

Exact solution of quasi two and three dimensional quantum spin models

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Abstract

A class of quasi two and three dimensional quantum lattice spin models with nearest and next nearest neighbour interactions is proposed. The basic idea of construction is to introduce interactions in an array of XXZ spin chains through twisting transformation. The models belong to quantum integrable systems allowing explicit R -matrix solution. The eigenvalue problem can be solved exactly using some symmetry of the models.

Quantum integrable coupled spin chains with XXZ as well as Hubbard interactions were introduced through twisting transformation in [1]. Following that exact eigenvalue solutions of such models and their extensions were obtained in [2] and [3]. We generalise here this concept by coupling neighbouring pairs of anisotropic XXZ spin- $\frac{1}{2}$ chains [5], which results to a family of quasi two (2d) and three dimensional (3d) quantum spin models with nearest neighbour (NN) and next NN interactions. Since the twisting transformation preserves integrability, all the 2d and 3d systems thus constructed turn out to be quantum integrable. Generalising further the useful technique of [2] for representing such interactions as the operator dependent unitary transformations, we can solve exactly the eigenvalue problem of the higher dimensional spin models. One should mention here that an integrable 2d quantum spin model, its corresponding classical statistical system as well as models with internal degrees of freedom were also obtained earlier following a different route [6]. However the twisting transformation due to its nice symmetry makes our approach much simpler, allowing easy construction of quasi two, three and in principle any arbitrary dimensional integrable spin models.

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Let us start from spin- $\frac{1}{2}$ operators $\sigma_{(i,m)}^a, a = 1, 2, 3; i = 1, 2, \dots, N; m = 1, 2, \dots, M;$ given at site (i, m) in a 2d $N \times M$ lattice with the commutation relations

$$[\sigma_{(j,m)}^a, \sigma_{(k,n)}^b] = i\epsilon^{abc}\sigma_{(j,m)}^c\delta_{jk}\delta_{nm}. \quad (1)$$

The simplest 2d quantum model we propose may be given by the Hamiltonian

$$\begin{aligned} H &= \sum_{j=1}^N \sum_{m=1}^M \sigma_{(j,m)}^+ \sigma_{(j+1,m)}^- \left(\rho_{m0} + i(\rho_{m1}^+ \sigma_{(j,m+1)}^3 - \rho_{m1}^- \sigma_{(j+1,m-1)}^3) + \rho_{m2} (\sigma_{(j,m+1)}^3 \sigma_{(j+1,m-1)}^3) \right) \\ &+ \Delta_m \sigma_{(j,m)}^3 \sigma_{(j+1,m)}^3 + h.c., \end{aligned} \quad (2)$$

where the parameters ρ_{ma} involve only two independent coupling constants θ_{m-1m} and θ_{mm+1} between the neighbouring chains as

$$\rho_{m0} = \cos 2\theta_{m-1m} \cos 2\theta_{mm+1}, \quad \rho_{m1}^\pm = \cos 2\theta_{m\mp 1m} \sin 2\theta_{mm\pm 1}, \quad \rho_{m2} = \sin 2\theta_{m-1m} \sin 2\theta_{mm+1}. \quad (3)$$

We see from the above Hamiltonian that it is asymmetric in x and y directions, which gives the model a quasi 2d structure. There are XXZ type NN spin interactions along the x -direction indicated by the sites j , while varieties of different interactions along the y direction denoted by the index m .

The terms with coefficients ρ_{m1}^\pm represent NN interaction involving three spins, while that with ρ_{m2} stands for the next to NN coupling involving four spin operators. Note that the anisotropy parameters $\frac{\Delta_m}{\rho_{m0}}$ are different for different chains and moreover depend on the neighbouring chain couplings $\theta_{mm+1}, \theta_{m-1m}$. Therefore $\Delta_m \rightarrow 1$ limit can not recover the isotropic case.

