemeister numbers $R(f_b) = R(f_{b'})$ [26].

Theorem 8 *Suppose that $f : E \to E$ admits a Fadell splitting in the sense that for some e in* Fix f *and $b = p(e)$ the following conditions are satisfied:*

1. *the sequence*

$$0 \longrightarrow \pi_1(F_b, e) \xrightarrow{i_*} \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, e) \longrightarrow 0$$

   is exact,

2. *p_* admits a right inverse (section) σ such that* Im σ *is a normal subgroup of $\pi_1(E, e)$ and $f_*(\text{Im } \sigma) \subset \text{Im } \sigma$.*

If we have a trace formulas of the type (7) for $R(\bar{f})$ and $R(f_b)$ then we have a trace formula of the same type for $R(f)$.

3 Trace formulas for the Nielsen numbers

3.1 The Jiang subgroup and trace formula

From the homotopy invariance theorem (see [26]) it follows that if a homotopy $\{h_t\} : f \cong g : X \to X$ lifts to a homotopy $\{\tilde{h}_t\} : \tilde{f} \cong \tilde{g} : \tilde{X} \to \tilde{X}$, then we have $\mathrm{Index}(f, p(Fix\ \tilde{f})) = \mathrm{Index}(g, p(Fix\ \tilde{g}))$. Suppose $\{h_t\}$ is a cyclic homotopy $\{h_t\} : f \cong f$; then this lifts to a homotopy from a given lifting $\tilde{f}$ to another lifting $\tilde{f}' = \alpha \circ \tilde{f}$, and we have

$$\mathrm{Index}(f, p(Fix\ \tilde{f})) = \mathrm{Index}(f, p(Fix\ \alpha \circ \tilde{f})).$$

In other words, a cyclic homotopy induces a permutation of lifting classes (and hence of fixed point classes); those in the same orbit of this permutation have the same index. This idea is applied to the computation of $N(f)$.

Definition 4 *The trace subgroup of cyclic homotopies (the Jiang subgroup) $I(\tilde{f}) \subset \pi$ is defined by*

$$I(\tilde{f}) = \left\{ \alpha \in \pi \;\middle|\; \begin{array}{l} \text{there exists a cyclic homotopy} \\ \{h_t\} : f \cong f \text{which lifts to} \\ \{\tilde{h}_t\} : \tilde{f} \cong \alpha \circ \tilde{f} \end{array} \right\}$$

(see [26]).

Let $Z(G)$ denote the centre of a group G, and let $Z(K, G)$ denote the centralizer of the subgroup $K \subset G$. The Jiang subgroup has the following properties: $1. I(\tilde{f}) \subset Z(\tilde{f}_*(\pi), \pi)$;
$2. I(id_{\tilde{X}}) \subset Z(\pi); 3. I(\tilde{g}) \subset I(\tilde{g} \circ \tilde{f}); 4. \tilde{g}_*(I(\tilde{f})) \subset I(\tilde{g} \circ \tilde{f}); 5. I(id_{\tilde{X}}) \subset I(\tilde{f})$.

The class of path-connected spaces X satisfying the condition $I(id_{\tilde{X}}) = \pi = \pi_1(X, x_0)$ is closed under homotopy equivalence and the topological product operation, and contains the simply connected spaces, generalized lens spaces, H-spaces and homogeneous spaces of the form G/G_0 where G is a topological group and G_0 a subgroup which is a connected, compact Lie group (for the proofs see [26]).

From theorem 1 and results of Jiang [26] it follows:

Theorem 9 *Suppose that there is an integer m such that $\tilde{f}^m_*(\pi) \subset I(\tilde{f}^m)$ and $L(f) \neq 0$. Then*

$$N(f) = R(f) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i tr(\Lambda^i f_{1*}{}^{\infty} \otimes A). \tag{11}$$

where k is $rgH_1(X, \mathbb{Z})^{\infty}$, A is linear map on the complex vector space of complex valued functions on the group $TorsH_1(X, \mathbb{Z})$, p the number of $\mu \in \mathrm{Spec}\ f^{\infty}_{1*}$ such that $\mu < -1$, and r the number of real eigenvalues of f^{∞}_{1*} whose absolute value is > 1.

Proof We have $\tilde{f}_*(\pi) \subset I(\tilde{f})$(see [26]) . For any $\alpha \in \pi$, $p(\mathrm{Fix}\ \alpha \circ \tilde{f}) = p(\mathrm{Fix}\ \tilde{f}_*(\alpha) \circ \tilde{f})$ by the fact (see [26]) that α and $\tilde{f}_*(\alpha)$ are in the same $\tilde{f}_*$-conjugacy class.

Since $\tilde{f}_*(\pi) \subset I(\tilde{f})$, there is a homotopy $\{h_t\} : f \cong f$ which lifts to $\{\tilde{h}_t\} : \tilde{f} \cong \tilde{f}_*(\alpha) \circ \tilde{f}$. Hence Index $(f, p(\mathrm{Fix}\ \tilde{f}))$ = Index $(f, p(\mathrm{Fix}\ \alpha \circ \tilde{f}))$. Since $\alpha \in \pi$ is arbitrary, any two fixed point classes of f have the same index. It immediately follows that $L(f) = 0$ implies $N(f) = 0$ and $L(f) \neq 0$ implies $N(f) = R(f)$. By property 1, $\tilde{f}(\pi) \subset I(\tilde{f}) \subset Z(\tilde{f}_*(\pi), \pi)$, so $\tilde{f}_*(\pi)$ is abelian. Hence $\tilde{f}_*$ is eventually commutative and $N(f) = R(f) = R(\tilde{f}_*) = R(f_{1*})$. The result now follows from theorem1.

Example 1 *Let $f : X \to X$ be a hyperbolic endomorphism of torus T^k. Then $H_1(X, \mathbb{Z})$ is torsion free and*

$$N(f) = R(f) = (-1)^{r+p} \sum_{i=0}^{k} (-1)^i \mathrm{Tr}\ (\Lambda^i f_{1*}). \tag{12}$$

3.2 Polyhedra with finite fundamental group.

For a compact polyhedron X with finite fundamental group, $\pi_1(X)$, the universal cover $\tilde{X}$ is compact, so we may explore the relation between $L(\tilde{f})$ and Index $(p(Fix\ \tilde{f}))$.

Definition 5 *The number $\mu([\tilde{f}]) = \#Fix\ \tilde{f}_*$, defined to be the order of the finite group $Fix\ \tilde{f}_*$, is called the multiplicity of the lifting class $[\tilde{f}]$, or of the fixed point class $p(Fix\ \tilde{f})$.*

Lemma 8 ([26])

$$L(\tilde{f}) = \mu([\tilde{f}]).\mathrm{Index}\ (f, p(Fix\ \tilde{f})).$$

Let W be the complex vector space of complex valued class functions on the fundamental group π.The map $\tilde{f}_*$ induces a linear map $B : W \to W$ defined by

$$B(f) := f \circ \tilde{f}_*.$$

Theorem 10 *Let X be a connected, compact polyhedron with finite fundamental group π. Suppose that the action of π on the rational homology of the universal cover $\tilde{X}$ is trival, i.e. for every covering translation $\alpha \in \pi$, $\alpha_* = id : H_*(\tilde{X}, \mathbb{Q}) \to H_*(\tilde{X}, \mathbb{Q})$. Let $L(f) \neq 0$. Then*

$$N(f) = R(f) = TrB, \tag{13}$$

Proof Under our assumption on X, any two liftings $\tilde{f}$ and $\alpha \circ \tilde{f}$ induce the same homology homomorphism $H_*(\tilde{X}, \mathbb{Q}) \to H_*(\tilde{X}, \mathbb{Q})$, and have thus the same value of $L(\tilde{f})$. From Lemma 8 it follows that any two fixed point classes f are either both essential or both inessential. Since $L(f) \neq 0$ there is at least one essential fixed point class of f. Therefore all fixed point classes of f are essential and $N(f) = R(f)$.The formula for $N(f)$ follows now from theorem 4

