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Injectivity of Local Diffeomorphisms from Nearly Spectral Conditions*

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INTRODUCTION

The 2-dimensional global asymptotic stability conjecture states: If f is a C^1 vector field on $\mathbb{R}^2$ with a singularity at o and the eigenvalues of $Df(x)$ have negative real parts for every $x \in \mathbb{R}^2$, then the stable manifold of o is $\mathbb{R}^2$.

The affirmative solution of this conjecture was independently arrived at by Fessler [4], Glutsyuk [5] and Gutierrez [6], the exact chronology being unknown to us. Moreover, Bernat and Llibre [2] have given an analytic counterexample to this conjecture of Markus–Yamabe in dimension 4 (and greater); their example has a bounded (periodic) orbit which does not tend to zero with time. Recently Anna Cima, Arno van den Essen, Armengol Gasull, Engelbert Hubbers and Francesco Mañosas have found extremely simple polynomial vector fields in dimension 3 and 4 (and, of course, higher), which are counterexamples to the Markus–Yamabe Conjecture; their polynomial examples do not have *periodic* orbits but rather orbits that tend to infinity with time. It is now an open question whether this is the general situation: Do bounded orbits of *polynomial* vector fields satisfying the Markus–Yamabe hypothesis necessarily tend to zero as $t \to \infty$, while unbounded orbits tend to infinity? What does the global stability hypothesis imply for polynomial, analytic and other vector fields? We do not concern ourselves further in this paper with the global asymptotic stability conjecture. In fact our results have no overlap with the aforementioned papers and our technique is new.

Gutierrez's paper [6] interested us especially for the following very strong injectivity theorem which it contains:

"A C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is injective if $[0, \infty) \cap \operatorname{Spec}(Df(x)) = \varnothing$ for all $x \in \mathbb{R}^2$."

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Here Spec A denotes the set of eigenvalues of the linear operator A. This result and a theorem of Olech [7] are the essential ingredients of the solution given by Gutierrez [6].

The aim of this paper is three-fold. We establish

(i) An injectivity theorem for local diffeomorphisms defined on $\mathbb{R}^n$, $n \geqslant 2$, with "nearly" spectral hypotheses (Theorem 1).

(ii) The existence of examples showing that the above injectivity result of Gutierrez fails spectacularly in high dimension (Theorem 4).

(iii) An injectivity theorem with spectral hypotheses for local diffeomorphisms defined on disks in $\mathbb{R}^2$ (Theorem 3); this result neither implies nor is implied by the injectivity result of Gutierrez.

The idea of our method can be roughly described as follows. In order to prove that a local diffeomorphism $f: \mathbb{R}^n \to \mathbb{R}^n$ is injective, we construct a new map $\psi: \mathbb{R}^n \to \mathbb{R}^n$ which is derived from f by a construction similar to the exponential map in Riemannian geometry. It is then shown that f is injective if ψ is surjective. One can think of ψ as a kind of "non-linear adjoint" of f. The advantage of our method is that injectivity questions are reduced to surjectivity questions and for the latter one can use the many tools of non-linear analysis. This approach has strong points of contact with algebraic geometry and also with geometries of negative curvature [10].

We were already working on injectivity questions when we heard from Gutierrez of his result. Our attention to references [2, 4–5], and to on-going work in the area was drawn by the referee, and we thank him wholeheartedly.

1. STATEMENT OF THE RESULTS

To fix notation, $S^{n-1} = \{v \in \mathbb{R}^n, |v| = 1\}$ stands for the unit sphere in $\mathbb{R}^n$ and, when $v \in S^{n-1}$, H_v stands for any slab region bounded by two hyperplanes orthogonal to v.

The next result is proved using degree theory.

THEOREM 1. *A C^1 map $f: \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 2$, is injective if*

(i) $[0, \infty) \cap \operatorname{Spec}(Df(x)) = \varnothing$ *for all* $x \in \mathbb{R}^n$ *and*

(ii) *for each* $v \in S^{n-1}$ *and each* v-slab H_v,

$$\int_0^\infty \left(1 - \left\langle v, \frac{Df(x(s))^* v}{|Df(x(s))^* v|} \right\rangle\right)^{1/2} ds = \infty$$

for each unit speed curve $x\colon [0, \infty) \to \mathbb{R}^n$ *with image in* H_v *that satisfies* $\lim_{s \to \infty} |x(s)| = \infty$.

The integrand in (ii) involving $Df(x)^*$ (the adjoint of the linearization $Df(x)$) is always non-negative and is positive precisely because of (i). This of course tends to make the integral infinite. Hence the result says that a local diffeomorphism is injective if the spectra of the operators $Df(x)^*$ miss $[0, \infty)$ both pointwise and in the asymptotic sense of the theorem. In this way the hypotheses of the theorem are "nearly" spectral.

