The Inviscid Limit and Boundary Layers for Navier-Stokes flows

Yasunori Maekawa
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto 606-8502, Japan
maekawa@math.kyoto-u.ac.jp

Anna Mazzucato
Mathematics Department
The Pennsylvania State University
University Park, PA, 16802 USA
alm24@psu.edu

Abstract

The validity of the vanishing viscosity limit, that is, whether solutions of the Navier-Stokes equations modeling viscous incompressible flows converge to solutions of the Euler equations modeling inviscid incompressible flows as viscosity approaches zero, is one of the most fundamental issues in mathematical fluid mechanics. The problem is classified into two categories: the case when the physical boundary is absent, and the case when the physical boundary is present and the effect of the boundary layer becomes significant. The aim of this chapter is to review recent progress on the mathematical analysis of this problem in each category.

1 Introduction

Determining the behavior of viscous flows at small viscosity is one of the most fundamental problems in fluid mechanics. The importance of the effect of viscosity, representing tangential friction forces in fluids, is classical and has been recognized for a long time. A well-known example is the resolution of D'Alembert' s paradox concerning the drag experienced by a body moving through a fluid, which is caused by neglecting the effect of viscosity in the theory of ideal fluids. For real flows like water and air, however, the kinematic viscosity is a very small quantity in many situation. Physically, the effect of viscosity is measured by a non-dimensional quantity, called the Reynolds number $Re := \frac{UL}{\nu}$, where U and L are the characteristic velocity and length scale in the flow, respectively, and ν is a kinematic viscosity of the fluid. Therefore, when U and L remain in a fixed range, the limit of vanishing viscosity is directly related to the behavior of high-Reynolds number flows. Hence, the theoretical treatment of the inviscid limit has great importance in applications and has been pursued extensively in various settings. The aim of this chapter is to give an overview of recent progress in the inviscid limit problem for incompressible Newtonian flows, although some results are also available in the important cases of compressible flows or non-Newtonian flows.

The governing equations for incompressible homogeneous Newtonian fluids are the Navier-Stokes equations

$$\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu = \nu \Delta u^\nu + f^\nu\,, \qquad \operatorname{div} u^\nu = 0\,, \qquad u^\nu|_{t=0} = u_0^\nu\,. \tag{1.1}$$

Here $u^\nu = (u_1^\nu, \cdots, u_n^\nu)$, $n = 2, 3$, and p^ν are the unknown velocity field and unknown pressure field. respectively at time t and position $x = (x_1, \cdots, x_n) \in \Omega$, respectively. In what follows, the domain $\Omega \subset \mathbb{R}^n$ will be either the whole space $\mathbb{R}^n$, or a domain with smooth boundary. Although many results can be stated for arbitrary dimension n, the physically relevant dimensions are $n = 2, 3$. The external force $f^\nu = (f_1^\nu, \cdots, f_n^\nu)$ is a given vector field, which will be typically taken as zero for simplicity, and $u_0^\nu = (u_{0,1}^\nu \cdots, u_{0,n}^\nu)$ is a given initial velocity field. Standard notation will be used throughout for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\nabla = (\partial_j, \cdots, \partial_n)$, $\Delta = \sum_{j=1}^n \partial_j^2$, $u^\nu \cdot \nabla = \sum_{j=1}^n u_j^\nu \partial_j$, and $\operatorname{div} f = \sum_{j=1}^n \partial_j f_j$. The symbol ν represents the kinematic viscosity of the fluid, which is taken as a small positive constant. The equations (1.1) are closed by imposing a suitable boundary condition, which will be specified in different context (see Section 3). The vorticity field is a fundamental physical quantity in fluid mechanics, especially for incompressible flows, and it is defined as the curl of the velocity field. Denoting $\omega^\nu = \operatorname{curl} u^\nu$, one has:

$$\omega^\nu = \partial_1 u_2^\nu - \partial_2 u_1^\nu \quad (n = 2)\,, \qquad\qquad \omega^\nu = \nabla \times u^\nu, quad(n = 3) \tag{1.2}$$

Above, we have identified the vector $omega^\nu = \varpi \mathbf{k}$, where $\mathbf{k} = (0, 0, 1)$, with the scalar ϖ, and called the latter also ω^ν with abuse of notation. For the reder's sake, we recall the vorticity equations:

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu - \omega^\nu \cdot \nabla u^\nu. \qquad u^\nu = K_\Omega[\omega^\nu], \tag{1.3}$$

where K_Ω stands for the Biot-Savart kernel in the domain Ω. The last term on the right is called the *vorticity stretching term* and it is absent in two space dimensions. This term depend quadratically in ω^ν and its presence precludes establishing long-time existence of Euler solutions in three space dimensions.