Lemma 9 *Let X be a polyhedron with finite fundamental group π and let $p : \tilde{X} \to X$ be its universal covering. Then the action of π on the rational homology of $\tilde{X}$ is trivial iff $H_*(\tilde{X}; \mathbb{Q}) \cong H_*(X; \mathbb{Q})$.*

Corollary 1 *Let $\tilde{X}$ be a compact 1-connected polyhedron which is a rational homology n-sphere, where n is odd. Let π be a finite group acting freely on $\tilde{X}$ and let $X = \tilde{X}/\pi$. Then theorem 10 applies.*

PROOF The projection $p : \tilde{X} \to X = \tilde{X}/\pi$ is a universal covering space of X. For every $\alpha \in \pi$, the degree of $\alpha : \tilde{X} \to \tilde{X}$ must be 1, because $L(\alpha) = 0$ (α has no fixed points). Hence $\alpha_* = id : H_*(\tilde{X}; \mathbb{Q}) \to H_*(\tilde{X}; \mathbb{Q})$.

Corollary 2 *If X is a closed 3-manifold with finite π, then theorem 10 applies.*

PROOF $\tilde{X}$ is an orientable, simply connected manifold, hence a homology 3-sphere. We apply corollary .

Corollary 3 *Let $X = L(m, q_1, \ldots, q_r)$ be a generalized lens space and $f : X \to X$ a continuous map with $f_{1*}(1) = d$ where $d \neq 1$. Then theorem 10 applies.*

PROOF By corollary 1 we see that theorem 10 applies for lens spaces. Since $\pi_1(X) = \mathbb{Z}/m\mathbb{Z}$, the map f is eventually commutative. A lens space has a structure as a CW complex with one cell e_i in each dimension $0 \leq i \leq 2l + 1$. The boundary map is given by $\partial e_{2k} = m.e_{2k-1}$ for even cells, and $\partial e_{2k+1} = 0$ for odd cells. From this we may calculate the Lefschetz numbers: $L(f) = 1 - d^{(l+1)} \neq 0$.

3.3 Other special cases

3.3.1 Self-map of a nilmanifold

Theorem 6 implies

Theorem 11 *Let f be any continious map of a nilmanifold M to itself.If $R(f)$ is finite then*

$$N(f) = R(f) = (-1)^{r+p} \sum_{i=0}^{m} (-1)^i \mathrm{Tr}\ \Lambda^i \tilde{F}, \tag{14}$$

where $\tilde{F}$, m,r, and p are the same as in theorem 6

3.3.2 Pseudo-Anosov homeomorphism of a compact surface

Let X be a compact surface of negative euler characteristic and $f : X \to X$ is a pseudo-Anosov homeomorphism,i.e. there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$. Fathi and Shub [9] has proved the existence of Markov partitions for a pseudo-Anosov homeomorphism.The existence of Markov partitions implies that there is a symbolic dynamics for (X, f).This means that there is a finite set N, a matrix $A = (a_{ij})_{(i,j) \in N \times N}$ with entries 0 or 1 and a surjective map $p : \Omega \to X$,where

$$\Omega = \{(x_n)_{n \in \mathbb{Z}} : a_{x_n x_{n+1}} = 1, n \in \mathbb{Z}\}$$

such that $p \circ \sigma = f \circ p$ where σ is the shift (to the left) of the sequence (x_n) of symbols.We have first [5]:

$$\# Fix \sigma^n = \mathrm{Tr}\ A^n.$$

In general p is not bijective.The non-injectivity of p is due to the fact that the rectangles of the Markov partition can meet on their boundaries.To cansel the overcounting of periodic points on these boundaries,we use Manning's combinatorial arguments [30] proposed in the case of Axiom A diffeomorphism (see also [32]) .Namely, we construct finitely many subshifts of finite type $\sigma_i, i = 0, 1, .., m$, such that $\sigma_0 = \sigma$, the other shifts semi-conjugate with restrictions of f [32] ,and signs $\epsilon_i \in \{-1, 1\}$ such that for each n

$$\#Fixf^n = \sum_{i=0}^{m} \epsilon_i.\#Fix\sigma_i^n = \sum_{i=0}^{m} \epsilon_i.\mathrm{Tr}\ A_i^n,$$

where A_i is transition matrix, corresponding to subshift of finite type σ_i. For pseudo-Anosov homeomorphism of compact surface $N(f^n) = \#Fix(f^n)$ for each $n > o$ [37].So we have following trace formula for Nielsen numbers

Theorem 12 *Let X be a compact surface of negative euler characteristic and $f : X \to X$ is a pseudo-Anosov homeomorphism.Then*

$$N(f^n) = \sum_{i=0}^{m} \epsilon_i.\mathrm{Tr}\ A_i^n.$$

3.3.3 Homeomorphisms of a hyperbolic 3-manifolds

Theorem 13 *[28] Suppose M is a orientable compact connected 3-manifold such that $intM$ admits a complete hyperbolic structure with finite volume and $f : M \to M$ is orientation preserving homeomorphism.Then*

$$N(f) = L(f) = \sum_{k=0}^{\dim M} (-1)^k \mathrm{Tr}\ \left[f_{*k} : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})\right]$$

4 The Reidemeister trace formula for generalised Lefschetz numbers

The results of this section are well known(see [33],[38],[8],[27]).We shall use this results later in section to estimate the radius of convergence of the Nielsen zeta function. The fundamental group $\pi = \pi_1(X, x_0)$ splits into $\tilde{f}_*$-conjugacy classes.Let π_f denote the set of $\tilde{f}_*$-conjugacy classes,and $\mathbb{Z}\ pi_f$ denote the abelian group freely generated by π_f .We will use the bracket notation $a \to [a]$ for both projections $\pi \to \pi_f$ and $\mathbb{Z}\pi \to \mathbb{Z}\pi_f$. Let x be a fixed point of f.Take a path c from x_0 to x.The $\tilde{f}_*$-conjugacy class in π of the loop $c.(f \circ c)^{-1}$,which is evidently independent of the choice of c, is called the coordinate of x.Two fixed points are in the same fixed point class F iff they have the same coordinates.This $\tilde{f}_*$-conjugacy class is thus called the coordinate of the fixed point class F and denoted $cd_\pi(F, f)$ (compare with discription in section 2).

The generalised Lefschetz number or the Reidemeister trace [33] is defined as

$$L_\pi(f) := \sum_F ind(F, f).cd_\pi(F, f) \in \mathbb{Z}\pi_f, \tag{15}$$

the summation being over all essential fixed point classes $F$of f.The Nielsen number $N(f)$ is the number of non-zero terms in $L_\pi(f)$,and the indices of the essential fixed point classes appear as the coefficients in $L_\pi(f)$.This invariant used to be called the Reidemeister trace because it can be computed as an alternating sum of traces on the chain level as follows ([33],[38]) . Assume that X is a finite cell complex and

$f : X \to X$ is a cellular map.A cellular decomposition e_j^d of X lifts to a π-invariant cellular structure on the universal covering $\tilde{X}$.Choose an arbitrary lift $\tilde{e}_j^d$ for each e_j^d .They constitute a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\tilde{X}$.The lift $\tilde{f}$ of f is also a cellular map.In every dimension d, the cellular chain map $\tilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\tilde{F}_d$ with respect to the above basis,i.e $\tilde{F}_d = (a_{ij})$ if $\tilde{f}(\tilde{e}_i^d) = \sum_j a_{ij}\tilde{e}_j^d$,where $a_{ij} \in \mathbb{Z}\pi$.Then we have the Reidemeister trace formula

$$L_\pi(f) = \sum_d (-1)^d [Tr\tilde{F}_d] \in \mathbb{Z}\pi_f. \tag{16}$$

4.1 The mapping torus approach to the Reidemeister trace formula.

Now we describe alternative approach to the Reidemeister trace formula proposed recently by Jiang [27]. This approach is useful when we study the periodic points of f,i.e. the fixed points of the iterates of f.

