Existence and Multiplicity Results for a non-Homogeneous Fourth Order Equation

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Abstract In this paper we investigate the problem of existence and multiplicity of solutions for a non-homogeneous fourth order Yamabe type equation. We exhibit a family of solutions concentrating at two points, provided the domain contains one hole and we give a multiplicity result if the domain has multiple holes. Also we prove a multiplicity result for vanishing positive solutions in a general domain.

1 Introduction and statements of the main results

In this paper we will study the existence and the multiplicity of positive solutions for a non-homogeneous problem of the form:

$$\begin{cases} \Delta^2 u = |u|^{p-1} u + f & \text{on} \quad \Omega \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{P}$$

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where Ω is a smooth bounded set of $\mathbb{R}^n$ and $p = \frac{n+4}{n-4}$ is the so-called critical exponent. These kind of problems were deeply studied in the case of the Laplacian (see for instance [1],[11], [19]). Let us recall that problem (P) was studied by Selmi [26] and Ben Ayed - Selmi [9] where the authors prove the existence of a one-bubble solution to the problem under assumptions on f. Here we will show that we can get two-bubble solutions if the domain contains small holes, and vanishing type solutions for a small generic perturbation f in the C^0 sense.

We recall that for $f = 0$, this problem has a deep geometrical meaning, in fact if (M, g) is an $n-$dimensional compact closed riemannian manifold with $n \geq 5$, we can define the $Q-$curvature

$$Q := \frac{n^3 - 4n^2 + 16n - 16}{8\,(n-2)^2\,(n-1)^2} R^2 - \frac{2}{(n-2)^2} \left|Ric\right|^2 + \frac{1}{2(n-1)} \Delta R,$$

where R is the scalar curvature and Ric is the Ricci curvature. After a conformal change of the metric one gets for $\widetilde{g} = u^{\frac{4}{n-4}} g$,

$$Q_{\widetilde{g}} u^{\frac{n+4}{n-4}} = P_g u, \tag{1}$$

where P_g is the Paneitz operator, defined by

$$P_g u := \Delta_g^2 u - div\left(\left(\frac{(n-2)^2+4}{2\,(n-2)\,(n-1)} Rg - \frac{4}{n-2} Ric\right) du\right) + \frac{n-4}{2} Qu.$$

This gives rise to the problem of prescribing the $Q-$curvature, as the analogous problem on the scalar curvature (see [12], [13] and [23]). We remark that in the flat case, for instance if we consider an open set of $\mathbb{R}^n$, the problem of prescribing constant Q-curvature coincides with (P) with $f = 0$, namely

$$\Delta^2 u = \left|u\right|^{p-1} u. \tag{2}$$

The variational formulation of (2) under Navier boundary conditions in a bounded set was deeply studied, especially with the methods of critical points at infinity theory, introduced by Bahri [3] (see [13], [18] and [17]). We also remark the fact that this problem is not compact, namely, for the case $f = 0$ it corresponds exactly to the limiting case of the Sobolev embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-4}}$, (see [27]), and thus we loose the compact embedding, so the variational setting in the classical spaces fails to show existence of solutions: in fact as in the case of the Laplacian, if the domain is star shaped we know that it has no positive solutions ([27], [28]). Finally we recall that in the recent paper [22], we studied the same Yamabe type problem, with a slightly super-critical exponent.

This work contains two main parts. In the first one we deal with a perturbation of the form εf, that is

$$\begin{cases} \Delta^2 u & = & |u|^{p-1} u + \varepsilon f & \text{on} & \Omega \\ u = \Delta u & = & 0 & \text{on} & \partial\Omega \end{cases}, \qquad (P_\varepsilon)$$

where f is a positive function in $C^\alpha(\Omega)$, $0 < \alpha < 1$, and $\Omega = \mathcal{D} - \overline{B(P, \mu)}$, for a given domain $\mathcal{D}$ and $P \in \mathcal{D}$. In this setting we have the following result:

Theorem 1.1. *There exists a constant $\mu_0 = \mu_0(\mathcal{D}, f) > 0$ such that for each $0 < \mu < \mu_0$ fixed, there exist $\varepsilon_0 > 0$ and a family of solutions u_ε of (P_ε) for $0 < \varepsilon < \varepsilon_0$, having exactly two concentration points, namely:*

$$u_\varepsilon(x) = c_n \left(\frac{\varepsilon^{\frac{2}{n-4}} \lambda_{1,\varepsilon}}{\varepsilon^{\frac{4}{n-4}} \lambda_{1,\varepsilon}^2 + |x - \xi_1^\varepsilon|^2} \right)^{\frac{n-4}{2}} + c_n \left(\frac{\varepsilon^{\frac{2}{n-4}} \lambda_{2,\varepsilon}}{\varepsilon^{\frac{4}{n-4}} \lambda_{2,\varepsilon}^2 + |x - \xi_2^\varepsilon|^2} \right)^{\frac{n-4}{2}} + \theta_\varepsilon(x)$$

and $\theta_\varepsilon(x) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ uniformly.

Indeed one gets more information about the solutions along the proof, for instance we will see that $\theta_{\varepsilon}(x) = \varepsilon w + o(\varepsilon)$, where w is the solution of:

$$\begin{cases} \Delta^2 w & = & f & \text{on} & \Omega \\ w = \Delta w & = & 0 & \text{on} & \partial\Omega \end{cases}.$$

And within the proof we have that the point $\left((\xi_1^{\varepsilon}, \xi_2^{\varepsilon}), (a_n(\lambda_1^{\varepsilon})^{n-4}, a_n(\lambda_2^{\varepsilon}))^{n-4}\right)$ is a critical point of the function Ψ defined by :

$$\Psi(\xi, \Lambda) = \frac{1}{2}\left(\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right) + \sum_{i=1}^{2} \Lambda_i w(\xi_i),$$

where G is the Green's function of the Ω and H its regular part.

Moreover if we consider a domain with multiple holes we obtain a multiplicity result. In fact, if $\Omega = \mathcal{D} - \cup_{1 \le i \le k} \overline{B}(P_i, \mu)$ with $P_1, ..., P_k \in \Omega$, the previous result can be generalized as in [14] and [22] to the following:

Theorem 1.2. *Let $1 \le m \le k$. There exists a constant $\mu_0 = \mu_0(\mathcal{D}, f) > 0$ such that for each $0 < \mu < \mu_0$ fixed, there exist $\varepsilon_0 > 0$ and a family of solutions u_{ε} of (P_{ε}) for $0 < \varepsilon < \varepsilon_0$, of the following form*

$$u_{\varepsilon}(x) = c_n \sum_{i=1}^{k} \sum_{j=1}^{2} \left(\frac{\varepsilon^{\frac{2}{n-4}} \lambda_{i,j,\varepsilon}}{\varepsilon^{\frac{4}{n-4}} \lambda_{i,j,\varepsilon}^2 + \left| x - \xi_{i,j}^{\varepsilon} \right|^2} \right)^{\frac{n-4}{2}} + \theta_{\varepsilon}(x)$$

and $\theta_{\varepsilon}(x) \longrightarrow 0$ *as* $\varepsilon \longrightarrow 0$ *uniformly.*

In particular for a domain with k holes we have at least $2^k - 1$ two-bubble solutions.

In the second part of the paper we deal with the problem

$$\begin{cases} \Delta^2 u & = & |u|^{p-1} u + f & \text{on} & \Omega \\ u = \Delta u & = & 0 & \text{on} & \partial\Omega \end{cases}, \qquad (P_f)$$

with no topological constraint on the domain Ω and $f \ge 0$ non identically zero. We prove the following:

Theorem 1.3. *There exist a residual subset $D \subset C^2\left(\overline{\Omega}\right)$ and $\varepsilon > 0$, such that if $f \in D$ and $|f|_{C(\bar{\Omega})} < \varepsilon$, the problem ($P_f$) has at least* $\sum_{i=0}^{\infty} \dim H_i(\Omega) + 1$ *positive solutions.*

Here $H_*(\Omega)$ denotes the singular homology of Ω. We have additional information for these solutions as well. In fact we will see that they vanish when $|f|_{C(\bar{\Omega})} \longrightarrow 0$, and they have energy smaller than the energy of a single bubble; in contrast with the solutions of the first theorem, where the energy of the solutions is greater than the one of the bubbles, and the solutions blow-up as $\varepsilon \longrightarrow 0$.

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2 Preliminaries and first estimates

Let us start by defining the following functions:

$$\overline{U}_{(\xi,\lambda)}(x) = \left(\frac{\lambda}{\lambda^2 + |x-\xi|^2}\right)^{\frac{n-4}{2}},$$

where $\lambda > 0$ and $\xi \in \Omega$. For $u \in D^2(\Omega)$, we will write Pu for the projection of u on $H^2(\Omega) \cap H_0^1(\Omega)$, defined as the unique solution of the problem

$$\begin{cases} \Delta^2 v = u & \text{on} \quad \Omega \\ v = \Delta v = 0 & \text{on} \quad \partial\Omega \end{cases},$$

We also recall that the Green's function of Δ^2 for a set Ω, with Navier boundary conditions is defined as the solution of

$$\begin{cases} \Delta_x^2 G(x,y) & = & \delta_y & \text{on} & \Omega \\ G(x,y) = \Delta_x G(x,y) & = & 0 & \text{on} & \partial\Omega \end{cases}.$$

This function can be written as

$$G(x,y) = \frac{a_n}{|x-y|^{n-4}} - H(x,y),\ \forall x,y \in \Omega \text{ and } x \neq y,$$

where a_n is a positive constant depending on n and H the positive smooth solution to

$$\begin{cases} \Delta_x^2 H(x,y) & = & 0 & \text{on} & \Omega \\ H(x,y) & = \frac{1}{|x-y|^{n-4}}, & \Delta H(x,y) = \Delta\frac{1}{|x-y|^{n-4}} & \text{on} & \partial\Omega \end{cases}.$$

Now let ξ_1, ξ_2 be two points in Ω, and $\lambda_1, \lambda_2 > 0$, we will write $\overline{U}_i = \overline{U}_{(\xi_i,\lambda_i)}$ and $U_i = P\overline{U}_i$. Then one has $U_i = \overline{U}_i - \theta_i$ and

$$\theta_i(x) = H(x,\xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy + o(\lambda_i^{\frac{n-4}{2}}).$$

Away from $x = \xi$, we have

$$U_i(x) = G(x,\xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy + o(\lambda_i^{\frac{n-4}{2}}).$$

For more details about these estimates we refer to the Appendix.