We intend to prove that the model represents an exactly integrable quantum system exhibiting a hierarchy of commuting conserved integrals with the Hamiltonian being one of them. For this we first find the associated quantum R -matrix satisfying the Yang-Baxter equation (YBE)

$$R_{\langle a \rangle \langle b \rangle}(\lambda - \mu) R_{\langle a \rangle \langle j \rangle}(\lambda) R_{\langle b \rangle \langle i \rangle}(\mu) = R_{\langle b \rangle \langle j \rangle}(\mu) R_{\langle a \rangle \langle j \rangle}(\lambda) R_{\langle a \rangle \langle b \rangle}(\lambda - \mu), \quad (4)$$

considering $R_{\langle a \rangle \langle j \rangle}^0(\lambda) = \prod_{m=1}^M R_{(a,m)(j,m)}^{xxz}(\lambda, \eta_m)$ as the R -matrix of M number of integrable noninteracting XXZ spin chains. $R_{(a,m)(j,m)}^{xxz}(\lambda, \eta_m)$ corresponds to the m -th chain acting nontrivially on the space $(I_1 \otimes \dots \otimes I_{m-1} \otimes V_a \otimes \dots \otimes I_m) \otimes (I_1 \otimes \dots \otimes I_{m-1} \otimes V_j \otimes \dots \otimes I_m)$ and given by its well known form [5]

$$R_{(a,m)(j,m)}^{xxz}(\lambda, \eta_m) = w_0(\lambda, \eta_m) I_m \otimes I_m + w_3(\lambda, \eta_m) \sigma_{(a,m)}^3 \sigma_{(j,m)}^3 + w(\eta_m) (\sigma_{(a,m)}^+ \sigma_{(j,m)}^- + \sigma_{(a,m)}^- \sigma_{(j,m)}^+), \quad (5)$$

where

$$w_0 + w_3 = \sin(\lambda + \eta_m), \quad w_0 - w_3 = \sin \lambda, \quad w = \sin \eta_m. \quad (6)$$

We perform twisting [8] plus similarity transformations to construct a new R -matrix

$$R_{\langle a \rangle \langle j \rangle}(\lambda) = F_{\langle a \rangle \langle j \rangle} G_{\langle a \rangle \langle j \rangle} R_{\langle a \rangle \langle j \rangle}^0(\lambda) G_{\langle a \rangle \langle j \rangle}^{-1} F_{\langle a \rangle \langle j \rangle} \quad (7)$$

where

$$F_{\langle a \rangle \langle j \rangle} = \prod_m F_{(a,m)(j,m+1)}, \quad G_{\langle a \rangle \langle j \rangle} = \prod_m G_{(a,m)(j,m+1)} \quad (8)$$

with

$$F_{(a,m)(j,m+1)} = f_{(a,m)(j,m+1)}^\theta (f_{(a,m+1)(j,m)}^\theta)^{-1} \quad \text{and} \quad G_{(a,m)(j,m+1)} = f_{(a,m)(a,m+1)}^\alpha f_{(j,m)(j,m+1)}^\alpha \quad (9)$$

and the explicit expression through the spin operators as

$$f_{(a,m)(j,m+1)}^\theta = \exp[i\theta_{mm+1} \sigma_{(a,m)}^3 \sigma_{(j,m+1)}^3]. \quad (10)$$

and with θ_{mm+1} replaced by α_{mm+1} for $f_{(a,m)(j,m+1)}^\alpha$. To show the new R -matrix (7) to be a solution of the YBE (4) we insert it in the equation and notice that due to the specially designed forms (9) and (10) of F and G , the twisting matrices can be pushed through all the R^0 's without spoiling their structures and canceled from both the sides. As a result we are left with the R^0 matrices only, which satisfy the YBE. For example, for shifting the F factors through R^0 in the term $R_{\langle a \rangle \langle b \rangle}^0 F_{\langle a \rangle \langle j \rangle} F_{\langle b \rangle \langle j \rangle}$ appearing in the equation, we notice that the only term in R^0 which does not commute with the F factors is $\prod_{m=1}^M \sigma_{(a,m)}^+ \sigma_{(b,m)}^-$. However using the obvious relation

$$\sigma_{(a,m)}^\pm \exp[i\theta \hat{X}(j, m+1) \sigma_{(a,m)}^3] = \exp[\pm i\theta \hat{X}(j, m+1)] \exp[i\theta \hat{X}(j, m+1) \sigma_{(a,m)}^3] \sigma_{(a,m)}^\pm \quad (11)$$