By formally taking the limit $\nu \to 0$ in (1.1), the Navier-Stokes equations are reduced to the Euler equations for incompressible flows

$$\partial_t u + u \cdot \nabla u + \nabla p = f\,, \qquad \operatorname{div} u = 0\,, \qquad\qquad u|_{t=0} = u_0\,. \tag{1.4}$$

The velocity field u of the Euler flows will be often written as u^0 in this chapter. Broadly speaking, a central theme of the inviscid limit problem is to understand when and in which sense the convergence of the Navier-Stokes flow u^ν to the Euler flow u is rigorously justified. Mathematically, the inviscid limit is a singular perturbation problem since the highest order term $\nu \Delta u^\nu$ is formally dropped from the equations of motion in the limit. For such a problem, in many cases the main issue is establishing enough *a priori* regularity for the solutions such that convergence is guaranteed via a suitable compactness argument. This singular perturbation for the Navier-Stokes equations provides a challenging mathematical problem, because of the fact the non-linearity contains derivatives of the unknown solutions, and the nonlocality coming from the pressure term. Indeed, even when the flow is two-dimensional and the fluid domain is the whole plane $\mathbb{R}^2$, the study of the inviscid limit problem becomes highly nontrivial if given data, such as initial data, possess little regularity. It should be emphasized here that working with nonsmooth data has an important motivation not only mathematically but also in applications, for typical structures of concentrated vorticities observed in turbulent flows, such as the vortex sheets, vortex filaments, vortex patches, are naturally modeled as flows with certain singularities. The problem for singular data but under the absence of physical boundary will be discussed in Section 2 of this chapter.

The inviscid limit problem becomes physically more important and challenging in the presence of a nontrivial boundary, where the viscosity effects are found to play a central role in general no matter how small the viscosity itself is, depending on the geometry and boundary conditions fort the flow. Recent developments of the mathematical theory in this case will be reviewed in Section 3. The main obstruction in analyzing the inviscid limit arises from the complicated structure of the flow close to the boundary. Indeed, due to the discrepancy between the boundary conditions in the Navier-Stokes equations and in the Euler equations, a boundary layer forms near the boundary where the effects of viscosity cannot be neglected even at very low viscosity. . The size and the stability property of the boundary layer crucially depend on the type of prescribed boundary conditions and also on the symmetry of the fluid domain and the flows, which are directly connected to the possibility of the resolution of the inviscid limit problem. These are discussed in details in Sections 3.1 - 3.3. The concept of a viscous boundary layer is first introduced by Prandtl [119] in 1904 under the no-slip boundary condition, that is, assuming the flow iadhere to the boundary. Since then, the theory of boundary layers has had a strong impact in fluid mechanics and also initiated fundamental approach in asymptotic analysis for singular perturbation problems in differential equations. The reader is referred to [122] for various aspects of the boundary layer theory in fluid mechanics. The basic idea is that the fluid region can be divided into two regions: a thin layer close to the boundary (boundary layer) where viscosity effect is significant, and the region outside this layer where the viscosity can be neglected and thus the fluid behaves like an Euler flow. As found by Prandtl, the thickness of the boundary layer is formally estimated as $\mathcal{O}(\sqrt{\nu})$, at least for no-slip boundary condition, which is a natural scale as a parabolic nature of the Navier-Stokes equations. In the case of the no-slip boundary condition the fundamental equations describing the boundary layer are the Prandtl equations, which will be focused in Section 3.4. Interestingly, the problem becomes most difficult in the case of the no-slip boundary condition, the most classical and physically justified type of boundary condition, due to the fact large gradients of velocity can form at the boundary, which my propagate in the bulk , giving rise to a strong instability mechanism for boundary layer in high frequencies. The rigorous description for the inviscid limit behavior of the Navier-Stokes flows is still largely open in many important situations, and has attracted several works in the literature. In this chapter standard notations in mathematical fluid mechanics will be used for spaces of functions and vector fields. For example, the spaces $C^\infty_{0,\sigma}(\Omega)$ and $L^p_\sigma(\Omega)$ are defined as