Let us set now J to be the functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u|^p,$$

and let us find an expansion of

$$J(U_1+U_2) = \frac{1}{2}\int_\Omega |\Delta(U_1+U_2)|^2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^p$$

For that we define the set

$$O_\delta(\Omega) = \{(\xi_1, \xi_2) \in \Omega \times \Omega; |\xi_1 - \xi_2| > \delta, \quad d(\xi_i, \partial\Omega) > \delta\},$$

where $\delta > 0$ is a small fixed number and we put

$$C_n = \frac{1}{2}\int_\Omega |\Delta \overline{U}|^2 - \frac{1}{p+1}\int_\Omega \overline{U}^p.$$

Then we have the following:

Lemma 2.1. *For (ξ_1, ξ_2) in $O_\delta(\Omega)$ we have*

$$\begin{aligned} J(U_1+U_2) &= 2C_n + \frac{1}{2}\left(\int_{\mathbb{R}^n} \overline{U}^p\right)\left(H(\xi_1,\xi_1)\lambda_1^{n-4} + H(\xi_2,\xi_2)\lambda_2^{n-4} - 2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2)\right) \\ &\quad + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right). \end{aligned}$$

Proof. The proof follows from the following estimates (see the Appendix):

$$\int_\Omega |\Delta U_i|^2 = \int_{\mathbb{R}^n} |\Delta \overline{U}|^2 - \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 H(\xi_i,\xi_i)\lambda_i^{n-4} + o(\lambda_i^{n-4})$$

and

$$\int_\Omega \Delta U_1 \Delta U_2 = \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2) + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right),$$

$$\frac{1}{p+1}\int_\Omega U_i^{p+1} = \frac{1}{p+1}\int_\Omega \overline{U}^{p+1} - \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 H(\xi_i,\xi_i)\lambda_i^{n-4} + o(\lambda_i^{n-4}),$$

$$\frac{1}{p+1}\int_\Omega (U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1} = 2\left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 \lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}} G(\xi_1,\xi_2) + o\left(\max(\lambda_1,\lambda_2)^{n-4}\right).$$

Therefore one has

$$J(U_1+U_2) = \frac{1}{2}\int_\Omega |\Delta(U_1+U_2)|^2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^p$$

$$= \sum\left(\frac{1}{2}\int_\Omega |\Delta U_i|^2 - \frac{1}{p+1}U_i^{p+1}\right) + \int_\Omega \Delta U_1 \Delta U_2 - \frac{1}{p+1}\int_\Omega (U_1+U_2)^{p+1} - U_1^{p+1} - U_2^{p+1}$$

$$= \sum \frac{1}{2}\left(\int_{\mathbb{R}^n} |\Delta \overline{U}|^2 - \left(\int_{\mathbb{R}^n} \overline{U}^p\right)^2 H(\xi_i,\xi_i)\lambda_i^{n-4}\right) - \frac{1}{p+1}\int_\Omega \overline{U}^{p+1} +$$

$$+\sum\left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2 H\left(\xi_i,\xi_i\right)\lambda_i^{n-4}+\left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}G\left(\xi_1,\xi_2\right)-$$

$$-2\left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}G\left(\xi_1,\xi_2\right)+o\left(\max(\lambda_1,\lambda_2)^{n-4}\right)$$

$$=2C_n+\frac{1}{2}\left(\int_{\mathbb{R}^n}\overline{U}^p\right)^2\left(H\left(\xi_1,\xi_1\right)\lambda_1^{n-4}+H\left(\xi_2,\xi_2\right)\lambda_2^{n-4}-2\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}G\left(\xi_1,\xi_2\right)\right)+$$

$$+o\left(\max(\lambda_1,\lambda_2)^{n-4}\right).$$

□

Now, we set $\Omega_\varepsilon=\varepsilon^{-\frac{2}{n-4}}\Omega$, and we put:

$$v(x')=\varepsilon u(\varepsilon^{\frac{2}{n-4}}x')$$

Then every solution u of (P_ε) corresponds to a solution v, by means of the previous rescaling, of the following problem:

$$\begin{cases}\Delta^2 v = |v|^{p-1}v+\varepsilon^{p+1}\widetilde{f} & \text{on } \Omega_\varepsilon\\ v=\Delta v = 0 & \text{on } \partial\Omega_\varepsilon\end{cases}$$

where $\widetilde{f}\left(x'\right)=f(\varepsilon^{\frac{2}{n-4}}x')$. Hence we define the following perturbed energy functional:

$$J_\varepsilon\left(u\right)=\frac{1}{2}\int_{\Omega_\varepsilon}|\Delta u|^2-\frac{1}{p+1}\int_{\Omega_\varepsilon}|u|^p-\varepsilon^{p+1}\int_{\Omega_\varepsilon}\widetilde{f}u.$$

We consider the function w defined by

$$\begin{cases}\Delta^2 w = f & \text{on } \Omega\\ w=\Delta w = 0 & \text{on } \partial\Omega\end{cases}, \tag{3}$$

and we obtain the following proposition. Set $\Lambda=(\Lambda_1,\Lambda_2)$ and $\lambda_i^2=\left(a_n^{-1}\Lambda_i\right)^{\frac{2}{n-4}}$,

Proposition 2.2. *Let V be the sum of U_1, U_2 rescaled on Ω_ε, then for $(\xi_1, \xi_2) \in O_\delta(\Omega)$, one has*

$$J_\varepsilon(V) = 2C_n + \varepsilon^2 \Psi(\xi, \Lambda) + o(\varepsilon^2),$$

where

$$\Psi(\xi, \Lambda) = \frac{1}{2}\left(\sum_{i=1}^{2} \Lambda_i^2 H(\xi_i, \xi_i) - 2\Lambda_1\Lambda_2 G(\xi_1, \xi_2)\right) + \sum_{i=1}^{2} \Lambda_i w(\xi_i).$$

Proof. The only term we need to estimate is

$$\begin{aligned}
\int_\Omega f(U_1 + U_2) &= \int_\Omega \left(\Delta^2 w\right)(U_1 + U_2) \\
&= \sum_{i=1}^{2} \int_\Omega \left(\Delta^2 w\right)\left(G(x, \xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy\right) + o(\lambda_i^{\frac{n-4}{2}}) \\
&= \sum_{i=1}^{2} w(\xi_i)\lambda_i^{\frac{n-4}{2}} \int_{\mathbb{R}^n} \overline{U}^p(y)dy + o(\lambda_i^{\frac{n-4}{2}}).
\end{aligned}$$

The conclusion follows. □

3 Reduction process

From now on let $\Omega_\varepsilon = \varepsilon^{-\frac{2}{n-4}}\Omega$. We will consider points $\xi_i' \in \Omega_\varepsilon$ and numbers $\Lambda_i > 0$, for $i = 1, 2$, such that $|\xi_1' - \xi_2'| > \delta\varepsilon^{-\frac{2}{n-4}}$, $d(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{2}{n-4}}$ and $\delta < \Lambda_i < \delta^{-1}$. Here we will adopt the same notations as in [14], that is $\overline{V}_i(x) = \overline{U}_{\xi_i', \Lambda_i^*}$ for $\Lambda_i^* = \left(c_n\Lambda_i^2\right)^{\frac{1}{n-4}}$; the related projections on $H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ will be denoted by V_i. Consider the functions

$$\overline{Z}_{ij} = \frac{\partial \overline{V}_i}{\partial \xi_{ij}},\ i = 1, ..., n \text{ and } \overline{Z}_{in+1} = \frac{\partial \overline{V}_i}{\partial \Lambda_i^*}$$

and their projections $Z_{ij} = P\overline{Z}_{ij}$. Let $V = V_1 + V_2$ and $\overline{V} = \overline{V}_1 + \overline{V}_2$.

For a given smooth function h, we want to solve the following linear problem:

$$\begin{cases} \Delta^2\varphi - pV^{p-1}\varphi & = & h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on} & \Omega_\varepsilon \\ \varphi = \Delta\varphi & = & 0 & \text{on} & \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1} Z_{ij}, \varphi \right\rangle & := & \int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\varphi = 0 & \text{for} & i = 1,2 \ ; \ j = 1,...,n+1 \end{cases} \tag{4}$$

We define the following weighted L^∞ norms : for a function u defined on Ω_ε

$$\|u\|_* = \left\| (w_1 + w_2)^{-\beta} u \right\|_{L^\infty} + \left\| (w_1 + w_2)^{-\beta - \frac{1}{n-4}} \nabla u \right\|_{L^\infty}$$

where $w_i = \left(\frac{1}{1 + |x - \xi_i'|^2} \right)^{\frac{n-4}{2}}$, $\beta = \frac{4}{n-4}$, and

$$\|u\|_{**} = \left\| (w_1 + w_2)^{-\gamma} u \right\|_{L^\infty}$$

where $\gamma = \frac{8}{n-4}$. We define also the set

$$O_\delta'(\Omega_\varepsilon) = \left\{ (\xi_1, \xi_2) \in \Omega_\varepsilon \times \Omega_\varepsilon ; |\xi_1 - \xi_2| > \delta\varepsilon^{-\frac{2}{n-4}}, \quad d(\xi_i, \partial\Omega) > \delta\varepsilon^{-\frac{2}{n-4}} \right\}.$$

We refer to [22] for the proof of the following :

Proposition 3.1. *There exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *and all* $h \in C^\alpha(\Omega_\varepsilon)$, *the problem (4) admits a unique solution* $\varphi = L_\varepsilon(h)$. *Moreover we have*

$$\|L_\varepsilon(h)\|_* \leq C \|h\|_{**}, \quad |c_{ij}| \leq C \|h\|_{**},$$

and

$$\left\| \nabla_{(\xi',\Lambda)} L_\varepsilon(h) \right\|_* \leq C \|h\|_{**}.$$

To split the difficulties, we start by finding a solution of

$$\begin{cases} \Delta^2(V+\eta) - (V+\eta)_+^p - \varepsilon^{p+1}\widetilde{f} & = & \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{on} & \Omega_\varepsilon \\ \eta = \Delta\eta & = & 0 & on & \partial\Omega_\varepsilon \\ \left\langle V_i^{p-1} Z_{ij}, \eta \right\rangle & = & -\left\langle V_i^{p-1} Z_{ij}, \varphi \right\rangle & \text{for} & i = 1,2 \ ; \ j = 1,...,n+1 \end{cases},$$

where φ is the solution of

$$\begin{cases} \Delta^2\varphi &= \varepsilon^{p+1}\widetilde{f} \quad \text{on} \quad \Omega_\varepsilon \\ \varphi = \Delta\varphi &= 0 \quad \text{on} \quad \partial\Omega_\varepsilon \end{cases}.$$

If we take $\eta = \overline{\eta} + \varphi$, then the equation on $\overline{\eta}$ reads as follows:

$$\Delta^2\overline{\eta} - pV^{p-1}\overline{\eta} = N_\varepsilon(\overline{\eta}) - R_\varepsilon + \sum_{i,j} c_{ij}V_i^{p-1}Z_{ij} \tag{5}$$

with

$$N_\varepsilon(\overline{\eta}) = |V + \overline{\eta} + \varphi|^{p-1}(V + \overline{\eta} + \varphi)_+ - pV^{p-1}(\overline{\eta} + \varphi) - V^p,$$

and

$$R_\varepsilon = V^p - \overline{U}_1^p - \overline{U}_2^p - p\,|V|^{p-2}\,\varphi.$$

Therefore, taking $\psi = -L_\varepsilon(R_\varepsilon)$ and $\overline{\eta} = \psi + v$, we get an equation on v of the following form :

$$\Delta^2 v - pV^{p-1}v = N_\varepsilon(\overline{\eta}) + \sum_{i,j} c_{ij}V_i^{p-1}Z_{ij}.$$

