A theory of regularity structures

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IMA, January 16, 2013

Introduction

Aim: construct solution theories for very singular SPDEs.
Examples:

$$\begin{aligned} \partial_t h &= \partial_x^2 h + (\partial_x h)^2 + \xi\;, && (d=1)\\ \partial_t u &= \Delta u + g_{ij}(u)\partial_i u\,\partial_j u + f(u)\eta\;, && (d=2)\\ \partial_t \Phi &= \Delta\Phi - \Phi^3 + \xi\;. && (d=3) \end{aligned}$$

Here ξ is space-time white noise and η is spatial white noise.

KPZ (h) universal model for weakly asymmetric interface growth. Dynamical Φ^4_3 universal model for ferromagnets near critical temperature. PAM (u with $g=0$ and $f(u)=u$) universal model for weakly killed diffusions.

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In general $(f,g) \mapsto fg$ well-posed on $\mathcal{C}^\alpha \times \mathcal{C}^\beta$ only if $\alpha + \beta > 0$.
(Write $\mathcal{C}^\alpha$ for $B^\alpha_{\infty,\infty}$ if $\alpha < 0$.)

One has $\xi \in \mathcal{C}^{-\frac{d}{2}-1-\kappa}$ and $\eta \in \mathcal{C}^{-\frac{d}{2}-\kappa}$ for every $\kappa > 0$.

Expectation: $h \in \mathcal{C}^{\frac{1}{2}-\kappa}$, $u \in \mathcal{C}^{1-\kappa}$ and $\Phi \in \mathcal{C}^{-\frac{1}{2}-\kappa}$.

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Existing techniques

1. Dirichlet forms (Albeverio, Ma, Röckner, ...) known IM
2. WN Analysis (Øksendal, Rozovsky, ...) unphysical solutions
3. Variational methods (Prévot, Röckner, ...) regular noise
4. Viscosity solutions (Lions, Souganidis, ...) maximum principle

5. Subtract SC and make sense of some terms "by hand" (Da Prato, Debussche, ...) additive noise
6. RP in time (Gubinelli, Tindel, Friz, ...) regularity in space
7. RP in space (Hairer, Maas, Weber, ...) one-dimensional
8. Paraproducts (Gubinelli, Imkeller, Perkowski) quite specific

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Results

Write ξ_ε and η_ε for mollified versions. Consider

$$\begin{aligned}
\partial_t h &= \partial_x^2 h + (\partial_x h)^2 - C_\varepsilon + \xi_\varepsilon \;, \\
\partial_t u &= \Delta u + g_{ij}(u)\big(\partial_i u\,\partial_j u - C_\varepsilon \delta_{ij}\big) + f(u)\big(\eta_\varepsilon + \bar C_\varepsilon f'(u)\big) \;, \\
\partial_t \Phi &= \Delta \Phi - \Phi^3 + C_\varepsilon \Phi + \xi_\varepsilon \;.
\end{aligned}$$

Theorem: There are choices of C_ε so that solutions converge to a limit independent of the choice of mollifier.

Corollary: Rates of convergence, precise local description of limit, suitable continuity, etc. In the case of u, one obtains a chain rule / Itô formula depending on choice of constants.

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Main idea

Problem: Solutions are not “regular” enough so some products make no sense...

Insight: Notion of “regularity” is not suitable: polynomials are not a good model for our solutions.

Solution: Replace polynomials by a family of functions / distributions adapted to the problem at hand. Requires complete rethinking of the notion of Taylor expansion!

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Abstract ingredients

First ingredient: graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$ in which coefficients of expansion take place. Intuition: T_α represents elements that are "homogeneous of order α". One does not impose $\alpha \in \mathbf{N}$ or even $\alpha \geq 0$! Typically, each T_α is finite-dimensional, but could be a Banach space.

Second ingredient: Group G of invertible linear transformations of T with the property that, for $\tau \in T_\alpha$ and $\Gamma \in G$,

$$\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta \;.$$

Terminology: The triple (A, T, G) is a regularity structure.

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Concrete ingredients

Realisation of a regularity structure over $\mathbf{R}^d$ with scaling $\mathfrak{s} = (\mathfrak{s}_1, \ldots, \mathfrak{s}_d)$ (set $|x|_\mathfrak{s} = \sum |x_i|^{1/\mathfrak{s}_i}$):

$$(x,\tau) \mapsto \Pi_x \tau \in \mathcal{S}'(\mathbf{R}^d)\;, \qquad (x,y) \mapsto \Gamma_{xy} \in G\;,$$

Structure equations: $\Pi_x \Gamma_{xy} = \Pi_y$ and $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$.