As a consequence we obtain at once

THEOREM 2. *A* C^1 *local diffeomorphism* $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geqslant 2$, *is injective if for each* $v \in S^{n-1}$ *and each* v-slab H_v

$$\sup_{x \in H_v} \left\langle v, \frac{Df(x)^* v}{|Df(x)^* v|} \right\rangle < 1.$$

Theorem 2 greatly improves the well-known sufficient condition for injectivity: $\langle Df(x)\, v, v \rangle < 0$ for all $v \in S^{n-1}$ and all $x \in \mathbb{R}^n$.

By a variation of the argument of Theorem 2 we can prove

THEOREM 3. *Let* $B(r)$ *be the open disk of radius* r *in* $\mathbb{R}^2$ *and* $f\colon B(r) \to \mathbb{R}^2$ *a* C^1 *map. If*

(i) $[0, \infty) \cap \operatorname{Spec}(Df(x)) = \varnothing$ *for all* $x \in B(r)$ *and*

(ii) $\mathbb{R} \cap \operatorname{Spec}(Df(x)) = \varnothing$ *for all* $x \notin B(r/\sqrt{2})$,

then f *is injective on* $B(r/\sqrt{2})$.

Finally, using results from algebraic geometry we can construct examples showing that Gutierrez's injectivity result fails dramatically in high dimension.

THEOREM 4. *There exist integers* $n > 2$ *and non-injective polynomial maps* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *with* $[0, \infty) \cap \operatorname{Spec}(Df(x)) = \varnothing$, $\forall x \in \mathbb{R}^n$.

In fact, it is possible to construct f as in Theorem 4 quite explicitly. The resulting integer n will be large.

2. PROOFS OF THEOREMS 1 AND 3

We first prove these results under the assumption that f is smooth. This restriction will be removed at the end of this section.

For $v \in S^{n-1} \subset \mathbb{R}^n$, we define the vector field G_v on $\mathbb{R}^n$ by

$$G_v(x) = v - \frac{Df(x)^* v}{|Df(x)^* v|}.$$

Consider also the following property:

(P) For every $a, b \in \mathbb{R}^n$ there exists $v \in S^{n-1}$ such that the G_v-trajectory through a contains b.

It is easy to see that if (P) holds then f is injective. Indeed, suppose that $f(a) = f(b)$ with $a \neq b$ and let v be as in the statement of (P), with corresponding G_v-trajectory $\phi_v(t)$, where $\phi_v(0) = a$ and $\phi_v(t_0) = b$. The derivative of $\langle f(\phi_v(t)), v\rangle$ with respect to t may be written, with $x = \phi_v(t)$, as

$$\langle G_v(x), Df(x)^* v\rangle = |Df(x)^* v| \left(-1 + \left\langle v, \frac{Df(x)^* v}{|Df(x)^* v|} \right\rangle \right).$$

The last expression is manifestly non-positive and, since $Df(x)$ has no positive eigenvalues, it is in fact strictly negative. Integration from $t = 0$ to $t = t_0$ gives $\langle f(b) - f(a), v\rangle < 0$, a contradiction to $f(b) = f(a)$.

We now proceed to show how property (P) is implied by the hypotheses of either Theorem 1 or Theorem 3.

Proof of Theorem 1. Noting that hypothesis (i) implies that $(1 - \langle v, Df(x)^* v/|Df(x)^* v|\rangle) \neq 0$, we can define the vector field $\bar{G}_v$ by

$$\bar{G}_v(x) = \left(1 - \left\langle v, \frac{Df(x)^* v}{|Df(x)^* v|} \right\rangle \right)^{-1} G_v(x).$$

We claim that $\bar{G}_v$ is a *complete* vector field, that is, its trajectories are defined for all times. Suppose, by contradiction, that $x(t)$, $0 \leqslant t < \tau < \infty$, is a maximal trajectory of $\bar{G}_v$ and let s denote its arc-length. Then $\tau = \int_0^\tau dt$ equals

$$\int_0^\infty \frac{dt}{ds}\, ds = \int_0^\infty \frac{1}{|\bar{G}_v(x(s))|}\, ds = \frac{1}{\sqrt{2}} \int_0^\infty \left(1 - \left\langle v, \frac{Df(x(s))^* v}{|Df(x(s))^* v|} \right\rangle \right)^{1/2} ds,$$