Lemma 3.2. *There exists $C > 0$ such that for $\varepsilon > 0$ small enough and $\|v\|_* \le \frac{1}{4}$, we have*

$$\|N_\varepsilon(\psi + v)\|_{**} \le \begin{cases} C\left(\|v\|_*^2 + \varepsilon\,\|v\|_* + \varepsilon^{p+1}\right) & \text{if } n \le 12 \\ C\left(\varepsilon^{2\beta-1}\,\|v\|_*^2 + \varepsilon^{2\beta}\,\|v\|_* + \varepsilon^{3p}\right) & \text{if } n > 12 \end{cases}$$

Proof. First, we recall that $\|\psi\|_* \le C\varepsilon^2$ and since $|\varphi| \le C\varepsilon^{p+1}$, we have

$$|\varphi|\,\overline{V}^{-\beta} \le C\varepsilon^{p+1}\overline{V}^{-\beta} \le C\varepsilon^2$$

hence $\|\varphi\|_* \le C\varepsilon^2$ and we can choose ε small enough so that

$$\|\overline{\eta}\|_* \le \|\psi\|_* + \|v\|_* < 1.$$

Now, we have

$$N_{\varepsilon}\left(\overline{\eta}\right)=\frac{p\left(p-1\right)}{2}\left(V+t\left(\overline{\eta}+\varphi\right)\right)^{p-2}\left(\overline{\eta}+\varphi\right)^{2},$$

for a certain $t\in(0,1)$ and hence if $n\leq 12$ we have

$$\begin{aligned}\left|\overline{V}^{-\frac{8}{n-4}}N_{\varepsilon}\left(\overline{\eta}\right)\right| &\leq C\overline{V}^{2\beta-\frac{8}{n-4}}\overline{V}^{p-2}\left\|\overline{\eta}+\varphi\right\|_{*}^{2}\\ &\leq C\left\|\overline{\eta}+\varphi\right\|_{*}^{2}\end{aligned}$$

If $n>12$ we have to distinguish two cases. First consider $\delta>0$ and take the region $d(y,\partial\Omega_{\varepsilon})>\delta\varepsilon^{-\frac{n+2}{n-4}}$, then one has the existence of $C_{\delta}>0$ such that $V>C_{\delta}\overline{V}$ and therefore we get

$$\begin{aligned}\left|N_{\varepsilon}\left(\overline{\eta}\right)\overline{V}^{-\frac{8}{n-4}}\right| &\leq C\overline{V}^{2\beta-\frac{8}{n-4}+p-2}\left\|\overline{\eta}+\varphi\right\|_{*}^{2}\\ &\leq C\varepsilon^{p-2}\left\|\overline{\eta}+\varphi\right\|_{*}^{2}\end{aligned}$$

If $d(y,\partial\Omega_{\varepsilon})\leq\delta\varepsilon^{-\frac{n+2}{n-4}}$ we have, by using Hopf lemma, that for δ sufficiently small $V\left(y\right)\sim\frac{\partial V}{\partial\nu}d(y,\partial\Omega_{\varepsilon})$, (recall that $\left|\nabla V\right|=\left|\nabla\overline{V}\right|+o\left(1\right)$) and $\left|\nabla V\right|\geq C\varepsilon^{\frac{n-3}{n-4}}$, for ε small enough. Thus $V\left(y\right)\geq C\varepsilon^{2\frac{n-3}{n-4}}d(y,\partial\Omega_{\varepsilon})$, therefore

$$\begin{aligned}\left|N_{\varepsilon}\left(\overline{\eta}\right)\overline{V}^{-\frac{8}{n-4}}\right| &\leq C\overline{V}^{-\frac{8}{n-4}}\left(\varepsilon^{2\frac{n-3}{n-4}}d(y,\partial\Omega_{\varepsilon})\right)^{p-2}\left(\overline{\eta}+\varphi\right)^{2}\\ &\leq C\overline{V}^{-\frac{8}{n-4}}\left(\varepsilon^{2\frac{n-3}{n-4}}d(y,\partial\Omega_{\varepsilon})\right)^{p-2}\left(\overline{\eta}+\varphi\right)^{2}\\ &\leq C\left(\varepsilon^{2\frac{n-3}{n-4}-\frac{n+2}{n-4}}\right)^{p-2}\left\|\overline{\eta}+\varphi\right\|_{*}^{2}\\ &\leq C\varepsilon^{2\beta-1}\left\|\overline{\eta}+\varphi\right\|_{*}^{2}.\end{aligned}$$

Finally

$$\left\|N_{\varepsilon}(\psi+v)\right\|_{**}\leq\begin{cases}C\left(\left\|\psi+v+\varphi\right\|_{*}^{2},\right) & \text{if } n\leq 12\\ C\left(\varepsilon^{2\beta-1}\left\|\psi+v+\varphi\right\|_{*}^{2}\right) & \text{if } n>12\end{cases}$$

Which finishes the proof. ☐

Now we want to find a solution to (5). The problem can be seen as a fixed point problem if we write it in the following way

$$v = -L_\varepsilon \left(N_\varepsilon \left(\psi + v \right) \right) = A_\varepsilon \left(v \right). \tag{6}$$

We have the following:

Proposition 3.3. *There exists $C > 0$ such that for $\varepsilon > 0$ small enough, the problem (6) has a unique solution v, with $\|v\|_* < C\varepsilon^2$. Moreover, the map $(\xi', \Lambda) \longrightarrow v$ is C^1 with respect to the norm $\|\cdot\|_*$, and $\left\|\nabla_{(\xi',\Lambda)} v\right\|_* \leq C\varepsilon^2$.*

Proof. Let

$$F = \left\{ u \in H^2\left(\Omega\right) \cap H_0^1\left(\Omega\right), \|u\|_* < \varepsilon^2 \right\},$$

and then consider $A_\varepsilon : F \longrightarrow H^2\left(\Omega\right) \cap H_0^1\left(\Omega\right)$. By using the previous lemma and proposition (3.1) we get

$$\begin{aligned}
\|A_\varepsilon\left(u\right)\|_* &\leq C\left\|N_\varepsilon(u+\psi)\right\|_{**} \\
&\leq \begin{cases} C\left(\|u\|_*^2 + \varepsilon\|u\|_* + \varepsilon^{p+1}\right) & \text{if } n \leq 12 \\ C\left(\varepsilon^{2\beta-1}\|u\|_*^2 + \varepsilon^{2\beta}\|u\|_* + \varepsilon^{3p}\right) & \text{if } n > 12 \end{cases} \\
&\leq \begin{cases} C\varepsilon^3 \text{ if } n \leq 12 \\ C\varepsilon^{2\beta+3} \text{ if } n > 12 \end{cases},
\end{aligned}$$

so for $\varepsilon > 0$ small enough, we have that A_ε maps F into itself. Now we estimate $\|A_\varepsilon\left(a\right) - A_\varepsilon\left(b\right)\|_*$ for $a, b \in F$. Since

$$\|A_\varepsilon\left(a\right) - A_\varepsilon\left(b\right)\|_* \leq C\left\|N_\varepsilon(a+\psi) - N_\varepsilon(b+\psi)\right\|_{**},$$

it suffices to show that N_ε is a contraction to finish the proof of the proposition. Note that by construction we have

$$D_u N_\varepsilon(u+\psi) = p\left|V + u + \psi + \varphi\right|^{p-2}\left(V + u + \psi + \varphi\right) - pV^{p-1}.$$

Then arguing as in [22], we obtain that N_ε is a contraction. Hence the existence and uniqueness of v follows. Next we prove that the map is C^1. We will apply the implicit function theorem to the map K defined by

$$K\left(\xi', \Lambda, v\right) = v - A_\varepsilon(v).$$

We recall that

$$D_{\xi'} N_\varepsilon(u) = p\left[|V+u+\varphi|^{p-2}(V+u+\varphi) - (p-1)V^{p-2}(u+\varphi) - V^{p-1}\right] D_{\xi'} V$$

Also,

$$D_u K\left(\xi', \Lambda, u\right) h = h + L_\varepsilon(D_u N_\varepsilon(u+\psi)h) = h + M(h).$$

Now

$$\begin{aligned} \|M(h)\|_* &\leq C\, \|D_u N_\varepsilon(u+\psi)h\|_{**} \\ &\leq C\left\|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right\|_\infty \|h\|_* \end{aligned}$$

and since

$$\left|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right| \leq C\overline{V}^{2\beta-1}\, \|u+\psi\|_*\,,$$

we get

$$\left\|\overline{V}^{-\frac{8}{n-4}+\beta} D_u N_\varepsilon(u+\psi)\right\|_\infty \leq C \begin{cases} \varepsilon^2 \text{ if } n \leq 12 \\ \varepsilon^{2\beta+1} \text{ if } n > 12 \end{cases},$$

hence

$$\|M(h)\|_* \leq C\varepsilon^{\min(2,2\beta+1)}\, \|h\|_*$$

Therefore by using the implicit function theorem, we have that φ depends continuously on the parameter (ξ', Λ). On the other hand if we differentiate with respect to ξ' we get

$$D_{\xi'} K\left(\xi', \Lambda, u\right) = D_{\xi'} u + D_{\xi'} L_\varepsilon(N_\varepsilon(u+\psi))$$

From proposition (3.1) we get that

$$\left\| D_{\xi'} L_{\varepsilon}(h) \right\|_{*} \leq C \left\| h \right\|_{**}$$

Thus we need to compute

$$D_{\xi'}\psi = (D_{\xi'}L_{\varepsilon})(R_{\varepsilon}) + L_{\varepsilon}(D_{\xi'}R_{\varepsilon}),$$

but

$$D_{\xi_1'}R_{\varepsilon} = pV^{p-1}D_{\xi_1'}V - p\overline{U}_1^{p-1}D_{\xi_1'}\overline{U}_1 - p\left(p-2\right)\left|V\right|^{p-3}D_{\xi_1'}V\varphi$$

which depends continuously on the parameters, and this is enough to prove that v is C^1 with respect to the parameters (ξ', Λ). Moreover we have

$$D_{\xi'}v = -\left(D_v K\left(\xi', \Lambda, v\right)\right)^{-1}\left[\left(D_{\xi'}L_{\varepsilon}\right)\left(N_{\varepsilon}\left(v+\psi\right)\right) + \right.$$
$$\left. + L_{\varepsilon}\left(D_{\xi'}(N_{\varepsilon}\left(v+\psi\right))\right) + L_{\varepsilon}\left(D_v(N_{\varepsilon})\left(v+\psi\right)D_{\xi'}\psi\right)\right],$$

hence

$$\left\|D_{\xi'}v\right\|_{*} \leq C\left(\left\|N_{\varepsilon}\left(v+\psi\right)\right\|_{**} + \left\|D_{\xi'}(N_{\varepsilon}\left(v+\psi\right))\right\|_{**} + \left\|D_v(N_{\varepsilon})\left(v+\psi\right)D_{\xi'}\psi\right\|_{**}\right).$$