The mapping torus T_f of $f : X \to X$ is the space obtained from $X \times [o, \infty)$ by identifiing $(x, s+1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$.On T_f there is a natural semi-flow $\phi : T_f \times [0, \infty) \to T_f, \phi_t(x, s) = (x, s+t)$ for all $t \geq 0$.Then the map $f : X \to X$ is the return map of the semi-flow ϕ.A point $x \in X$ and a positive number $\tau > 0$ determine the orbit curve $\phi_{(x,\tau)} := \phi_t(x)_{0 \leq t \leq \tau}$ in T_f.

Take the base point x_0 of X as the base point of T_f.It is known that the fundamental group $H := \pi_1(T_f, x_0)$ is obtained from π by adding a new generator z and adding the relations $z^{-1}gz = \tilde{f}_*(g)$ for all $g \in \pi = \pi_1(X, x_0)$.Let H_c denote the set of conjugacy classes in H. Let $\mathbb{Z}H$ be the integral group ring of H, and let $\mathbb{Z}H_c$ be the free abelian group with basis H_c.We again use the bracket notation $a \to [a]$ for both projections $H \to H_c$ and $\mathbb{Z}H \to \mathbb{Z}H_c$. If F^n is a fixed point class of f^n, then $f(F^n)$ is also fixed point class of f^n and $ind(f(F^n), f^n) = ind(F^n, f^n)$. Thus f acts as an index-preserving permutation among fixed point classes of f^n.By definition, an n-orbit class O^n of f to be the union of elements of an orbit of this action.In other words, two points $x, x' \in Fix(f^n)$ are said to be in the same n-orbit class of f if and only if some $f^i(x)$ and some $f^j(x')$ are in the same fixed point class of f^n.The set $Fix(f^n)$ splits into a disjoint union of n-orbits classes.Point x is a fixed point of f^n or a periodic point of period n if and only if orbit curve $\phi_{(x,n)}$ is a closed curve. The free homotopy class of the closed curve $\phi_{(x,n)}$ will be called the H -coordinate of point x,written $cd_H(x, n) = [\phi_{(x,n)}] \in H_c$.It follows that periodic points x of period n and x' of period n' have the same H-coordinate if and only if $n = n'$ and x,x' belong to the same n-orbits class of f. Thus it is possible equivalently define $x, x' \in Fixf^n$ to be in the same n-orbit class if and only if

Recently, Jiang [27] has considered generalised Lefschetz number with respect to H

$$L_H(f^n) := \sum_{O^n} ind(O^n, f^n).cd_H(O^n) \in \mathbb{Z}H_c, \tag{17}$$

and proved following trace formula:

$$L_H(f^n) = \sum_d (-1)^d [Tr(z\tilde{F}_d)^n] \in \mathbb{Z}H_c, \tag{18}$$

where $\tilde{F}_d$ be $\mathbb{Z}\pi$-matrices defined in (16) and $z\tilde{F}_d$ is regarded as a $\mathbb{Z}H$-matrix.

5 The Reidemeister zeta function of a group endomorphism

PROBLEM. For which groups and endomorphisms is the Reidemeister zeta function a rational function? When does it have a functional equation? Is $R_\phi(z)$ an algebraic function?

5.1 Reidemeister zeta functions of eventually commutative endomorphisms.

As we remarked in section 1 to find out about the Reidemeister zeta functions of eventually commutative endomorphisms, it is sufficient to study the zeta functions of endomorphisms of abelian groups. For the rest of this section G will be a finitely generated abelian group.

Theorem 14 *Let G is a finitely generated abelian group and ϕ an endomorphism of G .Then $R_\phi(z)$ is a rational function and is equal to*

$$R_\phi(z) = \left(\prod_{i=0}^{k} \det(1 - \Lambda^i\phi^\infty \otimes A.\sigma.z)^{(-1)^{i+1}}\right)^{(-1)^r} \tag{19}$$

where matrix A is defined in lemma 3, $\sigma = (-1)^p$,p , r and k are constants described in theorem 1.

PROOF If we repeat the proof of the theorem 1 for ϕ^n instead ϕ we receive that $R(\phi^n) = R((\phi^\infty)^n.R((\phi^{tor})^n)$.From this and lemmas 2 and 3 we have the trace formula for $R(\phi^n)$:

$$\begin{aligned} R(\phi^n) &= (-1)^{r+pn}\sum_{i=0}^{k}(-1)^i \mathrm{Tr}\ \Lambda^i(\phi^\infty)^n.\mathrm{Tr}\ A^n \\ &= (-1)^{r+pn}\sum_{i=0}^{k}(-1)^i \mathrm{Tr}\ (\Lambda^i(\phi^\infty)^n \otimes A^n) \\ &= (-1)^{r+pn}\sum_{i=0}^{k}(-1)^i \mathrm{Tr}\ (\Lambda^i\phi^\infty \otimes A)^n. \end{aligned}$$

We now calculate directly

$$\begin{aligned} R_\phi(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{R(\phi^n)}{n}z^n\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^r\sum_{i=0}^{k}(-1)^i \mathrm{Tr}\ (\Lambda^i\phi^\infty \otimes A)^n}{n}(\sigma.z)^n\right) \\ &= \left(\prod_{i=0}^{k}\left(\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\ (\Lambda^i\phi^\infty \otimes A)^n.(\sigma.z)^n\right)\right)^{(-1)^i}\right)^{(-1)^r} \\ &= \left(\prod_{i=0}^{k}\det\left(1 - \Lambda^i\phi^\infty \otimes A.\sigma.z\right)^{(-1)^{i+1}}\right)^{(-1)^r}. \end{aligned}$$

5.2 Endomorphisms of finite groups

Theorem 15 *Let ϕ be an endomorphism of a finite group G. Then $R_\phi(z)$ is a rational function and given by formula*

$$R_\phi(z) = \frac{1}{\det(1 - Bz)}, \tag{20}$$

Where B is defined in theorem 4

PROOF From theorem 4 it follows that $R(\phi^n) = \mathrm{Tr}\ B^n$ for every $n > o$.We now calculate directly

$$\begin{aligned} R_\phi(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Tr}\ B^n}{n} z^n\right) = \exp\left(\mathrm{Tr}\ \sum_{n=1}^{\infty} \frac{B^n}{n} z^n\right) \\ &= \exp\left(\mathrm{Tr}\ (-log(1 - Bz))\right) = \frac{1}{\det(1 - Bz)}. \end{aligned}$$

5.3 Endomorphisms of the direct sum of a free abelian and a finite group

Theorem 16 *Let G is the direct sum of free abelian and a finite group and ϕ an endomorphism of G .If the numbers $R(\phi^n)$ are all finite then $R_\phi(z)$ is a rational function and is equal to*

$$R_\phi(z) = \left(\prod_{i=0}^{k} \det(1 - \Lambda^i \phi^\infty \otimes B.\sigma.z)^{(-1)^{i+1}}\right)^{(-1)^r} \tag{21}$$

where matrix B is defined in theorem 4, $\sigma = (-1)^p$,p , r and k are constants described in theorem 5.

PROOF From lemma 4 it follows that $R(\phi^n) = R((\phi^\infty)^n.R((\phi^{tor})^n)$. From now on the proof repeat the proof of the theorem 14.