thus showing that the last integral is finite. Integrating the expression $(d/dt)\langle x(t), v\rangle = \langle \bar{G}_v(x), v\rangle = 1$ between 0 and τ we see that $x([0, \tau))$ lies in a v-slab, a contradiction to (ii) in the statement of the theorem. Hence the vector field $\bar{G}_v$ is complete. f being sufficiently differentiable, we can now define a continuous map $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ by $\psi(o) = a$ and $\psi(w) = \phi_{w/|w|}(|w|)$, if $w \neq o$; in this part of the argument ϕ_v stands for the integral curve of $\bar{G}_v$ passing through a at time zero. Clearly, property (P) will follow if the surjectivity of ψ can be established. This will be accomplished by showing that the map ψ is *properly* homotopic to the identity map. The desired conclusion then follows by the invariance of the topological degree under proper homotopies. Define $H\colon [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ by $H(s, w) = (1-s)\,\psi(w) + sw$. Integration of $\langle \bar{G}_v(x), v \rangle = 1$ gives $\langle \phi_v(t), v \rangle = \langle \phi_v(0), v \rangle + t$. In particular, $\langle \psi(w), w/|w| \rangle = \langle a, w/|w| \rangle + |w|$. From this we can easily obtain $\langle H(s, w), w/|w| \rangle \geqslant |w| - |a|$. It follows from Schwarz's inequality that H is proper, that is, $|H(s, w)| \to \infty$ as $|w| \to \infty$, uniformly in $s \in [0, 1]$. This concludes the proof of Theorem 2 under the assumption that f is smooth.

Proof of Theorem 3. The proof of Theorem 3 proceeds along similar lines. Again we must show (P) holds for each pair of distinct points $a, b \in B((r-\varepsilon)/\sqrt{2})$ with $\varepsilon > 0$ so small that (ii) in the statement of the theorem holds on the complement of this ball. For this part of the argument we modify the vector fields G_v used in the proof above by taking

$$\bar{G}_v(x) = \rho(x)\,\frac{G_v(x)}{|G_v(x)|},$$

where ρ is a smooth non-negative function on $B(r)$ which equals 1 on $B(r-\varepsilon)$ and whose support is a disk within $B(r)$. The integral curves $\phi_v(t)$ (passing through a at time $t=0$) of these new fields $\bar{G}_v$ are defined for all $t \geqslant 0$. If for some v we have $\phi_v(t_0) = b$, it follows as in the previous proof that $\langle f(b) - f(a), v \rangle < 0$, so that f cannot identify a and b.

By hypothesis (i) of the theorem we have $\langle \bar{G}_v(x), v \rangle \geqslant 0$ for all x and $\langle \bar{G}_v(x), v \rangle > 0$ on $\overline{B(r-\varepsilon)}$. If $k(\varepsilon) > 0$ denotes the infimum of these latter values on $\overline{B(r-\varepsilon)}$ over all $v \in S^1$, then $(d/dt)\langle \phi_v(t), v \rangle \geqslant k(\varepsilon)$ so long as $\phi_v(t) \in \overline{B(r-\varepsilon)}$. We see in this way that each trajectory $\phi_v(t)$ exits $\overline{B(r-\varepsilon)}$ at or before time $\tau(\varepsilon) = 2(r-\varepsilon)/k(\varepsilon)$.

Let J denote the complex structure of the plane. By hypothesis (ii) the expression $\langle \bar{G}_v(x), Jv \rangle$ has a fixed sign, independent of v, on the set $K = \overline{B(r-\varepsilon)} - B((r-\varepsilon)/\sqrt{2})$; without loss of generality we may assume this sign is positive.

For each $v \in S^1$ we let Σ_v denote the square tangent to $B((r-\varepsilon)/\sqrt{2})$ with sides parallel to v and Jv; this square lies in K. Suppose for some $v \in S^1$ the trajectory $\phi_v(t)$ (which we know exits $\overline{B(r-\varepsilon)}$) re-enters $B((r-\varepsilon)/\sqrt{2})$ (and therefore Σ_v). Since $\langle \bar{G}_v(x), Jv \rangle > 0$ on K, the trajectory $\phi_v(t)$ can only leave Σ_v through the "top" edge of Σ_v (which is parallel to v) and cannot re-enter the square Σ_v through this edge; since $\langle \bar{G}_v(x), v \rangle \geqslant 0$ on $B(r)$, this trajectory never re-enters Σ_v. In short, once outside $\overline{B(r-\varepsilon)}$ the trajectory will never return to $B((r-\varepsilon)/\sqrt{2})$.