Now, from Lemma (3.2), we know that

$$\left\|N_{\varepsilon}(v+\psi)\right\|_{**} \leq \begin{cases} C\varepsilon^3 \text{ if } n \leq 12 \\ C\varepsilon^{2\beta+3} \text{ if } n > 12 \end{cases}$$

and also

$$\begin{aligned} \left|D_{\xi'}(N_{\varepsilon}\left(u\right))\right| &= p\left[\left|V+u+\varphi\right|^{p-2}\left(V+u+\varphi\right) - \left(p-1\right)V^{p-2}\left(u+\varphi\right) - V^{p-1}\right]D_{\xi'}V \\ &\leq CV^{p-2}\left|D_{\xi'}V\right|\left|u\right| \leq C\overline{V}^{p-2+\frac{n-3}{n-4}+\beta}\left\|u\right\|_{*}. \end{aligned}$$

We get

$$\overline{V}^{-\frac{8}{n-4}}\left|D_{\xi'}(N_{\varepsilon}\left(u\right))\right| \leq C\overline{V}^{\frac{n-3}{n-4}+\beta-1}\left\|u\right\|_{*},$$

therefore

$$\left\| D_{\xi'}(N_\varepsilon\,(v+\psi)) \right\|_{**} \leq C\varepsilon^2$$

A similar estimate gives

$$\left\| D_v(N_\varepsilon)\,(v+\psi)\,D_{\xi'}\psi \right\|_{**} \leq C\varepsilon^2.$$

Since there is no difference in the case of the differentiation with respect to Λ, we omit it. □

4 Reduction of the functional

Here we want to go back to our original set Ω, therefore we will denote $\xi_i' = \varepsilon^{-\frac{2}{n-4}}\xi_i$ where $\xi_i \in \Omega$ and we remark that if we take ξ_i and Λ so that $c_{ij} = 0$, then we obtain a solution of our original problem. Let $\mathcal{I}_\varepsilon$ be the functional defined by

$$\mathcal{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \varepsilon\int_\Omega fu$$

so that $u = V + v + \varphi + \psi$ is a solution for our problem if and only if it is a critical point for this functional. Let us consider the functions defined on Ω by

$$\widehat{v}(\xi,\Lambda)\,(x) = \varepsilon^{-1} v\left(\varepsilon^{-\frac{2}{n-4}}\xi,\Lambda\right)\left(\varepsilon^{-\frac{2}{n-4}}x\right),$$

$$\widehat{\psi}\,(x) = \varepsilon^{-1}\psi\left(\varepsilon^{-\frac{2}{n-4}}x\right),$$

$$\widehat{\varphi}\,(x) = \varepsilon^{-1}\varphi\left(\varepsilon^{-\frac{2}{n-4}}x\right)$$

and

$$\widehat{U}_i(x) = \varepsilon^{-1}V_i\left(\varepsilon^{-\frac{2}{n-4}}x\right)$$

Therefore if we set $\widehat{U}(x) = \widehat{U}_2(x) + \widehat{U}_1(x)$, and

$$I\,(\xi,\Lambda) = \mathcal{I}_\varepsilon\left(\widehat{U} + \widehat{\psi} + \widehat{v}(\xi,\Lambda) + \widehat{\varphi}\right)$$

then

$$I(\xi,\Lambda)=J_\varepsilon\left(V+\psi+v+\varphi\right).$$

Next we state the following result and we refer to [22] for the proof.

Lemma 4.1. *$u=\widehat{U}+\widehat{\psi}+\widehat{v}(\xi,\Lambda)+\widehat{\varphi}$ is a solution of the problem (P) if and only if (ξ,Λ) is a critical point of I.*

Now we define

$$\sigma_f=\int_\Omega fw,$$

and we obtain

Proposition 4.2. *We have the following expansion:*

$$I(\xi,\Lambda)=2C_n+\varepsilon^2\left(\Psi\left(\xi,\Lambda\right)+\sigma_f\right)+o(\varepsilon^2),$$

where $o(\varepsilon^2)\longrightarrow 0$ as $\varepsilon\longrightarrow 0$ in the C^1 sense, uniformly in $O_\delta\left(\Omega\right)\times\left(\delta,\delta^{-1}\right)^2$.

Proof. Let us show first that

$$I(\xi,\Lambda)-\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)=o(\varepsilon^2),$$

and

$$\nabla_{(\xi,\Lambda)}\left(I(\xi,\Lambda)-\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)\right)=o(\varepsilon^2).$$

Indeed, using a Taylor expansion we have

$$J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{v}(\xi,\Lambda)+\widehat{\varphi}\right)-J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)=\int_0^1 tD^2J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}+t\widehat{v}\right)[\widehat{v},\widehat{v}]\,dt$$

and this holds since $DJ_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}+\widehat{v}\right)=0$. Therefore, we have

$$\int_0^1 tD^2J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}+t\widehat{v}\right)[\widehat{\varphi},\widehat{\varphi}]\,dt=\int_0^1 t\left[\int_{\Omega_\varepsilon}|\nabla v|^2-p\left(V+\psi+\varphi+tv\right)^{p-1}v^2\right]dt$$

$$=\int_0^1 t\int_{\Omega_\varepsilon}p\left[V^{p-1}-\left(V+\psi+\varphi+tv\right)^{p-1}\right]v^2+N_\varepsilon\left(v+\psi\right)v\,dt.$$

We have $\|v\|_* + \|\varphi\|_* + \|\psi\|_* = O\left(\varepsilon^2\right)$, and by using Lemma (3.2), we get

$$\int_{\Omega_\varepsilon} N_\varepsilon\left(v+\psi\right)v \leq \int_{\Omega_\varepsilon} \overline{V}^{p-1+\beta}\left\|N_\varepsilon\left(v+\psi\right)\right\|_{**}\|v\|_* \leq C\varepsilon^3 \int_{\Omega_\varepsilon} \overline{V}^{p-1+\beta} \leq C\varepsilon^3.$$

Now, the remaining part can be estimated as follows

$$\begin{aligned}\int_{\Omega_\varepsilon}\left[V^{p-1}-\left(V+\psi+\varphi+tv\right)^{p-1}\right]v^2 &\leq C\varepsilon^4\int_{\Omega_\varepsilon}\overline{V}^{2\beta}\left[V^{p-1}-\left(V+\psi+t\varphi\right)^{p-1}\right]\\ &\leq C\varepsilon^4,\end{aligned}$$

Same estimates hold if we differentiate with respect to ξ. In fact we have

$$D_\xi\left(J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{v}(\xi,\Lambda)+\widehat{\varphi}\right)-J_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)\right)=$$

$$\varepsilon^{-\frac{2}{n-4}}\int_0^1 t\int_{\Omega_\varepsilon} pD_{\xi'}\left(\left[V^{p-1}-\left(V+\psi+\varphi+tv\right)^{p-1}\right]v^2\right)+D_{\xi'}\left(N_\varepsilon\left(v+\psi\right)v\right)dt,$$

and the conclusion follows again from Lemma (3.2). Next step is to prove that

$$\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)-\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\varphi}\right)=o\left(\varepsilon^2\right)$$

and

$$D_\xi\left(\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)-\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\varphi}\right)\right)=o\left(\varepsilon^2\right),$$

So we start by writing

$$\begin{aligned}&\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\psi}+\widehat{\varphi})-\mathcal{I}_\varepsilon(\widehat{U}+\widehat{\varphi})=I_\varepsilon(U+\psi+\varphi)-I_\varepsilon(U+\varphi)\\ &=\int_0^1(1-t)\left(\left[p\int_{\Omega_\varepsilon}\left(V+\varphi+t\psi\right)^{p-1}\psi^2-\int_{\Omega_\varepsilon}\left|\Delta\psi\right|^2\right]-\right.\\ &\left.-\int_{\Omega_\varepsilon}\left(|V|^p-|V+\varphi|^p+p\left|V\right|^{p-1}\varphi\right)\psi+\int_{\Omega_\varepsilon}R^\varepsilon\psi\right).\end{aligned}$$

Also

$$D_\xi\left(\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\psi}+\widehat{\varphi}\right)-\mathcal{I}_\varepsilon\left(\widehat{U}+\widehat{\varphi}\right)\right)=\varepsilon^{-\frac{2}{n-4}}\left[\int_0^1(1-t)\left(D_{\xi'}\left[p\int_{\Omega_\varepsilon}\left(V+\varphi+t\psi\right)^{p-1}\psi^2-\right.\right.\right.$$

$$- \int_{\Omega_\varepsilon} |\Delta\psi|^2 \Big] dt - D_{\xi'} \int_{\Omega_\varepsilon} \left(|V|^p - |V+\varphi|^p + p\,|V|^{p-1}\varphi \right) \psi + D_{\xi'} \int_{\Omega_\varepsilon} R^\varepsilon \psi \Big) \Big]$$

Again, by using the fact that $\|\psi\|_* + \|R^\varepsilon\|_{**} + \left\|\nabla_{(\xi,\Lambda)}\psi\right\|_* + \left\|\nabla_{(\xi,\Lambda)}R^\varepsilon\right\|_{**} \leq C\varepsilon^2$, with $\|\varphi\|_* \leq C\varepsilon^p$ if $n \leq 12$ and $\|\varphi\|_* \leq C\varepsilon^2$ if $n > 12$, we get the desired result. The final steps, namely showing

$$\mathcal{I}_\varepsilon\left(\widehat{U} + \widehat{\varphi}\right) - \mathcal{I}_\varepsilon\left(\widehat{U}\right) = \varepsilon^2\sigma_f + o\left(\varepsilon^2\right),$$

and

$$D_\xi\left(\mathcal{I}_\varepsilon\left(\widehat{U} + \widehat{\varphi}\right) - \mathcal{I}_\varepsilon\left(\widehat{U}\right)\right) = o\left(\varepsilon^2\right),$$

are also obtained by using the same kind of estimates. □

5 Analysis of the exterior domain

Let us consider here $\Omega = \mathcal{D} - \overline{B(0,\mu)}$ for $\mu > 0$ small enough. Also for $E = \mathbb{R}^n - \overline{B(0,1)}$ define the set

$$\mathcal{V} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n; G_E(x,y) - H_E^{\frac{1}{2}}(x,x) H_E^{\frac{1}{2}}(y,y) < 0 \right\} \cap \left(\mu^{-1}\Omega\right),$$

where G_E and H_E are the Green's function and its regular part on the set E.