5.4 Endomorphisms of nilpotent groups

Theorem 17 *If G is a finitely generated torsion free nilpotent group and ϕ an endomorphism of G .Then $R_\phi(z)$ is a rational function and is equal to*

$$R_\phi(z) = \left(\prod_{i=0}^{m} \det(1 - \Lambda^i \tilde{F}.\sigma.z)^{(-1)^{i+1}}\right)^{(-1)^r} \tag{22}$$

where $\sigma = (-1)^p$,p , r, m and $\tilde{F}$ is defined in section 1.5.

PROOF If we repeat the proof of the theorem 6 for ϕ^n instead ϕ we receive that $R(\phi^n) = (-1)^{r+pn} \det(1 - \tilde{F})$(we suppose that Reidemeister numbers $R(\phi^n)$ are finite for all n).Last formula implies the trace formula for $R(\phi^n)$:

$$R(\phi^n) = (-1)^{r+pn} \sum_{i=0}^{m} (-1)^i \mathrm{Tr}\ (\Lambda^i \tilde{F})^n$$

From this we have formula 22 immedeately by direct calculation as in theorem 14.

Corollary 4 *Let the assumptions of theorem 17 hold. Then the poles and zeros of the Reidemeister zeta function are complex numbers which are reciprocal of an eigenvalue of one of the matrices*

$$\Lambda^i(\tilde{F}) : \Lambda^i(\tilde{G}) \longrightarrow \Lambda^i(\tilde{G}) \qquad 0 \leq i \leq \text{rank } G$$

5.4.1 Functional equation

Theorem 18 *Let $\phi : G \to G$ be an endomorphism of a finitely generated torsion free nilpotent group G. Then the Reidemeister zeta function $R_\phi(z)$ has the following functional equation:*

$$R_\phi\left(\frac{1}{dz}\right) = \epsilon_2 . R_\phi(z)^{(-1)^{\text{Rank } G}}. \tag{23}$$

where $d = \det \tilde{F}$ and ϵ_1 are a constants in $\mathbb{C}^\times$.

Proof Via the natural nonsingular pairing $(\Lambda^i \tilde{F}) \otimes (\Lambda^{m-i}\tilde{F}) \to \mathbb{C}$ the operators $\Lambda^{m-i}\tilde{F}$ and $d.(\Lambda^i \tilde{F})^{-1}$ are adjoint to each other.

We consider an eigenvalue λ of $\Lambda^i \tilde{F}$. By theorem 17, This contributes a term

$$\left((1 - \frac{\lambda\sigma}{dz})^{(-1)^{i+1}}\right)^{(-1)^r}$$

to $R_\phi\left(\frac{1}{dz}\right)$.

We rewrite this term as

$$\left(\left(1 - \frac{d\sigma z}{\lambda}\right)^{(-1)^{i+1}} \left(\frac{-dz}{\lambda\sigma}\right)^{(-1)^i}\right)^{(-1)^r}$$

and note that $\frac{d}{\lambda}$ is an eigenvalue of $\Lambda^{m-i}\tilde{F}$. Multiplying these terms together we obtain,

$$R_\phi\left(\frac{1}{dz}\right) = \left(\prod_{i=1}^{m} \prod_{\lambda^{(i)} \in \text{Spec } \Lambda^i \tilde{F}} \left(\frac{1}{\lambda^{(i)}\sigma}\right)^{(-1)^i}\right)^{(-1)^r} \times R_\phi(z)^{(-1)^m}.$$

The variable z has disappeared because

$$\sum_{i=0}^{m} (-1)^i \dim \Lambda^i \tilde{G} = \sum_{i=0}^{m} (-1)^i {C_k}^i = 0.$$

5.5 Some conjectures for wider classes of groups

For the case of almost nilpotent groups (ie. groups with polynomial growth, in view of Gromov's theorem [22]) we believe that some power of the Reidemeister zeta function is a rational function. We intend to prove this conjecture by identifying the Reidemeister number on the nilpotent part of the group with the number of fixed points in the direct sum of the duals of the quotients of successive terms in the central series. We then hope to show that the Reidemeister number of the whole endomorphism is a sum of numbers of orbits of such fixed points under the action of the finite quotient group (ie the quotient of the whole group by the nilpotent part). The situation for groups with exponential growth is very different. There one can expect the Reidemeister number to be infinite as long as the endomorphism is injective. This can be proved in the case of surface groups due to a theorem of C.Epstein (see [7]). He proves an estimate on numbers of geodesics, which when applied to the mapping torus of pseudo-Anosov map guarantees infinitely many loops which wrap around the mapping torus exactly once. This is equivalent (see section 4.1) to saying that the Reidemeister number is infinite. A rigid hyperbolic group has a finite outer automorphism group[23].This implies that the Reidemeister number of some iteration of endomorphism equals to the number of usual conjugacy classes which is in this case infinite. We believe this conjecture for the following reason. Let G be a group with exponential growth and let $l(g)$ be the length of an element of G. Then one might expect that most of the time

$$l(gx\phi(g)^{-1}) > (1+\epsilon)l(g)$$

for $g, x \in G$. This would imply that for fixed x,

$$\#\{gx\phi(g)^{-1} : l(gx\phi(g)^{-1}) < N\} < \#\{g \in G : l(gx\phi(g)^{-1}) < N/(1+\epsilon)\}$$

However since the group has exponential growth one can show : that this would imply

$$\frac{\#\{gx\phi(g)^{-1} : l(gx\phi(g)^{-1}) < N\}}{\#\{g \in G : l(gx\phi(g)^{-1}) < N\}} \to 0 \text{ as } N \to \infty.$$

If there were only finitely many twisted conjugacy classes then we could derive a contradiction by summing the left hand side of the above formula over a set of representatives x for the classes, and observing that this sum is always equal to 1.

6 The Reidemeister and Nielsen zeta functions of a continuous map.

Remark 2 *Using lemma 6 we may apply all theorems of section 5 to the Reidemeister zeta functions of continuos maps.*

6.1 The Jiang subgroup and the Nielsen zeta function

Theorem 19 *Suppose that there is an integer m such that $\tilde{f}_*^m(\pi) \subset I(\tilde{f}^m)$.If $L(f^n) \neq 0$ for every $n > o$,then*

$$N_f(z) = R_f(z) = \left(\prod_{i=0}^{k} \det(1 - \Lambda^i f_{1*}^{\infty} \otimes A.\sigma.z)^{(-1)^{i+1}}\right)^{(-1)^r} \tag{24}$$

If $L(f^n) = 0$ only for finite number of n,then

$$N_f(z) = \exp\left(P(z)\right).R_f(z) = \exp\left(P(z)\right).\left(\prod_{i=0}^{k} \det(1 - \Lambda^i f_{1*}^{\infty} \otimes A.\sigma.z)^{(-1)^{i+1}}\right)^{(-1)^r} \tag{25}$$

Where $P(z)$ is a polynomial ,$A, k, p,$ and r are as in theorem 14.

Proof

If $L(f^n) \neq 0$ for every $n > o$,then formula (19) follows from theorems 9 and 14. If $L(f^n) = 0$, then $N(f^n) = 0$ .If $L(f^n) \neq 0$, then $N(f^n) = R(f^n)$(see proof of theorem 9).So the fraction $N_f(z)/R_f(z) = \exp(P(z))$, where $P(z)$ is a polynomial whose degree equal to maximal n, such that $L(f^n) \neq 0$.

