

# An Order-Invariant Time Series Distance Measure

## [Position on Recent Developments in Time Series Analysis]

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Abstract: Although there has been substantial progress in time series analysis in recent years, time series distance measures still remain a topic of interest with a lot of potential for improvements. In this paper we introduce a novel *Order Invariant Distance* measure which is able to determine the (dis)similarity of time series that exhibit similar sub-sequences at arbitrary positions. Additionally, we demonstrate the practicality of the proposed measure on a sample data set of synthetic time series with artificially implanted patterns, and discuss the implications for real-life data mining applications.

## 1 INTRODUCTION

The distance between time series needs to be carefully defined in order to reflect the underlying (dis)similarity of such data. This is particularly desirable for similarity-based retrieval, classification, clustering, segmentation and other mining procedures of time series (Ding et al., 2008).

The choice of time series distance measure depends on the invariance required by the domain (Batista and Wang, 2011). Recent work has introduced techniques designed to efficiently measure similarity between time series with invariance to (various combinations of) the distortions of warping, uniform scaling, offset, amplitude scaling, phase, occlusions, uncertainty and wandering baseline (Batista and Wang, 2011).

In this study we propose a novel *Order-Invariant Distance (OID)* measure which is able to determine the (dis)similarity of time series that exhibit similar sub-sequences at arbitrary positions. We claim that order invariance is an important consideration for domains such as automotive engineering and smart home environments, where multiple sensors log contextual patterns in their naturally occurring order, and time series are compared to discriminate complex situations (Spiegel et al., 2011a; Spiegel et al., 2011b). To ensure the validity of our claim, we demonstrate the practicability and capabilities of the introduced *OID* measure on a sample data set of synthetic time series with artificially implanted patterns.

The rest of the paper is structured as followed. Section 2 presents popular distance measures which are frequently used to compare time series. Known invariance and important considerations for the design of time series distance measures are discussed in Section 3. Our proposed *Order-Invariant Distance* is introduced in Section 4. The capabilities of our presented *OID* measure are demonstrated in Section 5. Finally we conclude the paper in Section 6.

## 2 DISTANCE MEASURES

Suppose we have two time series,  $Q$  and  $C$ , of length  $n$ . To measure their similarity, we can use the *Euclidean Distance (ED)*:

$$ED(Q, C) = \sqrt{\sum_{i=1}^n (q_i - c_i)^2} \quad (1)$$

While *ED* is a simple measure, it is suitable for many problems (Lin et al., 2004). Nevertheless, in many domains the data is distorted in some way, and either the distortion must be removed before using *Euclidean Distance*, or a more robust measure must be used instead (Batista and Wang, 2011).

*Dynamic Time Warping (DTW)* allows a more intuitive distance measure for time series that have similar shape, but are not aligned in time (Keogh and Ratanamahatana, 2005). To align two sequences using *DTW* we construct a matrix which contains the

distances between any two points. A warping path  $W = w_1 \dots w_K$  is a contiguous set of matrix elements that defines a mapping between  $Q$  and  $C$  under several constraints (Keogh and Ratanamahatana, 2005):

$$DTW(Q, C) = \min \left\{ \sqrt[2]{\sum_{k=1}^K w_k} \right\} \quad (2)$$

This path can be found using dynamic programming to evaluate the following recurrence which defines the cumulative distance  $\gamma(i, j)$  as the distance  $d(i, j)$  found in the current cell and the minimum of the cumulative distances of the adjacent elements (Keogh and Ratanamahatana, 2005):

$$\gamma(i, j) = d(q_i, c_j) + \min \{ \gamma(i-1, j-1), \gamma(i-1, j), \gamma(i, j-1) \} \quad (3)$$

The *Euclidean Distance* between two sequences can be seen as a special case of *DTW* where both time series have same length and the warping path complies with the main diagonal of the distance matrix. However, the superiority of *DTW* over *ED* has been demonstrated by several authors (Ding et al., 2008; Keogh, 2003) for many data mining applications.

### 3 KNOWN INVARIANCE

Important considerations regarding time series distance include **amplitude invariance** and **offset invariance** (Batista and Wang, 2011). If we try to compare two time series measured on different scales they will not match well, even if they have similar shapes. Similarly, even if two time series have identical amplitudes, they may have different offsets. To measure the true underlying similarity we must first center and scale the time series (by trivial z-normalization).