Let us take $f = 1$ and $\mathcal{F}_a = \{x \in \mathbb{R}^n; 1 < |x| < a, a > 1\}$, then the solution of

$$\begin{cases} \Delta^2 w_a & = & f & \text{on} & \mathcal{F}_a \\ w_a = \Delta w_a & = & 0 & \text{on} & \partial\mathcal{F}_a \end{cases},$$

is given by

$$w_a(x) = -\frac{1}{8n(n+2)} \left(\frac{a^4 - 1}{a^{4-n} - 1} |x|^{4-n} - |x|^4 + a^{4-n} \frac{(1-a^n)}{a^{4-n} - 1} \right),$$

It is easy to see that it has a maximum for

$$|x_a| = \left(\frac{4\left(1 - a^{4-n}\right)}{(n-4)\left(a^4 - 1\right)} \right)^{\frac{-1}{n}},$$

and $|x_a| \longrightarrow \infty$ as $a \longrightarrow \infty$. Now we consider the function $\varphi_{\mathcal{F}_a}$ defined, on the set $\mathcal{F}_a$ by

$$\varphi_{\mathcal{F}_a}(x,y) = \frac{1}{2}\frac{H_{\mathcal{F}_a}(x,x)\,w_a(y)^2 + H_{\mathcal{F}_a}(y,y)\,w_a(x)^2 + 2G_{\mathcal{F}_a}(x,y)\,w_a(y)\,w_a(x)}{-H_{\mathcal{F}_a}(x,x)\,H_{\mathcal{F}_a}(y,y) + G^2_{\mathcal{F}_a}(x,y)},$$

we will extend it to the full exterior domain $E = \{x \in \mathbb{R}^n; 1 < |x|\}$, for that we just extend w_a by zero for $|x| > a$. Hence knowing that

$$H_E(x,y) = \frac{a_n}{||y|\,(x-\overline{y})|^{n-4}}$$

where $\overline{y} = \frac{y}{|y|^2}$, and since w_a is radially symmetric, we get that φ_E has a critical point (x,y) if and only if $\sin(\theta) = 0$ where θ is the angle between x and y. Now we set $x = se$ and $y = -te$, where e is a unit vector and s and t are real number greater than 1. we write

$$\widetilde{\varphi}_E(s,t) = \varphi_E(se,-te).$$

Explicitly :

$$2a_n\widetilde{\varphi}_E(s,t) = \left(\frac{\widetilde{w}_a(t)^2}{(s^2-1)^{n-4}} + \frac{\widetilde{w}_a(s)^2}{(t^2-1)^{n-4}} + 2\widetilde{w}_a(t)\,\widetilde{w}_a(s)\left(\frac{1}{(s+t)^{n-4}} - \frac{1}{(st+1)^{n-4}}\right)\right)$$
$$\left(\left(\frac{1}{(s+t)^{n-4}} - \frac{1}{(st+1)^{n-4}}\right)^2 - \left(\frac{1}{(t^2-1)^{n-4}(t^2-1)^{n-4}}\right)\right)^{-1}.$$

We recall now (see [22]) that the function defined by

$$\widetilde{\rho}(s,t) = a_n\left(-\frac{1}{(t^2-1)^{\frac{n-4}{2}}(s^2-1)^{\frac{n-4}{2}}} - \frac{1}{(1+st)^{n-4}} + \frac{1}{(s+t)^{n-4}}\right),$$

has a unique maximum point of the form (K,K), for $s,t > 1$ and a unique k satisfying $\widetilde{\rho}(k,k) = 0$. we can choose $a_0 > 0$, big enough, such that for $a > a_0$, we have $k < K < |x_a|$. Hence we can get the following :

Lemma 5.1. *The function $\widetilde{\varphi}_E$ admits a unique minimum, of the form (τ_a,τ_a). Moreover $\tau_a \in (k,K)$.*

Next we will work on the domain $\Omega = D - \overline{B(0,\mu)}$. We set m, (resp M) the radius of the largest (resp smallest) ball contained (resp containing) D, and set $\alpha = \min_\Omega f$, and $\beta = \max_\Omega f$. Thus, by using the maximum principle, we have $z_m \leq w \leq z_M$ for $\mu < |x| < m$, with w as defined in (3),

$$z_m(x) = \alpha\mu^4 w_{a_1}\left(\mu^{-1}x\right),$$

and

$$z_M(x) = \beta\mu^4 w_{a_2}\left(\mu^{-1}x\right),$$

here $a_1 = \mu^{-1}m$ and $a_1 = \mu^{-1}M$. we obtain the following

Lemma 5.2. *For $\mu > 0$ small enough the function φ_E has a relative minimum in a point $(\widetilde{x}_\mu, \widetilde{y}_\mu)$, with $|\widetilde{x}_\mu|$ and $|\widetilde{y}_\mu|$ belonging to $\left(k, \widetilde{k}\right)$, and $\widetilde{k}$ independent of μ.*

The proof of this lemma follows if we show that there exist $\widetilde{k} \geq K$ satisfying

$$\frac{\widetilde{\varphi}_{\mathcal{F}_{a_1}}\left(\widetilde{k}, \widetilde{k}\right)}{\widetilde{\varphi}_{\mathcal{F}_{a_2}}(K, K)} \geq 1,$$

the conclusion will follow from the fact that $\varphi_{\mathcal{F}_{a_1}} \leq \varphi_E \leq \varphi_{\mathcal{F}_{a_2}}$ and $\varphi_{\mathcal{F}_a}$ has a unique minimum point for a big enough.

Let us Define the set

$$\mathcal{X} = \left\{(x,y) \in \mathcal{V}, \text{ such that } k < |x|, |y| < \widetilde{k}\right\},$$

and call $c_\mu = \varphi_E(\widetilde{x}_\mu, \widetilde{y}_\mu)$. Now we choose $\delta_\mu > c_\mu$ in such way that the set $\{(x,y) \in \mathcal{X}, \varphi_E = \delta_\mu\}$ is a closed curve on which $\nabla\varphi_E \neq 0$. Observe then that if we call

$$\mathcal{J} = \{(x,y) \in \mathcal{X}, \text{ such that } \varphi_E \leq \delta_\mu\},$$

two situation might happen on $\partial\mathcal{J}$: either there exist a tangential direction τ such that $\nabla\varphi_E\cdot\tau\neq 0$, or x and y point in two different directions and $\nabla\varphi_E\left(x,y\right)\neq 0$ points in the normal direction to $\partial\mathcal{J}$.

Now if we look at $E_\mu=\mathbb{R}^n-\overline{B\left(0,\mu\right)}$, then we can easily see that G_{E_μ} and H_{E_μ}, are defined by

$$G_{E_\mu}\left(x,y\right)=\mu^{4-n}G_E\left(\mu^{-1}x,\mu^{-1}y\right)$$

and

$$H_{E_\mu}\left(x,y\right)=\mu^{4-n}H_E\left(\mu^{-1}x,\mu^{-1}y\right).$$

Note that $S_\mu=\mu\mathcal{J}$, corresponds exactly to the set $\left\{\varphi_E\left(\mu^{-1}x,\mu^{-1}y\right)\leq\delta_\mu\right\}$. Also

$$G\left(x,y\right)=G_{E_\mu}\left(x,y\right)+O(1)$$

on the set $\mu\mathcal{X}$. Therefore, it follows that:

$$\varphi_\Omega\left(x,y\right)=\mu^{n+4}\varphi_E\left(\mu^{-1}x,\mu^{-1}y\right)+o(1)$$

where

$$\varphi_\Omega\left(x,y\right)=\frac{1}{2}\frac{H_\Omega\left(x,x\right)w\left(y\right)^2+H_\Omega\left(y,y\right)w\left(x\right)^2+2G_\Omega\left(x,y\right)w\left(y\right)w\left(x\right)}{G_\Omega^2(x,y)-H_\Omega\left(x,x\right)H_\Omega\left(y,y\right)}$$

and $o(1)\longrightarrow 0$ as $\mu\longrightarrow 0$ in the C^1 sense.

6 Proof of Theorem 1.1

Since the function Ψ defined in section 2 is singular on the diagonal of $\Omega\times\Omega$, we replace the terms $G\left(\xi_1,\xi_2\right)$ by $G_M\left(\xi_1,\xi_2\right)=\min\left(G\left(\xi_1,\xi_2\right),M\right)$ for a constant $M>0$ to be fixed later. Hence Ψ is well defined on $S_\mu\times\mathbb{R}_+^2$.

We remark that in that set, we have $\rho(x,y) = H(x,x)^{\frac{1}{2}} H(y,y)^{\frac{1}{2}} - G(x,y) < 0$, therefore the principal part of Ψ which is a quadratic form, has a negative direction. We will set $\mathbf{e}(\xi_1,\xi_2)$ the vector defining the negative direction :

We have

$$\mathbf{e}(\xi_1,\xi_2) = \left(\frac{H(\xi_1,\xi_1)^{\frac{1}{2}}}{H(\xi_2,\xi_2)^{\frac{1}{2}} \rho(\xi_1,\xi_2)}, \frac{H(\xi_2,\xi_2)^{\frac{1}{2}}}{H(\xi_1,\xi_1)^{\frac{1}{2}} \rho(\xi_1,\xi_2)} \right),$$

Now we are going to consider the vector $\widetilde{\mathbf{e}}$ such that, for each (ξ_1,ξ_2), $\widetilde{\mathbf{e}}(\xi_1,\xi_2)$ is the critical point of $\Psi((\xi_1,\xi_2),\cdot)$. This vector can be written explicitly in the following form

$$\widetilde{\mathbf{e}}(\xi_1,\xi_2) = \left(\frac{H(\xi_2,\xi_2)\, w(\xi_1) + G(\xi_1,\xi_2))\, w(\xi_2))\, w(\xi_1))}{G^2(\xi_1,\xi_2) - H(\xi_2,\xi_2)\, H(\xi_1\xi_{2=1})}, \right.$$

$$\left. \frac{H(\xi_1,\xi_1)\, w(\xi_2) + G(\xi_1,\xi_2))\, w(\xi_2))\, w(\xi_1))}{G^2(\xi_1,\xi_2) - H(\xi_2,\xi_2)\, H(\xi_1\xi_{2=1})} \right).$$

Therefore we can check that $\Psi((\xi_1,\xi_2), \widetilde{\mathbf{e}}(\xi_1,\xi_2)) = \varphi_\Omega(\xi_1,\xi_2)$.

Now we can set the min-max scheme, in a similar way as in [1], [14] and [22]. Let us define

$$K_\mu = \{(x,y) \in \mathcal{X}, (|x|,|y|) = \mu(|\widetilde{x}_\mu|,|\widetilde{y}_\mu|)\},$$

We consider the family of curves $\mathcal{R}$, satisfying the following properties, $\gamma : K_\mu^2 \times [s,s^{-1}] \times [0,1] \longrightarrow A_\mu \times \mathbb{R}_+^2$ such that :

i) for $(\xi_1,\xi_2) \in K_\mu^2$, $t \in [0,1]$ it holds

$$\gamma(\xi_1,\xi_2,s,t) = (\xi_1,\xi_2,s\widetilde{\mathbf{e}}(\xi_1,\xi_2)),$$

and

$$\gamma\left(\xi_1,\xi_2,s^{-1},t\right) = \left(\xi_1,\xi_2,s^{-1}\widetilde{\mathbf{e}}(\xi_1,\xi_2)\right).$$

ii)$\gamma(\xi_1,\xi_2,t,0) = (\xi_1,\xi_2,t\widetilde{\mathbf{e}}(\xi_1,\xi_2))$, for all $(\xi_1,\xi_2,t) \in K_\mu^2 \times [s,s^{-1}]$.

Now arguing as in [22], the min-max value defined by

$$C(\Omega) = \inf_{\gamma \in \mathcal{R}} \sup_{(\xi_1, \xi_2, t) \in K_\mu^2 \times [s, s^{-1}]} \Psi\left(\gamma\left(\xi_1, \xi_2, t, 1\right)\right),$$

is a critical value of Ψ.