with arbitrary operator $\hat{X}(j, m+1)$, we find that the extra factor

$$\exp[i(\theta_{m-1m} \sigma_{(j,m-1)}^3 - \theta_{mm+1} \sigma_{(j,m+1)}^3)]$$

produced due to the transition of $F_{(a,m)(j,m+1)} F_{(a,m-1)(j,m)}$ is canceled exactly by the factor created due to $F_{(b,m)(j,m+1)} F_{(b,m-1)(j,m)}$. Thus $R_{\langle a \rangle \langle b \rangle}^0$ remains unchanged after taking the F factors through it. Similar reasoning holds for $R_{\langle a \rangle \langle j \rangle}^0$ and $R_{\langle b \rangle \langle j \rangle}^0$. The factors related to the similarity transformation: $G_{(a,m)(b,m+1)}^{-1} G_{(a,m)(j,m+1)}$ etc. on the other hand, are partially canceled among themselves and the remaining ones commute trivially with the R^0 matrices. This shows that the transformed R -matrix (7) is also a solution of the YBE, which proves the quantum integrability of the system and guarantees that the transfer matrix $\tau(\lambda) = \text{tr}_{\langle a \rangle} \left(\prod_{j=1}^N R_{\langle a \rangle \langle j \rangle}(\lambda) \right)$ would generate mutually commuting set of conserved operators $C_n = \left(\frac{\partial^n}{\partial \lambda^n} \log \tau(\lambda) \right)_{|\lambda=0}$ [4, 5].

We construct now the Hamiltonian in the explicit form supposing $H \equiv C_1 = \tau'(0)\tau^{-1}(0)$ and using the definition of $\tau(\lambda)$ along with the expressions (7) and (5). Notice an important property of the R -matrix (7):

$$\begin{aligned} R_{\langle a \rangle \langle j \rangle}(0) &= cF_{\langle a \rangle \langle j \rangle} G_{\langle a \rangle \langle j \rangle} P_{\langle a \rangle \langle j \rangle} G_{\langle a \rangle \langle j \rangle}^{-1} F_{\langle a \rangle \langle j \rangle} \\ &= cF_{\langle a \rangle \langle j \rangle} G_{\langle a \rangle \langle j \rangle} G_{\langle j \rangle \langle a \rangle}^{-1} F_{\langle j \rangle \langle a \rangle} P_{\langle a \rangle \langle j \rangle} = cP_{\langle a \rangle \langle j \rangle}, \end{aligned} \quad (12)$$

which follows easily from that of $R_{\langle a \rangle \langle j \rangle}^0(0) = \prod_{m=1}^M \sin \eta_m P_{(a,m)(j,m)} = cP_{\langle a \rangle \langle j \rangle}$ and the symmetry $F_{\langle j \rangle \langle a \rangle} = F_{\langle a \rangle \langle j \rangle}^{-1}$, $G_{\langle j \rangle \langle a \rangle} = G_{\langle a \rangle \langle j \rangle}$. Using this along with the property of the permutation operator like $P_{\langle a \rangle \langle j \rangle} R'_{\langle a \rangle \langle j+1 \rangle}(0) = R'_{\langle j \rangle \langle j+1 \rangle}(0) P_{\langle a \rangle \langle j \rangle}$, and $P_{\langle a \rangle \langle j \rangle}^2 = I$ we get

$$\tau(0) = c \left(\left(\prod_{m=1}^M P_{(j,m)(j+1,m)} \right) P_{\langle j \rangle \langle j+2 \rangle} \dots \right) tr_a(P_{\langle a \rangle \langle j \rangle}) \quad (13)$$

and

$$\begin{aligned} \tau'(0) &= c \sum_{j=1}^N \sum_{m=1}^M \left(\prod_{l=m-1}^m (F_{(j,l)(j+1,l+1)} G_{(j,l)(j+1,l+1)}) R_{(j,m)(j+1,m)}'^{xxz}(0) \prod_{n \neq m} P_{(j,n)(j+1,n)} \right. \\ &\quad \left. \prod_{l=m-1}^m F_{(j,l)(j+1,l+1)} G_{(j,l)(j+1,l+1)}^{-1} \right) P_{\langle j \rangle \langle j+2 \rangle} \dots \times tr_a(P_{\langle a \rangle \langle j \rangle}), \end{aligned} \quad (14)$$

yielding

$$H = \sum_{j=1}^N \sum_{m=1}^M S_{jj+1}^m R_{(j,m)(j+1,m)}'^{xxz}(0) P_{(j,m)(j+1,m)} (S_{jj+1}^m)^{-1}, \quad S_{jj+1}^m = \prod_{l=m-1}^m F_{(j,l)(j+1,l+1)} G_{(j,l)(j+1,l+1)}. \quad (15)$$