Furthermore, **local scaling invariance** or rather **warping invariance** (Batista and Wang, 2011) should be taken into account when comparing time series. This invariance is necessary in almost all biological signals, which match only when one is locally warped to align with the other. Recent empirical evidence strongly suggests that *Dynamic Time Warping* is a robust distance measure which works exceptionally well (Ding et al., 2008).

In contrast to the *localized scaling* that *DTW* deals with, in many data sets we must account for **uniform scaling invariance** (Batista and Wang, 2011), where we try to match a shorter time series against the prefix of a longer one. The main difficulty in creating uniform scaling invariance is that we typically do not

know the scaling factor in advance, and are thus condemned to testing all possibilities within a given range (Keogh, 2003).

**Phase invariance** (Batista and Wang, 2011) is important when matching periodic time series such as heart beats. By holding one time series fixed, and testing all circular shifts of the other, we can achieve phase invariance.

In domains where a small sub-sequence of a time series may be missing we must consider **occlusion invariance** (Batista and Wang, 2011). This form of invariance can be achieved by selectively declining to match subsections of a time series. However, most real-life problems require **multiple invariance**.

### 4 ORDER-INVARIANT DISTANCE

This paper introduces an *Order-Invariant Distance* (*OID*) measure which is able to determine the (dis)similarity of time series which have different shapes, but exhibit similar sub-sequences in arbitrary order. For example, we can imagine the speed recorded during two different car drives, from home to the convenience store and back, where the signals exhibit the same location-dependent traffic situations (e.g. crosswalk, intersection, driveway, traffic light) in reverse order (refer to Figure 1) and are therefore similar in regard of **order invariance**.

Although order invariance may be an important consideration for other real-life data mining applications, relevant literature (Batista and Wang, 2011) is lacking a time series distance measure which is able to determine the (dis)similarity of signals that contain multiple similar events at arbitrary positions in time. Commonly used measures like *ED* and *DTW* are not designed to deal with order invariance, because they discriminate time series according to their shapes and fail to recognize cross-alignments between unordered sub-sequences. To this end, we developed an *Order Invariant Distance* measure which matches similar sub-sequences regardless of their order or location.

Our proposed *OID* measure is based on the *Cross Recurrence Plot* (*CRP*) approach (Marwan, 2008; Marwan et al., 2007) which tests for occurrences of similar states in two different systems, or rather time series (with same physical units). The data length of both time series can differ, leading to a non-square recurrence matrix  $R$ :

$$R_{i,j} = \Theta(\epsilon - ||q_i - c_j||) \quad (4)$$

where  $\Theta$  represents the *Heaviside* function (i.e.  $\Theta(x) = 0$  if  $x < 0$ , and  $\Theta(x) = 1$  otherwise),  $||\cdot||$  is a

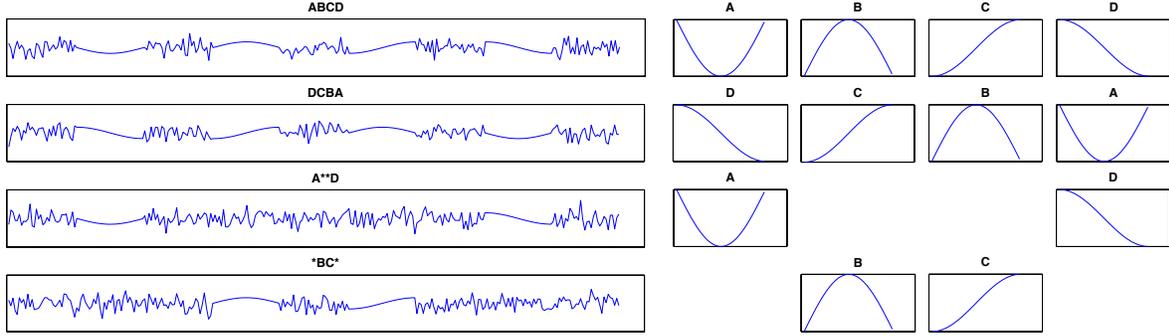


Figure 1: Sample data set of normally distributed pseudo-random time series (named as ABCD, DCBA, A\*\*D and \*BC\*, illustrated left) with artificially implanted sinus patterns (labeled as A to D, presented in their occurring order on the right).