Then the proof of theorem 1.1 follows as in ([15]).

7 Vanishing Solutions

In this section we will prove a multiplicity result concerning problem (P_f). Let us start by introducing a slightly different notation from the previous part. We set

$$\overline{U}_{(z,a)} = c_n \left(\frac{a}{1 + a^2 |x - z|^2}\right)^{\frac{n-4}{2}},$$

for every $z \in \Omega$ (it corresponds to $a = \frac{1}{\lambda}$ in the first part of the paper). Also, we set:

$$\overline{Z}_{(z,a),i} = \frac{\partial}{\partial z_i} \overline{U}_{(z,a)},$$

for $i = 1, \cdots, n$, and

$$\overline{Z}_{(z,a),n+1} = \frac{\partial}{\partial a} \overline{U}_{(z,a)}$$

Now we consider the functional I defined on $H^2(\Omega) \cap H_0^1(\Omega)$ by

$$I(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 - \frac{1}{p+1} \int_\Omega \left|u^+\right|^{p+1},$$

We know that critical points of this functional are positive solutions to the problem

$$\begin{cases} \Delta^2 u & = & u^p & \text{on} & \Omega \\ u = \Delta u & = & 0 & \text{on} & \partial\Omega \end{cases},$$

and, if $\Omega = \mathbb{R}^n$ then the solutions for

$$\begin{cases} \Delta^2 u & = & u^p & \text{on} & \mathbb{R}^n \\ u > 0 & \text{and} & u & \text{in} & D^{2,2}\left(\mathbb{R}^n\right) \end{cases},$$

are of the form $\overline{U}_{(z,a)}$. We define the set

$$S=\left\{u\in H^{2}\left(\Omega\right)\cap H_{0}^{1}\left(\Omega\right)-\{0\};\int_{\Omega}\left|\Delta u\right|^{2}=\int_{\Omega}\left|u^{+}\right|^{p+1}\right\},$$

It is easy to show that for every $u\in S$, we have $I(u)>\frac{C_n}{n}$. Now we take $0<d_0<1$ small enough so that, if $d(x,\partial\Omega)<d_0$, then there exists a unique $y\in\partial\Omega$ such that $|x-y|=d(x,\partial\Omega)$. We put $d(x)=\min(d_0,d(x,\partial\Omega))$, for every x in Ω. Next we set

$$\mathcal{O}\left(r\right)=\left\{(x,a)\in\Omega\times(1,\infty)\,;d(x)a=r\right\}$$

and

$$\overline{\mathcal{O}}\left(r\right)=\left\{(x,a)\in\Omega\times(1,\infty)\,;d(x)a\geq r\right\}.$$

If we consider the eigenvalue problem

$$\Delta^{2}v=\gamma p\overline{U}_{(z,a)}^{p}v \text{ on } D^{2}\left(\mathbb{R}^{n}\right),$$

then obviously $\overline{U}_{(z,a)}$ is an eigenfunction corresponding to $\gamma_1=\frac{1}{p}$. We take

$$T_{(z,a)}=span\left\{\overline{Z}_{(z,a),i},i=1,\ldots,n+1\right\},$$

and by using the classification in [21], we have that $T_{(z,a)}$ is exactly the eigenspace corresponding to the eigenvalue 1. We set T_0 the eigenspace corresponding to the eigenvalue γ_1 and

$$T_{(z,a)}^{+}=\left(T_{0}\oplus T_{(z,a)}\right)^{\perp},$$

where orthogonality here is with respect to the scalar product $(u,v)=\int_{\Omega}\Delta u\Delta v$, for every $u,v\in D^{2}\left(\Omega\right)$. Now by means of the sterographic projection from $\mathbb{R}^n$ to the sphere, we obtain a linear eigenvalue problem on a compact manifold, with operator (Paneitz) having compact resolvent. Therefore we have the following:

Lemma 7.1. *There exists $\gamma > 0$ such that for every $(z,a) \in \Omega \times (1,\infty)$, $v \in T^{+}_{(z,a)}$, we have*

$$\left\langle v, \Delta^2 v - p\overline{U}^p_{(z,a)} v \right\rangle \geq \gamma \int_\Omega p\overline{U}^p_{(z,a)} v^2.$$

We are going to find a particular solution to the problem (P_f):

Lemma 7.2. *There exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $\|f\|_{C(\bar{\Omega})} < \varepsilon_0$, the problem (P_f) has a unique solution $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, satisfying*

$$\|u_0\|_{C^1} \leq C_0 \|f\|_{C(\bar{\Omega})}.$$

Moreover:

$$\frac{1}{2}\int_\Omega (\Delta u_0)^2 - \frac{1}{p+1}\int_\Omega u_0^{p+1} - \int_\Omega u_0 f < \frac{C_n}{2n}.$$

Proof. Let λ_1 be the first eigenvalue of the operator Δ^2. For a fixed $0 < \lambda < \lambda_1$, consider the function

$$h(t) = \begin{cases} |t^+|^p & \text{if } t < t_0 \\ \lambda |t| & \text{if } t \geq t_0 \end{cases}$$

where t_0 is chosen such that h is continuous. Hence, since h has a linear growth at infinity and it is non-resonant, we can always find a solution to the problem

$$\begin{cases} \Delta^2 u &= h(u) + f \quad \text{on} \quad \Omega \\ u = \Delta u &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Moreover, using Schauder estimates we get that $\|u_0\|_{C^1} \leq C_0 \|f\|_{C(\bar{\Omega})}$. Thus by taking $\varepsilon_0 > 0$ small enough, we have the desired result. □

Let us consider $f \geq 0$ in $C\left(\overline{\Omega}\right)$ with $f \neq 0$. We get, by using Hopf's lemma, that there exists $c_1 > 0$ such that

$$\frac{c_1}{2} < -\frac{\partial u_0}{\partial \nu} < c_1, \ \forall x \in \partial\Omega.$$

Therefore, there exists $c_2 > 0$ such that

$$u_0(x) \geq c_2 d(x), \ \forall x \in \partial\Omega.$$

Next we want to find solutions of the form $u_0 + v$. We define on $H^2(\Omega) \cap H_0^1(\Omega)$ the functional

$$J(u) = \frac{1}{2}\int_\Omega (\Delta u)^2 - \frac{1}{p+1}\int_\Omega \left((u_0+u)^+\right)^{p+1} - (p+1)u_0^p v - u_0^{p+1}.$$

We note that v is a critical point of J if and only if $u_0 + v$ is a positive solution to (P_f).

Lemma 7.3. *There exists $\varepsilon_1 > 0$ such that for $\|f\|_{C(\bar{\Omega})} < \varepsilon_1$, and $v \in H^2(\Omega) \cap H_0^1(\Omega)$, $v^+ \neq 0$, there exists a unique $t_v > t_1 > 0$ such that $J(tv)$ is increasing on $(t_1.t_v]$, decreasing on (t_v, ∞), and $J(t_v v) = \max_{t>0} J(tv)$.*

Proof. We give a sketch of the proof: since we can pick ε_1 small enough, it suffices to prove the result for $u_0 = 0$ and then argue by continuity. The functional J is now equal to I. Let us consider then

$$I(tv) = t^2 a_1 - t^{p+1} a_2$$

where $a_1 = \frac{1}{2}\int_\Omega (\Delta v)^2$ and $a_2 = \frac{1}{p+1}\int_\Omega (v^+)^{p+1}$. This is just a polynomial equation to study. The result follows. □

Now we define the Nehari manifold

$$\mathcal{S} = \left\{t_v v; v \in H^2(\Omega) \cap H_0^1(\Omega) - \{0\}\right\}$$

We have that for v in S, $J(v) > 0$, and $\langle \nabla J(v), v\rangle = 0$ if and only if $v \in \mathcal{S} \cup \{0\}$. Therefore the critical points of J are in $\mathcal{S}$.

Lemma 7.4. *The functional J satisfies the Palais-Smale condition on $\left(0, \frac{C_n}{n}\right)$.*

Proof. Let $\{u_j\}$ be a (P-S) sequence at the level $0 < d < \frac{C_n}{n}$. Then we know by using the concentration compactness lemma, that there exists $\overline{u}$, $z_1, ..., z_k \in \Omega$, $a_1, .., a_k \in \mathbb{R}_+^*$ such that

$$u_j = \overline{u} + \sum_{i=1}^{k} \overline{U}_{(z_i,a_i)} + o(1)$$

in the weak sense. After localization of the blow-up points, namely by testing against a function with support around the z_i, we get that the energy $J(u_j) \geq k\frac{C_n}{n}$. This happens if and only if $k = 0$ since $d < \frac{C_n}{n}$, therefore the convergence holds. □

We will need the following estimates.

Lemma 7.5. *There exists $r_0 > 2$, such that for every $(z, a) \in \overline{\mathcal{O}}(r_0)$*

$$\int_\Omega u_0 U^p_{(z,a)} \geq O(d(z)a^{-\frac{n-4}{2}}),$$

$$\left\| U_{(z,a)} \right\|_{L^{\frac{n}{n-4}}} \leq O(a^{-\frac{n}{2}} \left|\ln(a)\right|),$$

and

$$\int_\Omega u_0^{\frac{n}{n-4}} U_{(z,a)}^{\frac{n}{n-4}} \leq O(d(z)^{\frac{n}{n-4}} a^{-\frac{n}{2}} \left|\ln(a)\right|).$$

Proof. We have (see Appendix):

$$\int_\Omega u_0 U^p_{(z,a)} \geq c \int_\Omega d(x) \left(\overline{U}^p_{(z,a)} - p\theta_{(z,a)} \overline{U}^{p-1}_{(z,a)} \right),$$

and

$$\begin{aligned} \int_\Omega d(x)\overline{U}^p_{(z,a)} &\geq \frac{d(z)}{2} \int_{2d(z)>d(x)>\frac{d(z)}{2}} \overline{U}^p_{(z,a)} \\ &\geq \frac{d(z)}{2} \int_0^{d(z)} r^{n-1} \left(\frac{a}{1+a^2r^2} \right)^{\frac{n+4}{2}} dr \\ &\geq C\frac{d(z)}{2} a^{\frac{n-4}{2}} \end{aligned}$$

Moreover:

$$\int_\Omega \theta_{(z,a)} \overline{U}_{(z,a)}^{p-1} = o\left(a^{-\frac{n-4}{2}}\right)$$

Then the first inequality is proved. For the second one, we get:

$$\begin{aligned} \left\|U_{(z,a)}\right\|_{L^{\frac{n}{n-4}}}^{\frac{n}{n-4}} &\leq \left\|\overline{U}_{(z,a)}\right\|_{L^{\frac{n}{n-4}}}^{\frac{n}{n-4}} \\ &\leq \left\|\overline{U}_{(0,a)}\right\|_{L^{\frac{n}{n-4}}(B(0,C)}^{\frac{n}{n-4}} \\ &\leq Ca^{-\frac{n}{2}} \left|\ln(a)\right|, \end{aligned}$$