Corollary 5 *Let the assumptions of theorem 19 hold. Then the poles and zeros of the Nielsen zeta function are complex numbers which are the reciprocal of an eigenvalue of one of the matrices*

$$\Lambda^i(f_{1*}^\infty \otimes A.\sigma)$$

Corollary 6 *Let $I(id_{\tilde{X}}) = \pi$ .If the assumptions of theorem 19 about Lefschetz numbers hold,then formulas (24) and (25) are valid.*

Corollary 7 *Suppose that X is asphericaland f is eventually commutative.If the assumptions of theorem 19 about Lefschetz numbers hold,then formulas (24) and (25) are valid.*

6.2 Polyhedra with finite fundamental group and Nielsen zeta function

Theorem 20 *Let X be a connected, compact polyhedron with finite fundamental group π. Suppose that the action of π on the rational homology of the universal cover $\tilde{X}$ is trivial, i.e. for every covering translation $\alpha \in \pi$, $\alpha_* = id : H_*(\tilde{X}, \mathbb{Q}) \to H_*(\tilde{X}, \mathbb{Q})$. If $L(f^n) \neq 0$ for every $n > o$,then*

$$N_f(z) = R_f(z) = \frac{1}{\det(1 - Bz)}, \tag{26}$$

If $L(f^n) = 0$ only for finite number of n,then

$$N_f(z) = \exp\left(P(z)\right).R_f(z) = \exp\left(P(z)\right).\frac{1}{\det(1 - Bz)}, \tag{27}$$

Where $P(z)$ is a polynomial, B is defined in theorem 4

Proof

If $L(f^n) \neq 0$ for every $n > o$,then formula (26) follows from theorems 10 and 15. If $L(f^n) = 0$, then $N(f^n) = 0$ .If $L(f^n) \neq 0$, then $N(f^n) = R(f^n)$(see proof of theorem 10).So the fraction $N_f(z)/R_f(z) = \exp(P(z))$, where $P(z)$ is a polynomial whose degree equal to maximal n, such that $L(f^n) \neq 0$.

Corollary 8 *Let $\tilde{X}$ be a compact 1-connected polyhedron which is a rational homology n-sphere, where n is odd. Let π be a finite group acting freely on $\tilde{X}$ and let $X = \tilde{X}/\pi$. Then theorem 20 applies.*

Corollary 9 *If X is a closed 3-manifold with finite π, then theorem 20 applies.*

Example 2 ([3]) *Let $f : S^2 \vee S^4 \rightarrow S^2 \vee S^4$ to be a continuous map of the bouquet of spheres such that the restriction $f/_{S^4} = id_{S^4}$ and the degree of the restriction $f/_{S^2} : S^2 \rightarrow S^2$ equal to -2. Then $L(f) = 0$, hence $N(f) = 0$ since $S^2 \vee S^4$ is simply connected. For $k > 1$ we have $L(f^k) = 2 + (-2)^k \neq 0$, therefore $N(f^k) = 1$. From this we have by direct calculation that*

$$N_f(z) = \exp(-z).\frac{1}{1-z}. \tag{28}$$

Remark 3 *We would like to mention that in all known cases the Nielsen zeta function is a nice function. By this we mean that it is a product of an exponential of a polynomial with a function some power of which is rational. May be this is a general pattern; it could however be argued that this just reflects our inability to calculate the Nielsen numbers in general case.*

6.3 Nielsen zeta function in other special cases

Theorem 17 implies

Theorem 21 *Let f be any continious map of a nilmanifold M to itself. If $R(f^n)$ is finite for all n then*

$$N_f(z) = R_f(z) = \left(\prod_{i=0}^{m} \det(1 - \Lambda^i \tilde{F}.\sigma.z)^{(-1)^{i+1}} \right)^{(-1)^r} \tag{29}$$

where $\sigma = (-1)^p$, p , r, m and $\tilde{F}$ is defined in section 5.4.

Theorem 12 implies

Theorem 22 *Let X be a compact surface of negative euler characteristic and $f : X \rightarrow X$ is a pseudo-Anosov homeomorphism. Then*

$$N_f(z) = \prod_{i=0}^{m} \det(1 - A_i z)^{-\epsilon_i} \tag{30}$$

where A_i and ϵ_i the same as in theorem 12.

Theorem 13 implies

Theorem 23 *Suppose M is a orientable compact connected 3-manifold such that $intM$ admits a complete hyperbolic structure with finite volume and $f : M \rightarrow M$ is orientation preserving homeomorphism. Then Nielsen zeta function is rational and*

$$N_f(z) = L_f(z)$$

6.4 Radius of convergence of the Nielsen zeta function

We denote by R the radius of convergence of the Nielsen zeta function $N_f(z)$, and by $h(f)$ the topological entropy of continuous map f. Let $h = \inf\{h(g)$ over all maps g of the same homotopy type as $f\}$.

Theorem 24 ([18]) *For any continuous map f of any compact polyhedron X into itself the Nielsen zeta function has positive radius of convergence R and*

$$R \geq \exp(-h) > 0. \tag{31}$$

In this section we propose another prove of positivity of R and give an exact algebraic lower estimation for the radius R using trace formulas (16) and (18) for generalised Lefschetz numbers.

For any set S let $\mathbb{Z}S$ denote the free abelian group with the specified basis S.The norm in $\mathbb{Z}S$ is defined by

$$\| \sum_i k_i s_i \| := \sum_i \mid k_i \mid \in \mathbb{Z}, \tag{32}$$

when the s_i in S are all different.

For a $\mathbb{Z}H$-matrix $A = (a_{ij})$,define its norm by $\|A\| := \sum_{i,j} \|a_{ij}\|$.Then we have inequalities $\|AB\| \leq \|A\ |.\|B\|$ when A, B can be multiplied, and $\|\text{Tr } A\| \leq \|A\|$ when A is a square matrix.For a matrix $A = (a_{ij})N(f^n)$ in $\mathbb{Z}S$, its matrix of norms is defined to be the matrix $A^{norm} := (\|a_{ij}\|)$ which is a matrix of non-negative integers.In what follows, the set S wiill be π, H or H_c.We denote by $s(A)$ the spectral radius of A, $s(A) = \lim_n(\|A^n\||)^{\frac{1}{n}}$, which coincide with the largest modul of an eigenvalue of A.

Theorem 25 *For any continious map f of any compact polyhedron X into itself the Nielsen zeta function has positive radius of convergence R,which admits following estimations*

$$R \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0 \tag{33}$$

and

$$R \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0, \tag{34}$$

where $\tilde{F}_d$ is the same as in section 4.

Proof By the homotopy type invariance of the invariants we can suppose that f is a cell map of a finite cell complex.By the definition the Nielsen number $N(f^n)$ is the number of non-zero terms in $L_\pi(f^n)$(see section 4).The norm $\|L_H(f^n)\|$ is the sum of absolute values of the indices of all the n-orbits classes O^n .It equals $\|L_\pi(f^n)\|$, the sum of absolute values of the indices of all the fixed point classes of f^n, because any two fixed point classes of f^n contained in the same n-orbit class O^n must have the same index.From this we have $N(f^n) \leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \|\sum_d(-1)^d[\text{Tr } (z\tilde{F}_d)^n]\| \leq \sum_d \|[\text{Tr } (z\tilde{F}_d)^n]\| \leq \sum_d \|\text{Tr } (z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)\|^n$(see [27]).The radius of convergence R is given by Caushy-Adamar formula:

$$\frac{1}{R} = \limsup_n (\frac{N(f^n)}{n})^{\frac{1}{n}} = \limsup_n (N(f^n))^{\frac{1}{n}}.$$

Therefore we have:

$$R = \frac{1}{\limsup_n (N(f^n))^{\frac{1}{n}}} \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0.$$

Inequalities:

$$N(f^n) \leq \|L_\pi(f^n)\| = \|L_H(f^n)\| = \|\sum_d (-1)^d [\text{Tr } (z\tilde{F}_d)^n]\| \leq \sum_d \|[\text{Tr } (z\tilde{F}_d)^n]\| \leq$$

$$\sum_d ||\text{Tr } (z\tilde{F}_d)^n|| \leq \sum_d \text{Tr } ((z\tilde{F}_d)^n)^{norm} \leq \sum_d \text{Tr } ((z\tilde{F}_d)^{norm})^n \leq \sum_d \text{Tr } ((\tilde{F}_d)^{norm})^n$$

and the definition of spectral radius give estimation:

$$R = \frac{1}{\limsup_n (N(f^n))^{\frac{1}{n}}} \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0.$$