Expressions (5),(10) through spin operators and their properties (11) reduce (15) finally to a family of 2d quantum spin models

$$\begin{aligned} H &= \sum_{j=1}^N \sum_{m=1}^M \sigma_{(j,m)}^+ \sigma_{(j+1,m)}^- \exp(i[(\theta_{mm+1} + \alpha_{mm+1})\sigma_{(j,m+1)}^3 - (\theta_{m-1m} + \alpha_{m-1m})\sigma_{(j+1,m-1)}^3 \\ &\quad + (\theta_{mm+1} - \alpha_{mm+1})\sigma_{(j+1,m+1)}^3 - (\theta_{m-1m} - \alpha_{m-1m})\sigma_{(j,m-1)}^3]) + \Delta_m \sigma_{(j,m)}^3 \sigma_{(j+1,m)}^3 + h.c., \end{aligned} \quad (16)$$

parametrised by the set $\{\theta_{mm+1}\}, \{\alpha_{mm+1}\}$. This also establishes that the family of models (16) belongs to the hierarchy of quantum integrable systems. Choosing the coupling parameters θ, α differently we can generate quasi 2d integrable models with rich varieties of interchain interactions. The simplest choice $\alpha_{mm+1} = \theta_{mm+1}$, using $(\sigma_{(a,m)}^3)^2 = I$ clearly yields the Hamiltonian (2), while $\alpha_{l+1} = (-1)^l \theta_{l+1}$ leads to the models like

$$H = \sum_{j=1}^N \sum_{m=1}^M (\sigma_{(j,m)}^+ \sigma_{(j+1,m)}^- S_m + \Delta_m \sigma_{(j,m)}^3 \sigma_{(j+1,m)}^3) + h.c., \quad (17)$$

with S_m taking alternate expressions as

$$\begin{aligned} S_m &= \exp(2i[\theta_{mm+1}\sigma_{(j,m+1)}^3 - \theta_{m-1m}\sigma_{(j,m-1)}^3]), \\ S_{m+1} &= \exp(2i[\theta_{m+1m+2}\sigma_{(j+1,m+2)}^3 - \theta_{mm+1}\sigma_{(j+1,m)}^3]). \end{aligned} \quad (18)$$

One may also choose further the M number of parameters θ_{mm+1} differently to get other types of interacting 2d models.

Next we address the eigenvalue problem: $H | \Psi \rangle = E | \Psi \rangle$ and show that by exploiting the symmetry of the operators $F_{(j,m)(j+1,m+1)}$, $G_{(j,m)(j+1,m+1)}$ appearing in (15) and using the known results for the XXZ spin chains one can solve this problem exactly. Indeed, using the relations like (11) one can conclude interestingly that our Hamiltonian (15) can be reduced to the form $H = SH_0S^{-1}$, where H_0 is a collection of exactly solvable XXZ chains and S is an unitary operator given by

$$S = \prod_{k,m,i,j(i<j)} f_{(k,m)(k,m+1)}^\alpha f_{(i,m)(j,m+1)}^\theta (f_{(i,m+1)(j,m)}^\theta)^{-1}, \quad (19)$$

with f^θ, f^α as in (10). Note that for the particular $m = 2$ case and $\alpha = 0$ the result is consistent with [2]. Therefore the eigenvalue problem of the quasi 2d Hamiltonian (2) can be reduced to that of H_0 with the same energy spectrum $E = \sum_m^M \sum_k^{M_m} \cos p_{k_m}$, where M_m are the number of spin excitations in the m -th chain. The eigenfunction on the other hand may be given by $S(\prod_m | \Psi_0^{(m)} \rangle)$, where $| \Psi_0^{(m)} \rangle$ is the known solution for a single XXZ chain solvable by the Bethe ansatz [7]. Note that, though our model can be solved through integrable XXZ chains, the results are not really equivalent. Indeed, the energy spectrum depends on the values of the unknown momentum parameters p_{k_m} , which in turn can be determined by the Bethe equations and these equations are not the same for the two models. In the 2d model due to the operator S extra factors like $\exp[i(\theta^{mm+1}M_{m+1} - \theta^{m-1m}M_{m-1})]$ appear in the determining Bethe equations signifying interactions between the neighbouring chains and this gives the set of momentum parameters a different value. The nature of the eigenfunction is also changed due to the same reason. At different sectors in the configuration space different phase factors arise in the wave functions due to the operator nature of S and at the boundaries of the sectors the wave functions suffer phase jumps resulting the occurrence of discontinuity at coinciding points. This unusual feature was observed also in case of the twisted Hubbard model [3].