Finally, for the last inequality we have:

$$\int_\Omega u_0^{\frac{n}{n-4}} U_{(z,a)}^{\frac{n}{n-4}} \leq \int_\Omega u_0^{\frac{n}{n-4}} \overline{U}_{(z,a)}^{\frac{n}{n-4}},$$

and by using the fact that there exists $c > 0$ such that $u_0(x) \leq cd(z)$ whenever $|x - z| \leq d(z)$, we get the desired result. □

Now we define the following sets :

$$\mathcal{M} = \left\{U_{(z,a)}; (z,a) \in \Omega \times (1,\infty)\right\},$$

$$\mathcal{N} = \left\{\lambda U_{(z,a)}; (z,a) \in \Omega \times (1,\infty), \lambda \in \left(\frac{1}{2}, 2\right)\right\}$$

and we call $\bar{T}_{(z,a)}$ the tangent space to $\mathcal{N}$ at $U_{(z,a)}$. We also set $F_{(z,a)}^- = \left\{\lambda U_{(z,a)}; \lambda \in \mathbb{R}\right\}$ and $F_{(z,a)}^+ = \bar{T}_{(z,a)}^\perp$. Finally, let $F_{(z,a)} = F_{(z,a)}^+ \oplus F_{(z,a)}^-$ and K be the linear operator defined by

$$Ku = u_1 - u_2,$$

for any $u = u_1 + u_2$, with $u_1 \in F_{(z,a)}^+$ and $u_2 \in F_{(z,a)}^-$. We have the following

Lemma 7.6. *There exist positive constants ε_2, r_1, δ and C_1 such that for $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$, $(z,a) \in \overline{\mathcal{O}}(r_1)$ and $w \in B_\delta\left(U_{(z,a)}\right)$, it holds:*

$$\left\langle \Delta^2 v - p\left(w + u_0\right)_+^p v, Kv\right\rangle \geq C_1 \int_\Omega \left(\Delta v\right)^2, \tag{7}$$

for every $v \in F_{(z,a)}$.

Proof. Again it is enough to show this inequality for $u_0 = 0$ and then argue by continuity. So let us take $u_0 = 0$ and by contradiction, let us assume that the inequality does not hold. Then there exists a sequence $(z_k, a_k) \in \overline{\mathcal{O}}(r_0)$, $v_k \in F_{(z_k,a_k)}$ with $\|v_k\| = 1$, $d(z_k)a_k = r_k \longrightarrow \infty$, and $w_k \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\left\|w_k - U_{(z_k,a_k)}\right\| \longrightarrow 0$ as $k \longrightarrow \infty$, verifying

$$\limsup \left\langle \Delta^2 v_k - p\,(w_k)_+^p\, v_k, K v_k \right\rangle \leq 0.$$

We can always write $v_k = v_{k,1} + v_{k,2}$ according to the splitting of $F_{(z_k,a_k)}$. Since $r_k \longrightarrow \infty$, we have $\left\|\overline{U}_{(z_k,a_k)} - U_{(z_k,a_k)}\right\| \longrightarrow 0$. Therefore it is easy to see that

$$dist(F_{(z_k,a_k)}, span\left\{T_{(z_k,a_k)}, U_{(z_k,a_k)}\right\}) \longrightarrow 0.$$

Thus,

$$\lim_{k \longrightarrow \infty} dist\left(v_{k,1}, F^+_{(z_k,a_k)}\right) = 0$$

and by using Lemma (7.1.) we have for k big enough

$$\left\langle v_{k,1}, \Delta^2 v_{k,1} - p\left(w_k^+\right)^{p-1} v_{k,1}\right\rangle \geq \frac{\gamma}{2}\int_\Omega p\left(w_k^+\right)^{p-1} v_{k,1}^2.$$

Now let us assume for instance that $\|v_{k,1}\| > c$, for k big enough. Then there exists $\widetilde{c} > 0$, such that $\left\langle v_{k,1}, \Delta^2 v_{k,1} - p\left(w_k^+\right)^{p-1} v_{k,1}\right\rangle > \widetilde{c}$, and hence

$$\limsup \left\langle v_{k,1}, \Delta^2 v_{k,1} - p\left(w_k^+\right)^{p-1} v_{k,1}\right\rangle > \widetilde{c}.$$

By definition of $v_{k,2}$ we have

$$\left\langle v_{k,2}, \Delta^2 v_{k,2} - p\left(w_k^+\right)^{p-1} v_{k,2}\right\rangle \leq \|v_{k,2}\|\,(1-p)\,.$$

Therefore, knowing also that

$$\lim_{k \longrightarrow \infty} dist\left(v_{k,2}, F^-_{(z_k,a_k)}\right) = 0$$

we get that either $\|v_{k,1}\| = \|v_{k,2}\| = 0$, that is $\|v_k\| = 0$, or

$$\limsup \left\langle \Delta^2 v_k - p\,(w_k)_+^p\, v_k, K v_k \right\rangle > 0$$

which is a contradiction. Then the lemma holds. □

Proposition 7.7. *There exist $r_2 > 0$ and $C_2 > 0$ satisfying: for every $f \in C\left(\overline{\Omega}\right)$, $|f|_{C(\bar{\Omega})} < \varepsilon_2$, and each $(z,a) \in O\,(r_2)$, there exists $w_{(a,z)} \in S \cap B_{\frac{\delta}{2}}\left(U_{(z,a)}\right)$ such that*

$$\left\|w_{(a,z)} - U_{(z,a)}\right\| \leq C_2 \left\|\nabla J\left(U_{(z,a)}\right)\right\| \tag{8}$$

and

$$J\left(w_{(a,z)}\right) = \min_{u \in F^+_{(z,a)} \cap B_{\frac{\delta}{2}}(0)} \ \max_{v \in F^-_{(z,a)} \cap B_{\frac{\delta}{2}}(0)} J(U_{(z,a)} + u + v),$$

that is

$$J\left(w_{(a,z)} + v\right) \leq J\left(w_{(a,z)}\right) \leq J(w_{(a,z)} + u),$$

for every $u \in F^+_{(z,a)} \cap B_\delta\,(0)$ and $v \in F^-_{(z,a)} \cap B_\delta\,(0)$.

Proof. The existence of $w_{(a,z)}$ follows from the fact that $\left\|\nabla J\left(U_{(z,a)}\right)\right\| \longrightarrow 0$ as $d(z)a \longrightarrow \infty$ and (7): by Taylor expansion we see that the functional is convex in the direction of $F^+_{(z,a)}$ and concave in the direction of $F^-_{(z,a)}$. We have a saddle point, therefore $w\,(a,z)$ exists as in [2] and it is in $F_{(z,a)}$. Now we want to prove that

$$\left\|w_{(a,z)} - U_{(z,a)}\right\| \leq C_2 \left\|\nabla J\left(U_{(z,a)}\right)\right\|$$

We note first that since $w_{(a,z)}$ is a saddle point, we have $\langle \nabla J\,(w\,(a,z))\,, w\,(a,z)\rangle = 0$, then $w\,(a,z) \in S$. Using again a Taylor expansion we have

$$\left\langle \nabla J\left(w_{(z,a)}\right), K\left(w_{(z,a)} - U_{(z,a)}\right)\right\rangle =$$

$$= \left\langle \nabla J\left(U_{(z,a)}\right) + J''\left(U_{(z,a)}\right)\left(w_{(z,a)} - U_{(z,a)}\right), K\left(w_{(z,a)} - U_{(z,a)}\right)\right\rangle + o\left(\left\|w_{(a,z)} - U_{(z,a)}\right\|^2\right)$$

By noticing that

$$J''\left(U_{(z,a)}\right)h = \Delta^2 h - p\left|U_{(z,a)}\right|^{p-1} h,$$

and by using (7), we get

$$\left\langle \nabla J\left(w_{(z,a)}\right), K\left(w_{(z,a)} - U_{(z,a)}\right)\right\rangle \geq \left\langle \nabla J\left(U_{(z,a)}\right), K\left(w_{(z,a)} - U_{(z,a)}\right)\right\rangle +$$

$$+C_1\left\|w_{(a,z)} - U_{(z,a)}\right\|^2 + o\left(\left\|w_{(a,z)} - U_{(z,a)}\right\|^2\right)$$

But $\left\langle \nabla J\left(w_{(z,a)}\right), K\left(w_{(z,a)} - U_{(z,a)}\right)\right\rangle = 0$ by construction of $w_{(z,a)}$, therefore we obtain the desired result by a simple application of Cauchy-Schwartz inequality. □

Lemma 7.8. *Let $f = 0$. There exists $r_2 > 0$ such that for every $r > r_2$, there exists $c_r > \frac{C_n}{n}$ verifying*

$$J(w_{(z,a)}) > c_r,$$

for every $(z,a) \in \mathcal{O}(r)$.

Proof. By using the expansion of $\left\|U_{(z,a)}\right\|^2$ (see Appendix), we have the existence of $m > 0$, such that $\left\|U_{(z,a)}\right\| > m$ for $(z,a) \in \overline{O}(r_2)$. Let now $r \geq r_2$. Since $f = 0$ and $w_{(z,a)} \in S$, then $J(w_{(z,a)}) > \frac{C_n}{n}$ for all $(z,a) \in O(r)$. So let us assume by contradiction that

$$\inf_{(z,a)\in O(r)} J(w_{(z,a)}) = \frac{C_n}{n}.$$

Then there exists a sequence $(z_k, a_k) \in O(r)$, such that

$$\left\|w_{(z_k,a_k)} - \overline{U}_{(z'_k,a'_k)}\right\| \longrightarrow 0$$

where $(z'_k, a'_k) \in \Omega \times (1, \infty)$ is such that $d(z'_k)\, a'_k \longrightarrow \infty$. Thus

$$\left\| w_{(z_k,a_k)} - U_{\left(z'_k,a'_k\right)} \right\| \longrightarrow 0.$$

Using (8), we have $\left\| w_{(z_k,a_k)} - U_{(z_k,a_k)} \right\| < \frac{m}{4}$, since $(z_k, a_k) \in \overline{O}(r_2)$. This leads to $\left\| U_{(z_k,a_k)} - U_{\left(z'_k,a'_k\right)} \right\| \le \frac{m}{4}$. But we know that $d(z'_k)\, a'_k \longrightarrow \infty$ and $d(z_k)\, a_k = r$, therefore

$$\lim_{k\longrightarrow\infty} \left\| U_{(z_k,a_k)} - U_{\left(z'_k,a'_k\right)} \right\| \ge 2m$$

which is a contradiction. □

Lemma 7.9. *Let $f \in C\left(\overline{\Omega}\right)$, such that $|f|_{C\left(\overline{\Omega}\right)} < \varepsilon_2$, then there exist $r_3 > 0$, $C_3, C_4 > 0$ such that*

$$J(w_{(z,a)}) \le \frac{C_n}{n} + C_3\, (d(z)a)^{-(n-4)} - C_4 d(z) a^{\frac{n-4}{2}}$$

for every $(z,a) \in \overline{\mathcal{O}}(r_3)$.