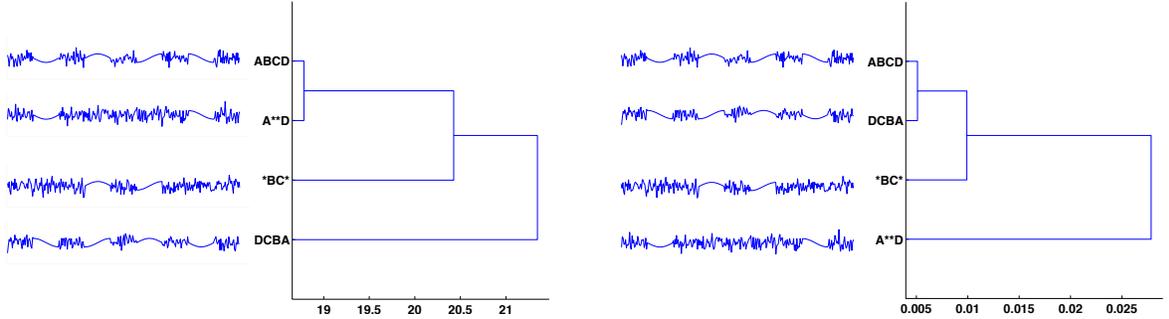


Figure 2: Agglomerative hierarchical cluster tree (dendrogram) of synthetic time series data (introduced in Figure 1) according to the Dynamic Time Warping (left) and our proposed Order-Invariant Distance (right), where the x-axis reveals the distance between the time series being merged and the y-axis illustrates the corresponding name and shape of the signal.

norm (i.e.  $L_2$ -norm) and  $\epsilon$  is a threshold distance that determines the radius of the similarity neighborhood.

A closer inspection of the *Cross Recurrence Plot* (matrix  $R$ ) reveals small-scale structures, which can be classified in single dots and lines of different direction (Marwan, 2008; Marwan et al., 2007). We are especially interested in lines that run parallel to the main diagonal, which occur when the trajectories of two sub-sequences are similar.

In order to go beyond the visual impressions yielded by *Cross Recurrence Plots*, several measures of complexity which quantify the small-scale structures have been introduced and are known as *Recurrence Quantification Analysis (RQA)* (Marwan, 2008; Marwan et al., 2007). These measures are based on the recurrence matrix  $R(\epsilon)$  considering a certain  $\epsilon$ -neighborhood. Commonly used measures include recurrence rate, determinism, entropy as well as average diagonal line length  $L$ :

$$L(\epsilon, l_{min}) = \frac{\sum_{l=l_{min}}^N l \cdot P(\epsilon, l)}{\sum_{l=l_{min}}^N P(\epsilon, l)} \quad (5)$$

where

$$P(\epsilon, l) = \sum_{i,j=1}^N \{ (1 - R_{i-1,j-1}(\epsilon)) \cdot (1 - R_{i+l,j+l}(\epsilon)) \cdot \prod_{k=0}^{l-1} R_{i+k,j+k}(\epsilon) \} \quad (6)$$

is the histogram of diagonal line of length  $l$ . The frequency and length of the diagonal lines are obviously related to a certain similarity between the dynamics of both time series. A measure based on the lengths of such lines can be used to find non-linear interrelations between two time series, which cannot be detected by the common cross-correlation function (Marwan et al., 2007). To this end, we propose the **reciprocal** average diagonal line length of a *Cross Recurrence Plot* as an *Order-Invariant Distance (OID)* measure for time series that exhibit similar sub-sequences at arbitrary positions in time.