$$
\begin{aligned}
C^\infty_{0,\sigma}(\Omega) &= \{f \in C^\infty_0(\Omega)^n \mid \operatorname{div} f = 0 \ \text{ in } \Omega\}\,, \\
L^p_\sigma(\Omega) &= \overline{C^\infty_{0,\sigma}(\Omega)}^{\|f\|_{L^p(\Omega)}}\,, \qquad 1 < p < \infty\,.
\end{aligned}
$$

The Bessel potential spaces $H^s(\mathbb{R}^n)$, the Sobolev spaces $W^{s,p}(\Omega)$, $s \geq 0$, $1 \leq p \leq \infty$, and the space of Lipschitz continuous functions $\mathrm{Lip}\,(\mathbb{R}^n)$ are also defined as usual, and the n product space X^n will be often written as X for simplicity.

2 Inviscid limit problem without physical boundary

This section is devoted to the analysis of the inviscid limit problem for the Navier-Stokes equations when the fluid domain has no physical boundary. In this case the effect of the boundary layer is absent at least from the physical boundary, and the problem is more tractable and has been analyzed in various functional settings. Typically the Navier-Stokes flow is expected to converge to the Euler flow in the inviscid limit. Then the main interest here is the class (regularity) of solutions for which this convergence is verified and its rate of convergence in a suitable topology. To simplify the presentation, only the Cauchy problem in $\mathbb{R}^n$ will be the focus in this section, and therefore the external force in (1.1) or (1.4) will be taken as zero. To give an overview of known results, it will be convenient to classify the solutions depending on their regularity as follows.

(I) Regular solutions

(II) Singular solutions: (II-1) Bounded vorticity · Vortex patch (II-2) Vortex sheet · Vortex filament · Point vortices

Since the external force is assumed to be zero, the above classification is essentially for the initial data. Then, the typical case of "(I) Regular solutions" is that the initial data u^ν_0 and u_0 belong to the Sobolev space $H^s(\mathbb{R}^n)$ with $s > \frac{n}{2} + 1$, which is embedded in the space $C^1(\mathbb{R}^n)$. On the other hand, the category "(II) Singular solutions" corresponds to the case when the initial data are less regular than the C^1 class.

(I) Regular solutions. The verification of the inviscid limit in $\mathbb{R}^n$ for smooth initial data (e.g., better than the C^1 class) is classical. Indeed, it is proved in [55, 112] for $\mathbb{R}^2$ and in [127, 70] for $\mathbb{R}^3$, and in [38] for a compact manifold without boundary of any dimension. The following result is given in [105, Theorem 2.1], and provides the relation between the regularity of solutions and the rate of convergence. For simplicity, the result is stated here only for the case $u_0^\nu = u_0$ in (1.1) and (1.4), though the case $u_0^\nu \neq u_0$ is also given in [105].

Theorem 2.1. *Let $u_0^\nu = u_0 \in H^s(\mathbb{R}^n)$ with some $s > \frac{n}{2} + 1$. Let $u \in C_{loc}([0, T^*); H^s(\mathbb{R}^n))$ be the (unique) solution to the Euler equations, where $T^* > 0$ is the time of existence of the solution. Then, for all $T \in (0, T^*)$ there exists $\nu_0 > 0$, such that for all $\nu \in [0, \nu_0]$ there exists a unique solution $u^\nu \in C([0, T]; H^s(\mathbb{R}^n))$ to the Navier-Stokes equations. Moreover, it follows that*

$$\lim_{\nu \to 0} \|u^\nu - u\|_{L^\infty(0,T;H^s)} = 0\,, \qquad \|u^\nu - u\|_{L^\infty(0,T;H^{s'})} \leq C(\nu T)^{\frac{s-s'}{2}}\,,$$

for $s - 2 \leq s' \leq s$. Here, C depends only on u and T.

Since the time T^* in Theorem 2.1 is the time of existence for the Euler flow, the inviscid limit holds on any time interval in two space dimensions. The estimate in $H^{s'}(\mathbb{R}^n)$ for the case $s' = s - 2$ is obtained by a standard energy method, and the case $s - 2 < s' < s$ is derived from interpolation. The convergence in $H^s(\mathbb{R}^n)$ is more delicate, and a regularization argument for the initial data is needed in the proof.