Proof. For $(z,a) \in \overline{\mathcal{O}}(r_2)$, we take $\widetilde{U}_{(z,a)} = t_{U_{(z,a)}} U_{(z,a)}$ as in [19]. So we have $J\left(\widetilde{U}_{(z,a)}\right) = \max_{t\ge 0}\left(tU_{(z,a)}\right)$. Hence by construction of $w_{(z,a)}$, we have

$$J(w_{(z,a)}) \le J\left(\widetilde{U}_{(z,a)}\right).$$

We see that in fact, $t_1 < t_{U_{(z,a)}} < t_2$ for every $(z,a) \in \overline{O}(r_2)$ with t_1 and t_2 two fixed real numbers. Now

$$J\left(\widetilde{U}_{(z,a)}\right) \le \max_{t\ge 0}\left\{\frac{1}{2}\int_\Omega t^2\left(\Delta U_{(z,a)}\right)^2 - \frac{1}{p+1}\int_\Omega t^{p+1} U_{(z,a)}^{p+1}\right\} -$$

$$- \min_{t_1\le t\le t_2}\left\{\frac{1}{p+1}\int_\Omega \left(\left(u_0 + tU_{(z,a)}\right)^+\right)^{p+1} - t^{p+1}U_{(z,a)}^{p+1} - (p+1)t u_0^p U_{(z,a)} - u_0^{p+1}\right\},$$

after studying the polynomial equation

$$\frac{1}{2}\int_\Omega t^2\left(\Delta U_{(z,a)}\right)^2 - \frac{1}{p+1}\int_\Omega t^{p+1} U_{(z,a)}^{p+1},$$

and using the estimate in the Appendix, one can see that

$$\max_{t\geq 0}\left\{\frac{1}{2}\int_\Omega t^2\left(\Delta U_{(z,a)}\right)^2-\frac{1}{p+1}\int_\Omega t^{p+1}U_{(z,a)}^{p+1}\right\}=\frac{C_n}{n}+O(a^{-(n-4)})\leq c+O((ad(z))^{-(n-4)})$$

By using a Taylor expansion near zero and at infinity, we find that

$$\frac{1}{p+1}\int_\Omega\left(\left(u_0+tU_{(z,a)}\right)^+\right)^{p+1}-t^{p+1}U_{(z,a)}^{p+1}-(p+1)tu_0^pU_{(z,a)}-u_0^{p+1}\geq\int_\Omega u_0t^pU_{(z,a)}^p-$$

$$-C\int_\Omega t^{\frac{n}{n-4}}u_0^{\frac{n}{n-4}}U_{(z,a)}^{\frac{n}{n-4}}$$

Therefore

$$-\min_{t_1\leq t\leq t_2}\left\{\frac{1}{p+1}\int_\Omega\left(\left(u_0+tU_{(z,a)}\right)^+\right)^{p+1}-t^{p+1}U_{(z,a)}^{p+1}-\right.$$

$$\left.-(p+1)tu_0^pU_{(z,a)}-u_0^{p+1}\right\}\leq C\int_\Omega t_2^{\frac{n}{n-4}}u_0^{\frac{n}{n-4}}U_{(z,a)}^{\frac{n}{n-4}}-\int_\Omega u_0t_1^pU_{(z,a)}^p$$

By using the estimates in Lemma (7.5), we get

$$C\int_\Omega t_2^{\frac{n}{n-4}}u_0^{\frac{n}{n-4}}U_{(z,a)}^{\frac{n}{n-4}}-\int_\Omega u_0t_1^pU_{(z,a)}^p\leq O(d(z)^{\frac{n}{n-4}}a^{-\frac{n}{2}}\left|\ln(a)\right|)-O(d(z)a^{-\frac{n-4}{2}}),$$

therefore

$$\begin{aligned}J\left(\widetilde{U}_{(z,a)}\right)&\leq\frac{C_n}{n}+O((ad(z))^{-(n-4)})+O(d(z)^{\frac{n}{n-4}}a^{-\frac{n}{2}}\left|\ln(a)\right|)-O(d(z)a^{-\frac{n-4}{2}})\\&\leq\frac{C_n}{n}+O\left(ad(z)\right)^{-(n-4)}+Ad(z)^{\frac{n}{n-4}}a^{-\frac{n}{2}}\left|\ln(a)\right|-Bd(z)a^{-\frac{n-4}{2}}\end{aligned}$$

for A and B two positive constants. The conclusion follows. □

Now we define the set:

$$\mathcal{R}=\left\{(z,a)\in\overline{\mathcal{O}}(r_3);\ C_3\left(d(z)a\right)^{-(n-4)}<C_4d(z)a^{\frac{n-4}{2}}\right\}.$$

In this set we have $J\left(w_{(z,a)}\right)<\frac{C_n}{n}$ and thus Palais-Smale holds.

Proof. of Theorem (1.3.)

Now the proof of the theorem follows straightforward. In fact, using a minmax argument on the homology classes of $\mathcal{R}$, we obtain critical points of $(z,a) \longmapsto J\left(w_{(z,a)}\right)$, namely for each $[\alpha] \in H_*(\mathcal{R}) \cong H_*(\Omega)$, we have that the values c_α defined by

$$c_\alpha = \min_{\alpha\in[\alpha]} \max_{(z,a)\in\alpha} J\left(w_{(z,a)}\right)$$

are critical values of the function defined before. Moreover, these critical values corresponds to critical points belonging to the inside of the set $\overline{\mathcal{O}}(r_3)$, by Lemma (7.8). Now we use a transversality theorem (see Appendix) on the map defined by

$$\Psi(u,f) = \Delta^2 u - |u|^{p-1} u - f,$$

to show that these critical points are non-degenerate. This ends the proof.

□

8 Appendix

Here we will give a list of estimates that we used in some of the proofs. Here the O is for $\frac{d_i}{\lambda_i} \longrightarrow \infty$ and $\varepsilon_{12} \longrightarrow 0$. Let $\overline{U}_{(\xi,\lambda)}(x) = \left(\frac{\lambda}{1+\lambda^2|x-\xi|^2}\right)^{\frac{n-4}{2}}$, and for $i=1,2$, we will set $\overline{U}_i = \overline{U}_{(\xi_i,\lambda_i)}$. By using the same notation as in section 1, we set $U_i = P\overline{U}_i$, $\varepsilon_{12} = \frac{1}{\frac{\lambda_2}{\lambda_1}+\frac{\lambda_1}{\lambda_2}+\lambda_1\lambda_2|\xi_1-\xi_2|^2}$ and $d_i = dist(\xi_i,\partial\Omega)$.

Lemma 8.1. *Let $\theta_1 = \overline{U}_1 - U_1$, then :*

i) $0 \le \theta_1 \le \overline{U}_1$,

ii) $\theta_1(x) = H(\xi_1,x)\,\lambda_1^{\frac{n-4}{2}} + f_1(x)$

iii) $f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}}}{d_1^{n-2}}\right)$, $\frac{\partial}{\partial\lambda_1} f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}+1}}{d_1^{n-2}}\right)$

iv) $\frac{\partial}{\partial\xi_1} f_1(x) = O\left(\frac{\lambda_1^{\frac{n}{2}}}{d_1^{n-1}}\right)$

Lemma 8.2. *It holds:*

$$i)\ \|U_1\|^2 = \langle U_1, U_1\rangle = C_n - c_1 H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$ii)\ \langle U_2, U_1\rangle = c_1\left(\varepsilon_{12} - H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-2}{n-4}} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$$

$$iii)\int_\Omega U_1^{\frac{2n}{n-4}} = C_n - \frac{2n}{n-4}H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$iv)\ \int_\Omega U_1^{\frac{n+4}{n-4}}U_2 = \langle U_2, U_1\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n\geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n\leq 7\end{cases}.$$

Lemma 8.3. *We have the following estimates on* $\frac{\partial}{\partial\lambda}U_1$.

$$i)\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = \frac{n-4}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-4} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$ii)\int_\Omega U_1^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$iii)\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda_1}\varepsilon_{12} + \frac{n-4}{2}H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-2}{n-4}} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$$

$$iv)\int_\Omega U_2^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n\geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n\leq 7\end{cases}$$

$$v)\int_\Omega U_2\frac{1}{\lambda_1}\left(\frac{\partial}{\partial\lambda}U_1\right)^{\frac{n+4}{n-4}} = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\lambda}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n\geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n\leq 7\end{cases}$$

Lemma 8.4. *We have the following estimates on* $\frac{\partial}{\partial\xi}U_1$

$$i)\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = -\frac{1}{2}c_1 H(\xi_1,\xi_1)\lambda_1^{n-3} + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$ii)\int_\Omega U_1^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = 2\left\langle U_1, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + O\left(\frac{\lambda_1^{n-2}}{d_1^{n-2}}\right)$$

$$iii)\left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle = c_1\left(\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}\varepsilon_{12} - \frac{\partial}{\partial\xi_1}H(\xi_1,\xi_2)\lambda_1^{\frac{n-4}{2}}\lambda_2^{\frac{n-4}{2}}\right) + O\left(\varepsilon_{12}^{\frac{n-1}{n-4}}\frac{|\xi_1-\xi_2|}{\lambda_2} + \frac{\lambda_1^{n-2}}{d_1^{n-2}} + \frac{\lambda_2^{n-2}}{d_2^{n-2}}\right)$$

$$iv)\int_\Omega U_2^{\frac{n+4}{n-4}}\frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1 = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n\geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n\leq 7\end{cases}$$

$$v)\int_\Omega U_2\frac{1}{\lambda_1}\left(\frac{\partial}{\partial\xi_1}U_1\right)^{\frac{n+4}{n-4}} = \left\langle U_2, \frac{1}{\lambda_1}\frac{\partial}{\partial\xi_1}U_1\right\rangle + \begin{cases} O\left(\varepsilon_{12}^{\frac{n}{n-4}}\ln\left(\varepsilon_{12}^{-1}\right) + \frac{\lambda_1^n}{d_1^n}\ln\left(\frac{d_1}{\lambda_1}\right)\right) & \text{if } n\geq 8 \\ O\left(\varepsilon_{12}\ln\left(\varepsilon_{12}^{-1}\right)^{\frac{n-4}{n}}\frac{\lambda_1^{n-4}}{d_1^{n-4}}\right) & \text{if } n\leq 7\end{cases}$$

The proof of these estimates are similar to the ones in [3]. For more details we refer also to [7], [8] and [17].

Next we state a Transversality Theorem: see [] for the proof.

Theorem 8.5. *Let* X, Y *and* Z *be three Banach spaces, and* $\Psi : X \times Y \longrightarrow Z$ *be a* C^1 *map satisfying the following conditions: given* $z \in Z$

i) for every $(x, y) \in \Psi^{-1}(z)$, *the map* $D_x\ \Psi(x, y) : X \longrightarrow Z$ *is a Fredholm operator of index* 0.

ii)for every $(x, y) \in \Psi^{-1}(z)$, *the map* $D\ \Psi(x, y) : X \times Y \longrightarrow Z$ *is surjective.*

Then the set of $y \in Y$, *satisfying that* z *is a regular value of* $\Psi(\cdot, y)$, *is a residual set in* Y.

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