Example 3 *Let X be surface with boundary, and $f : X \to X$ be a map.Fadell and Husseini [8] devised a method of computing the matrices of the lifted chain map for surface maps.Suppose $\{a_1, ..., a_r\}$ is a free basis for $\pi_1(X)$. Then X has the homotopy type of a bouquet B of r circles which can be decomposed into one 0-cell and r 1-cells corresponding to the a_i,and f has the homotopy type of a cellular map $g : B \to B$. By the homotopy type invariance of the invariants,we can replace f with g in computations.The homomorphism $\tilde{f}_* : \pi_1(X) \to \pi_1(X)$ induced by f and g is determined by the images $b_i = \tilde{f}_*(a_i), i = 1, .., r$.The fundamental group $\pi_1(T_f)$ has a presentation $\pi_1(T_f) =< a_1, ..., a_r, z | a_i z = z b_i, i = 1, .., r >$.Let*

$$D = (\frac{\partial b_i}{\partial a_j})$$

be the Jacobian in Fox calculus(see [4]).Then,as pointed out in [8], the matrices of the lifted chain map $\tilde{g}$ are

$$\tilde{F}_0 = (1), \tilde{F}_1 = D = (\frac{\partial b_i}{\partial a_j}).$$

Now, we can find estimations for the radius R from (33) and (34).

Remark 4 *Let X be a compact connected surface and $f : X \to X$ be a homeomorphism.Let $\chi(X) < 0$.By Thurston's classification theorem [37] f is isotopic to a homeomorphism ϕ such that either (1) ϕ is a periodic map;or (2) ϕ is a pseudo-Anosov map with stretching factor $\lambda > 1$(see section 3.3.1);or (3) ϕ is a reducible map,i.e. there is a system of disjoint simple closed curves in int$X$ which is invariant by ϕ and which has aϕ-invariant tubular neighborhood U such that each component of $X - U$ has negative euler characteristic and on each ϕ-component of $X - U$, ϕ satisfies 1 or 2.Then, as follows from [18],radius $R = \frac{1}{\lambda}$, where $\lambda > 1$ is the largest stretching factor of the pseudo-Anosov pieses(if there is no pseudo-Anosov piece) $\lambda = 1$.*

7 Connection with Reidemeister Torsion

7.1 Preliminaries

7.1.1 Reidemeister torsion

Like the Euler characteristic, the Reidemeister torsion is algebraically defined. Roughly speaking, the Euler characteristic is a graded version of the dimension, extending the dimension from a single vector space to a complex of vector spaces. In a similar way, the Reidemeister torsion is a graded version of the absolute value of the determinant of an isomorphism of vector spaces. Let $d^i : C^i \to C^{i+1}$ be a cochain complex C^* of finite dimensional vector spaces over $\mathbb{C}$ with $C^i = 0$ for $i < 0$ and large i. If the cohomology $H^i = 0$ for all i we say that C^* is *acyclic*. If one is given positive densities Δ_i on C^i then the Reidemeister torsion $\tau(C^*, \Delta_i) \in (0, \infty)$ for acyclic C^* is defined as follows:

Definition 6 *Consider a chain contraction $\delta^i : C^i \to C^{i-1}$, ie. a linear map such that $d \circ \delta + \delta \circ d = id$. Then $d + \delta$ determies a map*

$(d+\delta)_+ : C^+ := \oplus C^{2i} \to C^- := \oplus C^{2i+1}$ and a map $(d+\delta)_- : C^- \to C^+$. Since the map $(d+\delta)^2 = id + \delta^2$ is unipotent, $(d+\delta)_+$ must be an isomorphism. One defines $\tau(C^*, \Delta_i) :=\mid \det(d+\delta)_+ \mid$ (see [21]).

Reidemeister torsion is defined in the following geometric setting. Suppose K is a finite complex and E is a flat, finite dimensional, complex vector bundle with base K. We recall that a flat vector bundle over K is essentially the same thing as a representation of $\pi_1(K)$ when K is connected. If $p \in K$ is a basepoint then one may move the fibre at p in a locally constant way around a loop in K. This

defines an action of $\pi_1(K)$ on the fibre E_p of E above p. We call this action the holonomy representation $\rho : \pi \to GL(E_p)$. Conversely, given a representation $\rho : \pi \to GL(V)$ of π on a finite dimensional complex vector space V, one may define a bundle $E = E_\rho = (\tilde{K} \times V)/\pi$. Here $\tilde{K}$ is the universal cover of K, and π acts on $\tilde{K}$ by covering tranformations and on V by ρ. The holonomy of E_ρ is ρ, so the two constructions give an equivalence of flat bundles and representations of π.

If K is not connected then it is simpler to work with flat bundles. One then defines the holonomy as a representation of the direct sum of π_1 of the components of K. In this way, the equivalence of flat bundles and representations is recovered.

Suppose now that one has on each fibre of E a positive density which is locally constant on K. In terms of ρ_E this assumption just means $\mid \det \rho_E \mid = 1$. Let V denote the fibre of E.

Then the cochain complex $C^i(K; E)$ with coefficients in E can be identified with the direct sum of copies of V associated to each i-cell σ of K. The identification is achieved by choosing a basepoint in each component of K and a basepoint from each i-cell. By choosing a flat density on E we obtain a preferred density Δ_i on $C^i(K, E)$. One defines the R-torsion of (K, E) to be $\tau(K; E) = \tau(C^*(K; E), \Delta_i) \in (0, \infty)$.

7.1.2 Unitary dual of the direct sum of a free abelian and a finite group

In this section let again the group G will be $G = \mathbb{Z}^m \oplus F$,where F is a finite group and m is a natural number.Let $\phi : G \to G$ be an endomorphism.One defines the unitary dual $\hat{G}$ of G to be the set of all equivalence classes of irreducible, unitary representations of G. If $\rho : G \to U(V)$ is a unitary representation of G on V then $\rho\circ\phi : G \to U(V)$ is also a representation, which we denote $\hat{\phi}(\rho)$. If the representations ρ_1 and ρ_2 are equivalent then representations $\hat{\phi}(\rho_1)$ and $\hat{\phi}(\rho_2)$ are also equivalent. Therefore the endomorphism ϕ induces a dual map $\hat{\phi} : \hat{G} \to \hat{G}$ from the unitary dual to itself. Now consider the dual $\hat{G}$ for $G = \mathbb{Z}^r \oplus F$. This is the cartesian product of the duals of $\mathbb{Z}^m$ and $\hat{F}$: $\hat{G} = \hat{\mathbb{Z}}^m \otimes \hat{F}$. From this it follows that topologically the dual $\hat{G}$ is a uni! on of finitely many disjoint tori

Lemma 10 *[25] If ϕ is any endomorphism of G where G is the direct sum of a finite group with a free abelian group, then*

$$R(\phi) = \#Fix(\hat{\phi})$$

7.2 The Reidemeister zeta Function and the Reidemeister Torsion of the Mapping Tori of the dual map.

Let $f : X \to X$ be a homeomorphism of a compact polyhedron X.