We extend now this idea to construct quasi 3d quantum spin models by defining the R -matrix (7) with transforming operators

$$F_{\langle a \rangle \langle j \rangle} = \prod_{m,p} F_{(a,m,p)(j,m+1,p)}^{(m)} F_{(a,m,p)(j,m,p+1)}^{(p)}, \quad G_{\langle a \rangle \langle j \rangle} = \prod_{m,p} G_{(a,m,p)(j,m+1,p)}^{(m)} G_{(a,m,p)(j,m,p+1)}^{(p)} \quad (20)$$

Here as before the index $j = 1, \dots, N$ denotes the site number in the x -direction, $m = 1, \dots, M$ stands for the chain number along the y -direction, while the additional index $p = 1, \dots, L$ indicates the layer number along the z -direction. Therefore we may extend our above definitions to the 3d case as

$$F_{(a,m,p)(j,m,p+1)}^{(p)} = f_{(a,m,p)(j,m,p+1)}^{\theta_{pp+1}} (f_{(a,m,p+1)(j,m,p)}^{\theta_{pp+1}})^{-1}$$

and

$$G_{(a,m,p)(j,m,p+1)}^{(p)} = f_{(a,m,p)(a,m,p+1)}^{\alpha_{pp+1}} f_{(j,m,p)(j,m,p+1)}^{\alpha_{pp+1}} \quad (21)$$

with

$$f_{(a,m,p)(j,m,p+1)}^{\theta_{pp+1}} = \exp[i\theta_{pp+1}\sigma_{(a,m,p)}^3\sigma_{(j,m,p+1)}^3]. \quad (22)$$

for fixed chain index m and θ_{pp+1} as the interlayer coupling. Analogous relations as (21) hold also for $F_{(a,m,p)(j,m+1,p)}^{(m)}$ with fixed layer index p . All the above arguments for proving the R -matrix as the YBE solution also go through for the present extension and one can generate the related Hamiltonian of a family of quantum integrable 3d spin models. In the simplest case of $\alpha_{ab} = \theta_{ab}$ the explicit form of such models may be given as

$$H = \sum_{j=1}^N \sum_{m=1}^M \sum_{p=1}^L (\sigma_{(j,m,p)}^+ \sigma_{(j+1,m,p)}^- S_{m,p}^{jj+1} + \Delta_m \sigma_{(j,m,p)}^3 \sigma_{(j+1,m,p)}^3) + h.c. \quad (23)$$

with the interchain and interlayer interactions given as

$$S_{m,p}^{jj+1} = \exp(2i[(\theta_{mm+1}\sigma_{(j,m+1,p)}^3 - \theta_{m-1m}\sigma_{(j+1,m-1,p)}^3) + (\theta_{pp+1}\sigma_{(j,m,p+1)}^3 - \theta_{p-1p}\sigma_{(j+1,m,p-1)}^3)]). \quad (24)$$

Note that due to the factorised form of

$$S_{m,p}^{jj+1} = S_{(m+1,p)}^j (S_{(m-1,p)}^{j+1})^{-1} S_{(m,p+1)}^j (S_{(m,p-1)}^{j+1})^{-1}$$

with

$$S_{(m,p+1)}^j = \exp(2i[\theta_{pp+1}\sigma_{(j,m,p+1)}^3]) = \cos 2\theta_{pp+1} + i\sigma_{(j,m,p+1)}^3 \sin 2\theta_{pp+1} \quad (25)$$

etc. such interactions take place between NN and next NN chains in the same plane (with fixed p) as well as between NN and next NN layers in the same chain (with m fixed) and finally between such terms themselves.

The eigenvalue problem of such 3d models can also be solved similar to the 2d case by transforming the Hamiltonian to the noninteracting array of XXZ spin chains through unitary transformation.

Thus we have constructed and solved a class of quasi 2d and 3d quantum spin models and shown also their exact integrability. It should be obvious from our construction that the approach allows further extension of such integrable models to any arbitrary dimension.

Unlike the long ranged systems [9], the present class of 2d models includes only nearest and next nearest neighbour interactions. Therefore such models, possibly with more interesting twisting operators, might be helpful in constructing integrable higher dimensional physically significant models, which could not be achieved through the approach of tetrahedron equation [10]. It would also be important to construct higher dimensional Hubbard like models through similar technique [11].

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