Analytical bounds: For ψ a generic smooth test function, write

$$\psi_x^\lambda(y) = \lambda^{-|\mathfrak{s}|}\psi\Big(\frac{y_1 - x_1}{\lambda^{\mathfrak{s}_1}}, \ldots, \frac{y_d - x_d}{\lambda^{\mathfrak{s}_d}}\Big)\;.$$

One imposes that for $\tau \in T_\alpha$ and $x \in \mathbf{R}^d$ the model (Γ, Π) satisfies $\big(\Pi_x \tau\big)(\psi_x^\lambda) \lesssim \lambda^\alpha$ and $\|\Gamma_{xy}\tau\|_\beta \lesssim |x-y|_\mathfrak{s}^{\alpha-\beta}$.

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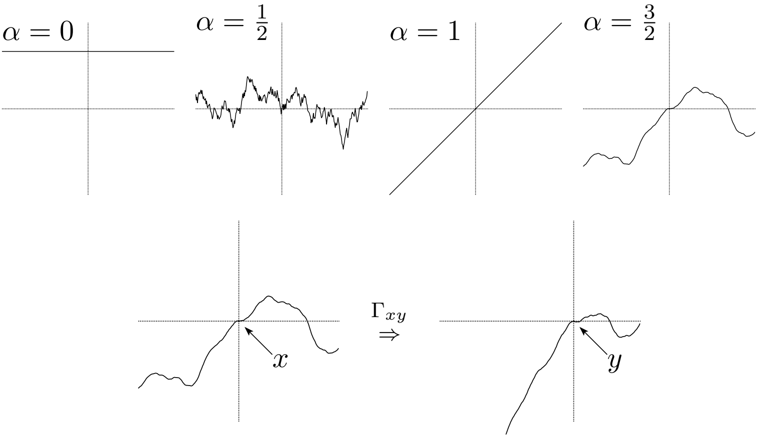
Example



$\alpha = 0$
$\alpha = \frac{1}{2}$
$\alpha = 1$
$\alpha = \frac{3}{2}$
x
Γ_{xy}
$\Rightarrow$
y

Measuring regularity

Take $f\colon \mathbf{R}^d \to T_\gamma^-$ ("jet" of order γ at each point), say $f \in \mathcal{D}^\gamma$ if

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \lesssim |x-y|_{\mathfrak{s}}^{\gamma-\alpha} \;.$$

Note: Space depends on the choice of model (Γ, Π)!

Theorem: There exists $\mathcal{R}\colon \mathcal{D}^\gamma \to \mathcal{S}'$ such that

$$\big|\big(\mathcal{R}f - \Pi_x f(x)\big)(\psi_x^\lambda)\big| \lesssim \lambda^\gamma \;.$$

If $\gamma > 0$, then $\mathcal{R}$ is unique. Terminology: the map $\mathcal{R}$ is a "reconstruction operator".

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Multiplication

Assume there is a bilinear product $(\tau, \bar{\tau}) \mapsto \tau \star \bar{\tau}$ on T such that

$$T_\alpha \star T_\beta \subset T_{\alpha+\beta}\;, \qquad \Gamma(\tau \star \bar{\tau}) = (\Gamma\tau) \star (\Gamma\bar{\tau})\;.$$

Write $f \in \mathcal{D}_\alpha^\gamma$ if $f \in \mathcal{D}^\gamma$ and $f(x) \in \bigoplus_{\beta \geq \alpha} T_\beta$.

Theorem: If $f_1 \in \mathcal{D}_{\alpha_1}^{\gamma_1}$ and $f_2 \in \mathcal{D}_{\alpha_2}^{\gamma_2}$, then $f_1 \star f_2 \in \mathcal{D}_\alpha^\gamma$ with

$$\alpha = \alpha_1 + \alpha_2\;, \qquad \gamma = (\alpha_1 + \gamma_2) \wedge (\alpha_2 + \gamma_1)\;.$$

Extension Theorem: Always possible to extend structure + model to satisfy assumptions.

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Composition

Take $f_1, \ldots, f_n$ in $\mathcal{D}_0^\gamma$ and $F\colon \mathbf{R}^n \to \mathbf{R}$ smooth. Write

$$f_i(x) = \bar{f}_i(x)\mathbf{1} + \tilde{f}_i(x)\ .$$

Idea: $\tilde{f}_i(x)$ describes small-scale fluctuations of f around $\bar{f}(x)$. With $\star$ as before, define $\hat{F}$ by

$$\hat{F}(f_1, \ldots, f_n)(x) = \sum_k \frac{D^k F(\bar{f}_1(x), \ldots, \bar{f}_n(x))}{k!} \tilde{f}(x)^{\star k}\ .$$

Theorem: $\hat{F}$ maps $\left(\mathcal{D}_0^\gamma\right)^n$ into $\mathcal{D}_0^\gamma$ and

$$\mathcal{R}\hat{F}(f_1, \ldots, f_n) = F(\mathcal{R}f_1, \ldots, \mathcal{R}f_n)\ .$$

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Schauder estimate

Take some kernel $K\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}$ with

$$K(x,y) \lesssim |x-y|_{\mathfrak{s}}^{\beta-|\mathfrak{s}|}\;,$$

(In suitable sense + similar bounds on derivatives.) Example: heat kernel with $\beta = 2$.