Let $T_f := (X \times I)/(x, 0) \sim (f(x), 1)$ be the mapping tori of f. We shall consider the bundle $p : T_f \to S^1$ over the circle S^1. We assume here that E is a flat, complex vector bundle with finite dimensional fibre and base S^1. We form its pullback p^*E over T_f. Note that the vector spaces $H^i(p^{-1}(b), c)$ with $b \in S^1$ form a flat vector bundle over S^1, which we denote H^iF. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iF$ is acyclic for all i. Under these conditions D. Fried [21] has shown that the bundle p^*E is acyclic, and we have

$$\tau(T_f; p^*E) = \prod_i \tau(S^1; E \otimes H^iF)^{(-1)^i}. \tag{35}$$

Let g be the prefered generator of the group $\pi_1(S^1)$ and let $A = \rho(g)$ where $\rho : \pi_1(S^1) \to GL(V)$. Then the holonomy around g of the bundle $E \otimes H^iF$ is $A \otimes f_i^*$.

Since $\tau(E) = \mid \det(I - A) \mid$ it follows from (20) that

$$\tau(T_f; p^*E) = \prod_i \mid \det(I - A \otimes f_i^*) \mid^{(-1)^i}. \tag{36}$$

We now consider the special case in which E is one-dimensional, so A is just a complex scalar λ of modulus one. Then in terms of the rational function $L_f(z)$ we have [21]:

$$\tau(T_f; p^*E) = \prod_i \mid \det(I - \lambda.f_i^*) \mid^{(-1)^i} = \mid L_f(\lambda) \mid^{-1} \tag{37}$$

Theorem 26 *Let $\phi : G \to G$ be an automorphism of G,where G is the direct sum of a finite group with a finitely generated free abelian group, then*

$$\tau\left(T_{\hat{\phi}}; p^*E\right) = \mid L_{\hat{\phi}}(\lambda) \mid^{-1} = \mid R_\phi(\sigma\lambda) \mid^{(-1)^{r+1}},$$

where λ is the holonomy of the one-dimensional flat complex bundle E over S^1, r and σ are the constants described in theorem 5 .

Proof We know from lemma 10 that $R(\phi^n)$ is the number of fixed points of the map $\hat{\phi}^n$. In general it is only necessary to check that the number of fixed points of $\hat{\phi}^n$ is equal to the absolute value of its Lefschetz number. We assume without loss of generality that $n = 1$. We are assuming that $R(\phi)$ is finite, so the fixed points of $\hat{\phi}$ form a discrete set. We therefore have

$$L(\hat{\phi}) = \sum_{x \in \mathrm{Fix}\ \hat{\phi}} \mathrm{Index}\ (\hat{\phi}, x).$$

Since ϕ is a group endomorphism, the trivial representation $x_0 \in \hat{G}$ is always fixed. Let x be any fixed point of $\hat{\phi}$.Since $\hat{G}$ is union of tori $\hat{G}_0, ..., \hat{G}_t$ and $\hat{\phi}$ is a linear map, we can shift any two fixed points onto one another without altering the map $\hat{\phi}$. This gives us for any fixed point x the equality

$$\mathrm{Index}\ (\hat{\phi}, x) = \mathrm{Index}\ (\hat{\phi}, x_0)$$

and so all fixed points have the same index. It is now sufficient to show that Index $(\hat{\phi}, x_0) = \pm 1$. This follows because the map on the torus

$$\hat{\phi} : \hat{G}_0 \to \hat{G}_0$$

lifts to a linear map of the universal cover, which is an euclidean space. The index is then the sign of the determinant of the identity map minus this lifted map. This determinant cannot be zero, because $1 - \hat{\phi}$ must have finite kernel by our assumption that the Reidemeister number of ϕ is finite (if $\det(1 - \hat{\phi}) = 0$ then the kernel of $1 - \hat{\phi}$ is a positive dimensional subspace of $\hat{G}$, and therefore infinite).

Corollary 10 *Let $f : X \to X$ be a homeomorphism of a compact polyhedron X. If $\pi_1(X)$ is the direct sum of a finite group with a free abelian group, then then*

$$\tau\left(T_{\widehat{(f_{1*})}}; p^*E\right) = \left| \ L_{\widehat{(f_{1*})}}(\lambda) \ \right|^{-1} = \left| \ R_f(\sigma\lambda) \ \right|^{(-1)^{r+1}},$$

where r and σ are the constants described in theorem 9.

8 Concluding remarks

8.1 Reidemeister and Nielsen numbers and zeta functions modulo a normal subgroup

In the theory of (ordinary) fixed point classes, we work on the universal covering space. The group of covering transformations plays a key role. It is not surprising that this theory can be generalized to work on all regular covering spaces. Let K be a normal subgroup of the fundamental group $\pi_1(X)$. Consider the regular covering $p_K : \tilde{X}/K \to X$ corresponding to K. A map $\tilde{f}_K : \tilde{X}/K \to \tilde{X}/K$ is called a lifting of $f : X \to X$ if $p_K \circ \tilde{f}_K = f \circ p_K$. We know from the theory of covering spaces that such liftings exist if and only if $f_*(K) \subset K$. If K is a fully invariant subgroup of $\pi_1(X)$ (in the sense that every endomorphism sends K into K) such as, for example the commutator subgroup of $\pi_1(X)$, then there is a lifting of any continuous map.

We may define the mod K-Reidemeister and Nielsen numbers (see [26]) and zeta functions (see [12]) and develop a similar theory for them by simply replacing $\tilde{X}$ and $\pi_1(X)$ by $\tilde{X}/K$ and $\pi_1(X)/K$ in every definition, every theorem and every proof, since everything was done in terms

of liftings and covering translations.

8.2 Minimal dynamical zeta function

8.2.1 Radius of convergence of the minimal zeta function

In the Nielsen theory for periodic points, it is well known that $N(f^n)$ is often poor as a lower bound for the number of fixed points of f^n.A good homotopy invariant lower bound $NF_n(f)$,called the Nielsen type number for f^n,is defined in [26].Consider any finite set of periodic orbit classes $\{O^{k_j}\}$ of varied period k_j such that every essential periodic m-orbit class, $m|n$, contains at least one class in the set.Then $NF_n(f)$ is the minimal sum $\sum_j k_j$ for all such finite sets. Halpern(see [26]) has proved that for all n $NF_n(f) = \min\{\#Fix(g)|g$ has the same homotope type as f }.Recently, Jiang [27] found that as far as asymptotic growth rate is concerned,these Nielsen type numbers are no better than the Nielsen numbers.

Lemma 11 ([27])

$$\limsup_n (N(f^n))^{\frac{1}{n}} = \limsup_n (NF_n(f))^{\frac{1}{n}} \tag{38}$$

Let us consider minimal dynamical zeta function

$$M_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{NF_n(f)}{n} z^n\right),$$

In paper [13] it was proved that $M_f(z)$ has positive radius of convergence R. Below we propose another prove of this fact and give an exast algebraic lower estimation for the radius R.

Theorem 27 *For any continious map f of any compact polyhedron X into itself the minimal zeta function has positive radius of convergence R,which admits following estimations*

$$R \geq \exp(-h) > 0, \tag{39}$$

$$R \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0, \tag{40}$$

and

$$R \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0, \tag{41}$$

where $\tilde{F}_d$ and h is the same as in section 6.1.

PROOF The theorem follows from Caushy-Adamar formula,lemma 11 and theorems 24 and 25.