Letting Ω denote the closed disk of radius $\tau(\varepsilon)$ (defined above) with centre $o \in \mathbb{R}^2$, we define a (continuous) map $\psi: \Omega \to \mathbb{R}^2$ by $\psi(o) = a$ and $\psi(w) = \phi_{\omega/|\omega|}(|\omega|)$ if $\omega \neq o$. The previous paragraph shows that $\psi(\partial\Omega) \cap B((r-\varepsilon)/\sqrt{2}) = \emptyset$. Since $\langle \phi_v(t), v \rangle > 0$ near o we see that a is assumed precisely once by ψ so that its index with respect to the curve $\psi \mid \partial\Omega$ is non-zero; since the line segment ab does not meet $\psi(\partial\Omega)$, from the previous sentence it follows that b must also have non-zero index with respect to this curve. If $b \notin \psi(\Omega)$ then it follows from the obvious homotopy $\psi_s = \psi\,|_{|z|=s}$ for $0 \leqslant s \leqslant \tau(\varepsilon)$ that b has the same (non-zero) index with respect to each of the curves ψ_s; since ψ_0 is the constant curve a, the point b has index zero relative to ψ_0. This contradiction means $\psi(\omega) = b$ for some $\omega \in \Omega$, i.e. $\phi_v(t) = b$ for some $v \in S^1$ and $t > 0$. By property (P) we now have a contradiction.

Finally we remark that in the proofs given above of Theorems 1 and 3 we needed f to be sufficiently smooth to guarantee that the map ψ is well-defined. We now show how to drop the smoothness assumption in Theorem 1. Similar remarks apply to Theorem 3. Suppose f, as in Theorem 1, is only C^1. We can approximate f in the Whitney topology of $C^1(\mathbb{R}^n, \mathbb{R}^n)$ by smooth maps f_n verifying the hypothesis and such that f_n converges to f uniformly on compact sets. Our previous work then implies that f_n is injective. To conclude that f is also injective we use some simple facts about the topological degree. Suppose $f(p_1) = f(p_2) = q$ with $p_1 \neq p_2$. Since f is a local C^1 diffeomorphism, there exist disjoint bounded neighborhoods U_i of p_i $(i = 1, 2)$ which are mapped homeomorphically onto a neighborhood V of q. In particular, $\deg(f, U_i, q)$ is either $+1$ or -1, according to whether f preserves or reverses orientation. Since the restriction of f_n to $\overline{U_i}$ converges uniformly to the restriction of f to $\overline{U_i}$, we have $\deg(f_n, U_i, q) = 1$ or -1 for all sufficiently large n. But this contradicts the injectivity of f_n. This concludes the proofs of both theorems.

As remarked in the introduction, Theorem 3 neither implies nor is implied by Gutierrez's injectivity result. We record here a special case of his result that follows from the proof of Theorem 3. For the moment this is as close as our method brings us to Gutierrez's injectivity result.

THEOREM 5. *A C^1 map $f: \mathbb{R}^2 \to \mathbb{R}^2$ is injective if, for every $x \in \mathbb{R}^2$, $[0, \infty) \cap \operatorname{Spec}(Df(x)) = \emptyset$ and in each strip the set of all points y where $(-\infty, 0) \cap \operatorname{Spec}(Df(y)) \neq \emptyset$ is bounded.*

We refer to [10] for a formulation of the injectivity problem as a transversality phenomenon in geometries of negative curvature.

3. PERTURBATIONS OF THE IDENTITY BY HOMOGENEOUS MAPS AND THE PROOF OF THEOREM 4

Let $H\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 homogeneous map of degree $r > 1$, i.e. $H(\tau x) = \tau^r H(x)$ for all $\tau > 0$ and $x \in \mathbb{R}^n$. Then $F = I - H$, where I is the identity map, is a perturbation of the identity by a homogeneous map. While, at first sight, such maps appear rather special they are important artifacts of all polynomial mappings. This is so because of the result in algebraic geometry due to Bass–Connell–Wright [1] and Druzkowski [3] cited as Theorem 7 below.

From Euler's formula $DH(x)x = rH(x)$ we see that F satisfies the functional equation

$$DF(x)x = rF(x) + (1 - r)x. \tag{1}$$

Suppose furthermore that

$$F \text{ is a local diffeomorphism.} \tag{2}$$

It is an easy consequence of homogeneity that (2) is equivalent to

$$(0, \infty) \cap \operatorname{Spec}(DH(x)) = \varnothing \qquad \forall x \neq 0. \tag{2$'$}$$

Setting $F(x) = o$ in (1) gives $DH(x)x = rx$, which contradicts (2′) unless $x = o$. This shows that o is covered once by F. This observation hints at the possibility that, under fairly general conditions, a (locally invertible) perturbation of the identity by a homogeneous map is injective (see also [9]).