Classical Schauder: if $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$, then $Kf \in \mathcal{C}_{\mathfrak{s}}^{\alpha+\beta}$.

Wanted: Replace $\mathcal{C}^{\alpha}$ by $\mathcal{D}^{\alpha}$? Certainly requires sufficiently "rich" structure / model.

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Requirements

First requirement: Some abstract version of an integration map of order β. Take $\mathcal{I}\colon T \to T$ with the property that $\mathcal{I}T_\alpha \subset T_{\alpha+\beta}$. Also need $\bar{T} \subset T$, where $\bar{T}$ denotes abstract polynomials in d variables. (i.e. usual Taylor expansions). Then, one wants

$$\mathcal{I}\bar{T} = 0 , \qquad \big(\mathcal{I}\Gamma - \Gamma\mathcal{I}\big)\tau \in \bar{T} ,$$

for every $\tau \in T$ and $\Gamma \in G$.

Second requirement: The model Π should be adapted to $\mathcal{I}$ and K:

$$\big(\Pi_x \mathcal{I}\tau\big)(y) = \int K(y,z)\,\big(\Pi_x\tau\big)(dz) - \sum_{|k|_\mathfrak{s} < \alpha+\beta} \frac{(y-x)^k}{k!} D^k(\ldots)$$

Multi-level Schauder

Theorem: Under previous assumptions, one can build an operator $\mathcal{K}_\gamma \colon \mathcal{D}^\gamma_\alpha \to \mathcal{D}^{\gamma+\beta}_{(\alpha+\beta)\wedge 0}$ such that

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R} f \;.$$

Consequence: all of the above SPDEs can be lifted to a fixed point problem in $\mathcal{D}^\gamma_\alpha$ for suitable α and large enough γ.

One obtains unique short-time solutions for all of them (in bounded periodic domains). Solutions depend continuously on the model.

Non-trivial fact: One can build structures and models satisfying all relevant assumptions! Nice Hopf algebra structures appearing, but slightly different from Connes-Kreimer...

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Renormalisation

All above example: one has a canonical lift $\xi_\varepsilon \mapsto Z_\varepsilon = (\Gamma^{(\varepsilon)}, \Pi^{(\varepsilon)})$.
Problem: typically, Z_ε does not converge as $\varepsilon \to 0$.

Idea: canonical construction of a finite-dimensional "renormalisation group" $\mathfrak{R}$ acting on the space $\mathcal{X}$ of all realisations. Find sequence $M_\varepsilon \in \mathfrak{R}$ such that $\hat{Z}_\varepsilon = M_\varepsilon Z_\varepsilon \to \hat{Z}$.

Typically, one can choose $M_\varepsilon \in \mathfrak{R}_0 \subset \mathfrak{R}$. More symmetries $\Rightarrow$ smaller $\mathfrak{R}_0$. Limit unique modulo elements in $\mathfrak{R}_0$. (Examples: $\mathfrak{R}_0$ is a one-parameter subgroup of $\mathfrak{R}$.)

Eventually, identify solutions with renormalised model $\hat{Z}_\varepsilon$ with classical solutions to a modified SPDE.

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General picture

$$
\begin{tikzcd}
\mathcal{X} \arrow[loop left, "\mathfrak{R}"] & \times & \mathcal{C}^\alpha \arrow[r, "\mathcal{S}_R"] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\cdot"] & \mathcal{D}^\gamma \arrow[d, "\mathcal{R}"] \\
\mathcal{C} \arrow[u, "\Psi"] & \times & \mathcal{C}^\alpha \arrow[r, "\mathcal{S}_C"] & \mathcal{C}(\mathbf{R}_+, \mathcal{C}^\alpha) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\
\xi_\varepsilon & & u_0 &
\end{tikzcd}
$$

$\mathcal{S}_C$: Classical solution to the PDE with smooth input.
$\mathcal{S}_R$: Abstract fixed point using suitable regularity structure: locally jointly continuous!

Outlook

Many open questions remain:

1. Systematic way of choosing renormalisation procedure.
2. Obtain convergence results for discrete approximations (see H., Maas, Weber).
3. Equations in domains and / or non-constant coefficients.
4. Show convergence of near-critical Ising model to dynamical Φ^4_3. (Easier for Kac interactions.)
5. Show convergence of weakly interacting PAM to continuous PAM in dimensions 2 and 3.
6. Characterisation of the renormalisation group. Does it always yield a modified equation in closed form?

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