8.3 Dold congruenses

Definition 7 *We shall say that a sequence of integers $a(n)$ satisfies the Dold congruences iff for all $n \in N$ one has*

$$\sum_{d|n} \mu(d)a(n/d) \equiv 0 \bmod n.$$

Dold [6] proved that the sequence $L(f^n)$ of Lefschetz numbers has this property. The following easy lemma can be proved purely combinatorially:

Lemma 12 *Let X is any set and $f : X \to X$ any function. If for any n, f^n has only finitely many fixed points in X then the sequence of numbers $\#Fix(f^n)$ satisfies the Dold congruences. Any sequence of natural numbers satisfying the Dold congruences arises in this way.*

In [15] we have proved that if $f : X \to X$ is eventually commutative map of compact polyhedron,or any self-map if $\pi_1(X)$ is finite, then the numbers $R(f^n)$ satisfy the Dold congruences.
Let $g : M \to M$ be an expanding map [36] of the orientable smooth compact manifold M.Then M is aspherical and is a $K(\pi_1(M), 1)$, and $\pi_1(M)$ is torsion free [36].According to Shub [36] any lifting $\tilde{g}$ of g has exactly one fixed point.From this and the covering homotopy theorem it follows that the fixed point of g are pairwise inequivalent.The same is true for all iterates g^n.Therefore $N(g^n) = NF_n(g) = \#Fix(g^n)$ for all n.So the sequence of the Nielsen and Nielsen type numbers $N(g^n) = NF_n(g)$ of an expanding map satisfies the Dold congruences.
For a hyperbolic endomorphism of torus T^n and for a pseudo-Anosov homeomorphism of a compact surface we also have equalities $N(g^n) = NF_n(g) = \#Fix(g^n)$ for all n (see [13] and [37]), and ,therefore, the Dold congruences for the sequence of the Nielsen and Nielsen type numbers .

8.4 Open questions

Question 1 *For which spases and maps in the inequalities (33) - (34) does the equality hold ?*

Sometime, the minimal zeta function coincide with the Nielsen zeta function,for example, for hyperbolic endomorphism of torus, for an expanding map of the orientable smoth manifold (see [13]).This motivate the following

Question 2 *For which spaces and maps minimal zeta function is a rational function,when it is a meromorphic function?When does it have a functional equation?Which zeros and poles it has?*

Question 3 *For which spases and maps we have trace formula for Nielsen type numbers? When we have the Dold congruences for this numbers?When in the inequalities (37)-(38) does equalities hold?*

The trace formulae which we recieve in this article appear to be very similar to formulae arising in thermodynamical formalism. The relation of the radius of convergence with the entropy (Theorem 24), Markov partition in the pseudo-Anosov case(Theorem 22) and the relation of the Reidemeister zeta function to Artin-Mazur zeta function on the unitary dual space(lemma 10) also indicate a connection with this theory. It is another open question to understand this connection.

References

[1] D. Anosov, The Nielsen number of maps of nilmanifolds, *Russian Math. Surveys* **40** (1985), 149-150.

[2] M. Artin and B. Mazur, On periodic points, Annals of Math., 81 (1965), 82-99.

[3] I. Babenko, S. Bogatyi, Private communication.

[4] J. S. Birman, Braids, links and mapping class groups, Ann. Math. Studies vol. 82, Princeton Univ. Press, Princeton, 1974.

[5] R. Bowen and O. Lanford, Zeta functions of restrictions of the shift transformation, Proc. Global Anal., 1968, 43-49.

[6] A. Dold, Fixed point indices of iterated maps, Inventiones Math. 74 (1983), 419-435.

[7] C. Epstein, The spectral theory of geometrically periodic hyperbolic 3-manifolds, Memoirs of the AMS, vol. 58, number 335, 1985.

[8] E. Fadell, S. Husseini, The Nielsen number on surfaces, Topological methods in nonlinear functional analysis, Contemp. Math. vol. 21, AMS, Providence, 1983, 59-98.

[9] A. Fathi and M. Shub, Some dynamics of pseudo-Anosov diffeomorphisms, Asterisque 66-67 (1979), 181-207.

[10] A. L. Fel'shtyn, New zeta function in dynamics, in Tenth Internat. Conf. on Nonlinear Oscillations, Varna, Abstracts of Papers, Bulgar. Acad. Sci., 1984, 208

[11] -, A new zeta function in Nielsen theory and the universal product formula for dynamic zeta functions, Functsional Anal. i Prilozhen 21 (2) (1987), 90-91 (in Russian); English transl.: Functional Anal. Appl. 21 (1987), 168-170.

[12] -, Zeta functions in Nielsen theory, Functsional Anal. i Prilozhen 22 (1) (1988), 87-88 (in Russian); English transl.: Functional Anal. Appl. 22 (1988), 76-77.

[13] -, New zeta functions for dynamical systems and Nielsen fixed point theory, in : Lecture Notes in Math. 1346, Springer, 1988, 33-55.

[14] -, The Reidemeister zeta function and the computation of the Nielsen zeta function, Colloquium Mathematicum 62 (1) (1991), 153-166.

[15] A. L. Fel'shtyn and R. Hill, Dynamical zeta functions, Nielsen theory and Reidemeister torsion, Nielsen Theory and Dynamical Systems, Contemporary Mathematics 152 (1993), 43-69.

[16] A. L. Fel'shtyn and R. Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, K-theory, to appear,

[17] A. L. Fel'shtyn, R. Hill, P. Wong, Reidemeister numbers of equivariant maps, Preprint IHES, December 1993.

[18] A. L. Fel'shtyn and V. B. Pilyugina , The Nielsen zeta function, Funktsional. Anal. i Prilozhen. 19 (4) (1985), 61-67 (in Russian); English transl.: Functional Anal. Appl. 19 (1985), 300-305.

[19] J. Franks, Homology and dynamical systems, Regional Conf. Ser. Math, 49 (1982), 1-120.

[20] D. Fried, Periodic points and twisted coefficients, Lect. Notes in Math., 1007 (1983), 261-293.

[21] D. Fried, Lefschetz formula for flows, The Lefschetz centennial conference, Contemp. Math., 58 (1987), 19-69.

[22] M. Gromov, Groups of polynomial growth and expanding maps, Publicationes Mathematiques, 53, 1981, 53-78.

[23] M. Gromov, Hyperbolic groups, Essays in Group theory, Mathematical Sciences Research Institute Publications, 8, 1987, 75-265.

[24] P. R. Heath, Product formulae for Nielsen numbers of fibre maps, Pacific J. Math. 117 (2) (1985), 267-289.

[25] R. Hill, Some new results on Reidemeister numbers from group theoretical point of view, Preprint, 1993.

[26] B. Jiang, Nielsen Fixed Point Theory, Contemp. Math. 14, Birkhäuser, 1983.

[27] B. Jiang, Estimation of the number of periodic orbits, Preprint of Universität Heidelberg, Mathematisches Institut, Heft 65, Mai 1993.

[28] B. Jiang, S. Wang, Lefschetz numbers and Nielsen numbers for homeomorphisms on aspherical manifolds, Topology - Hawaii, World Scientific , Singapore (1992), 119-136.

[29] A. Mal'cev, On a class of homogeneous spaces, *Izvestiya Akademii Nauk SSSR. Seriya Matematičeskaya* **13** (1949), 9-32.

[30] A. Manning, Axiom A diffeomorphisms have rational zeta function, Bull. London Math. Soc. 3 (1971), 215-220.

[31] B. Norton-Odenthal, Ph. D Thesis, University of Wisconsin, Madison, 1991.

[32] W. Parry and M. Pollicot, Zeta functions and the periodic structure of hyperbolic dynamics, Asterisque, vol.187-188, 1990

[33] K. Reidemeister, Automorphism von Homotopiekettenringen, Math. Ann. 112 (1936), 586-593.

[34] W. Rudin : Fourier Analysis on Groups, Interscience tracts in pure and applied mathematics number 12, 1962.

[35] D. Ruelle, Zeta function for expanding maps and Anosov flows, Invent. Math, 34 (1976), 231- 242.

[36] M. Shub, Endomorphisms of compact differentiable manifolds, Amer. J. Math. 91 (1969), 175-179.

[37] W. P.Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. AMS 19 (1988), 417-431.

[38] F.Wecken, Fixpunktklassen, II, Math. Ann. 118 (1942), 216-234.

[39] A. Weil, Numbers of solutions of equations in finite fields, ibid. 55 (1949), 497-508.

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