Before stating such a theorem we remark that the equivalent conditions (2) and (2′) imply local injectivity of F. Since $\operatorname{Spec} A = \operatorname{Spec} A^*$, we have, where H_v denotes a v-slab,

$$v \notin \{DH(x)^* v \colon x \in H_v\} \qquad \forall v \in S^{n-1}, \forall H_v \rightrightarrows F \text{ is loc. invertible.}$$

This should be compared to the hypothesis of the next theorem, in which $\operatorname{cl} A$ stands for the closure of a set A and H_v stands for a v-slab.

Theorem 6. *Let $H\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 homogeneous map of degree $r > 1$. Then*

$$v \notin \operatorname{cl} \{DH(x)^* v \colon x \in H_v\} \qquad \forall v \in S^{n-1}, \forall H_v \rightrightarrows F \textit{ is invertible.}$$

Proof. We will show that $f = -F = -I + H$ satisfies the hypothesis of Theorem 2 and so is injective. Fix $v \in S^{n-1}$. Since the hypothesis implies (2′), it is obvious that $Df(x)^* v \neq 0$. Then the quantity $\langle v, Df(x)^* v / |Df(x)^* v| \rangle$ is the cosine of the angle formed by v and

$Df(x)^* v = -v + DH(x)^* v$. If this quantity is positive (i.e. angle $< \pi/2$) then $\langle DH(x)^* v, v \rangle > 0$ and from the previous equation we have $\angle(v, DH(x)^* v) < \angle(v, Df(x)^* v)$, as a simple picture shows; by the hypothesis we know that for each slab H_v there exists a constant $\delta \in (0, \pi/2)$ such that $\angle(v, DH(x)^* v) > \delta$ for all $x \in H_v$. Hence $\angle(v, Df(x)^* v) > \delta > 0$ for all $x \in H_v$. We have then shown that for each $v \in S^{n-1}$,

$$\sup_{x \in H_v} \left\langle v, \frac{Df(x)^* v}{|Df(x)^* v|} \right\rangle < 1$$

and the result follows from Theorem 2.

Next we record the following result from algebraic geometry due to Bass–Connell–Wright [1] and Druzkowski [3].

THEOREM 7 [1], [3, p. 305]. *Let $p\colon \mathbb{R}^m \to \mathbb{R}^m$ be a polynomial map. Then it is possible to effectively construct another polynomial map $F\colon \mathbb{R}^n \to \mathbb{R}^n$ (for some $n = n(p) > m$) with the following properties*

(i) *F is a perturbation of the identity by a homogeneous polynomial of degree $\leqslant 3$.*

(ii) *p is a local diffeomorphism if and only if F is*; $\det Dp \equiv 1$ *if and only if* $\det DF \equiv 1$.

(iii) *p is injective if and only if F is.*

Theorem 7 is often referred to in the literature as a *reduction* theorem. To understand the terminology we recall the Jacobian Conjecture in algebraic geometry. It states that every polynomial map $p\colon \mathbb{R}^n \to \mathbb{R}^n$ with constant non-zero jacobian determinant must be injective (the conjecture is usually stated for polynomial maps $p\colon \mathbb{C}^n \to \mathbb{C}^n$). Theorem 7 *reduces* the proof of this conjecture to establishing injectivity for those polynomial maps of constant non-zero jacobian determinant that are perturbations of the identity by homogeneous maps of degree 3, at the cost of increasing the dimension.

It was even believed that any polynomial map $p\colon \mathbb{R}^n \to \mathbb{R}^n$ with everywhere non-zero jacobian determinant (not necessarily constant) would be injective. A counterexample was recently constructed by Pinchuk [8].

THEOREM 8 [8]. *There exists a non-injective polynomial local diffeomorphism $p\colon \mathbb{R}^2 \to \mathbb{R}^2$.*

Proof of Theorem 4. Let $p\colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in Theorem 8 and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ the associated map given by Theorem 7. Then F is a non-injective local diffeomorphism of the form $F = I - H$ where H is a homogeneous map. We claim that the (non-injective) map $f = -F$ also satisfies the conclusion of Theorem 4. Indeed, if $Df(x)v = \lambda v$ with $\lambda \geqslant 0$ for some $v \in S^{n-1}$, then $DH(x)v = (\lambda + 1)v$, in contradiction to the fact that (2′) and (2) are equivalent. This concludes the proof of Theorem 4. We also observe that this example f shows that the closure in the hypothesis in Theorem 6 cannot be dropped.

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