$$OID = 1/L \quad (7)$$

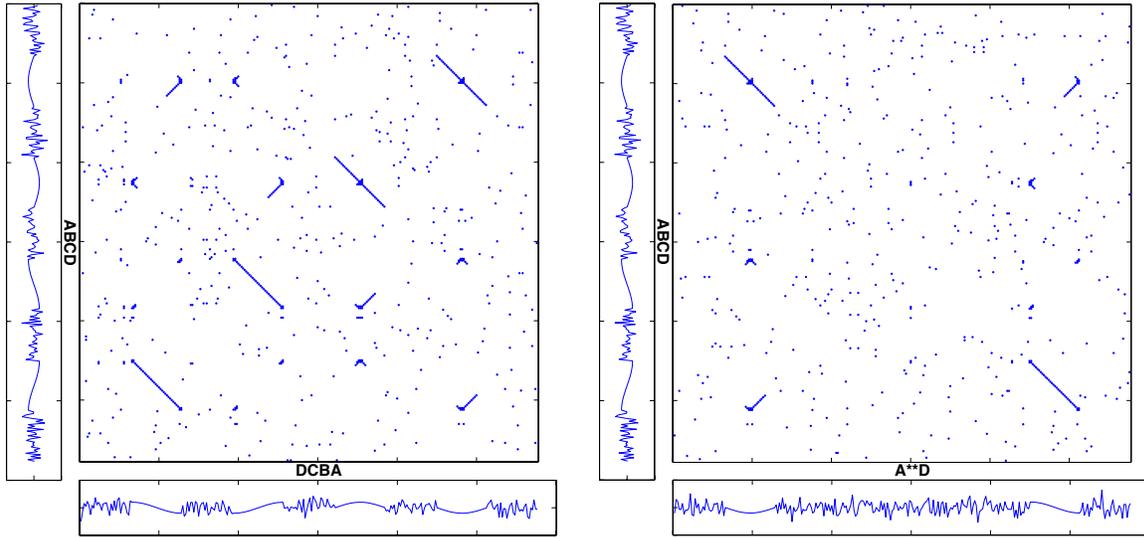


Figure 3: Cross Recurrence Plot (CRP) of synthetic time series ABCD and DCBA (left) as well as ABCD and A\*\*D (right) introduced in Figure 1. Note that the main diagonal runs from upper left to bottom right.

## 5 CASE STUDY

In this section we demonstrate the practicality of our proposed *Order-Invariant Distance* measure on a sample data set of synthetic time series. As illustrated in Figure 1, we consider four different normally distributed pseudo-random time series with artificially implanted sinus patterns. The first two time series comprise the same sub-sequences in reverse order, whereas the last two time series contain a subset of the artificially implanted signals.

Figure 2 shows a direct comparison of *Dynamic Time Warping* and our introduced *Order-Invariant Distance* measure. As expected, the hierarchical cluster tree generated by means of *DTW* indicates a relatively small distance between the time series ABCD, A\*\*D and \*BC\*, because they exhibit similar sub-sequences at the same positions. However, *DTW* treats the time series DCBA as an outlier, due to the artificially implanted patterns occurring in reverse order. In contrast, the *OID* measure considers the time series ABCD and DCBA as most similar, because the order of the matched patterns is disregarded. Furthermore, the dendrogram generated by means of *OID* reveals that the time series A\*\*D and \*BC\* are dissimilar to ABCD and DCBA, which is due to the fact that the overlap of same or similar sub-sequences is relatively small ( $\leq 50\%$ ).

Figure 3 illustrates the *Cross Recurrence Plots* of the time series ABCD and DCBA as well as ABCD and A\*\*D introduced in Figure 1. Lines parallel to the main diagonal (from upper left to bottom right) in-

dicate similar sub-sequences in both time series. The average diagonal line length  $L$  is higher for the *Cross Recurrence Plot* of the time series ABCD and DCBA than for the *CRP* of the pair ABCD and A\*\*D. Since we want similar time series to have a small distance, our proposed *Order Invariant Distance* measure is defined as the reciprocal of the average diagonal line length (refer to Equation 7).

The presented results serve to demonstrate the capabilities of the proposed *Order-Invariant Distance* measure, rather than to draw any conclusions without full evaluation. However, we strongly believe that the introduced *OID* measure is suitable to determine the (dis)similarity of time series which exhibit same or similar sub-sequences at arbitrary positions in time.

## 6 CONCLUSION

In this paper we introduced order invariance for time series, which has, to our knowledge, been missed by the community. Hence, we proposed a novel *Order Invariant Distance* measure which is able to determine the (dis)similarity of time series with similar sub-sequences at arbitrary positions in time.

In addition, we demonstrated the practicality of our proposed *OID* measure on a sample data set of synthetic time series with artificially implanted patterns. We strongly believe that order invariance is an important consideration for many real-life data mining applications. Our future work will include a full evaluation on publicly available data.

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