(II) Singular solutions. (II-1) Bounded vorticity · Vortex patch: The Euler equations are uniquely solvable, at least locally in time, when the initial vorticity is bounded [145] or nearly bounded [146, 135]. Hence it is expected that the Navier-Stokes flow converges to the Euler flow in the vanishing viscosity limit also for such class of initial data. This class includes some important solutions, called vortex patches, which are typically vorticity fields defined as characteristic functions of bounded domains with smooth boundary. Theorem 2.1 cannot be applied for this class of solutions, since the condition that the vorticity be bounded is not enough to ensure the local-in-time H^s regularity of the velocity for $s > \frac{n}{2} + 1$. The next result is established in [26] for the two-dimensional case, where the effect of the singularity appears in the rate of convergence.

Theorem 2.2. *Let $u_0^\nu = u_0$ be an L^2 perturbation of a smooth stationary solution to the Euler equations (see [26, Definition 1.1] for the precise definition). If in addition $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ for some $1 < p < \infty$, then*

$$\|u^\nu - u\|_{L^\infty(0,T;L^2)} \leq C\|\omega_0\|_{L^\infty \cap L^2} (\nu T)^{\frac{1}{2}\exp(-C\|\omega_0\|_{L^\infty \cap L^2} T)}\,.$$

Thus, although the inviscid limit is still verified in $L^\infty(0, T; L^2(\mathbb{R}^2))$ even when the initial vorticity is merely in $L^\infty \cap L^2$, the upper bound on the rate of convergence in Theorem 2.2 decreases in time. This decrease, in fact, reflects the lack of Lipschitz regularity for the velocity field of the Euler flow, e.g. $u \in L^\infty(0, T; \mathrm{Lip}(\mathbb{R}^2))$. On the other hand, the Lipschitz regularity of the velocity is ensured for a class of vortex patches. Then, mentioned below, the rate of convergence can be estimated uniformly in time for this class. First, the definition of vortex patches is given as follows.

Definition 2.3. *Let $n = 2, 3$ and $0 < r < 1$. A vector field $u \in L^p_\sigma(\mathbb{R}^n)$, $2 < p < \infty$, is called a C^r vortex patch if the vorticity $\omega = \operatorname{curl} u$ has the form*

$$\omega = \chi_A \omega_i + \chi_{A^c} \omega_e\,, \tag{2.1}$$

where $A \subset \mathbb{R}^n$ is an open set of class C^{1+r} and ω_i, ω_e are compactly supported C^r functions (vector fields when $n = 3$). Here, χ_A and χ_{A^c} denote the characteristic functions of A and $A^c = \mathbb{R}^n \setminus A$, respectively, and the condition $\omega_i \cdot \mathbf{n} = \omega_e \cdot \mathbf{n}$ is assumed on ∂A, which is always valid when $n = 2$.

In Definition 2.3 the condition $\omega_i \cdot \mathbf{n} = \omega_e \cdot \mathbf{n}$ is assumed on ∂A so that the divergence free condition $\operatorname{div} \omega = 0$ is satisfied in the sense of distributions, which is necessary since $\omega = \operatorname{curl} u$. The simplest vortex patch in the two-dimensional case is the constant vortex patch introduced in [98], where $\omega_i = 1$, $\omega_e = 0$, and A is a bounded domain with C^{1+r} boundary. In the three-dimensional case, the constant vortex patch cannot exist because of the requirement $\operatorname{div} \omega = 0$. The classical reference for constant vortex patches is [25], where it is proved that if the initial vorticity is a C^r constant vortex patch then it remains to be a C^r constant vortex patch for all time; see also [14]. Moreover, the velocity is bounded in $L^\infty(0,T;\mathrm{Lip}\,(\mathbb{R}^n))$ for all $T > 0$. The regularity of the vorticity field up to the boundary of general two-dimensional vortex patch is shown in [34, 63]. The result for the two dimensional case is extended to the three dimensional C^{1+r} vortex patches by [44] but for a bounded time interval, as due to possible vortex stretching mechanism, the existence of the Euler solutions is only local in time. The reader is also referred to [64] for a Lagrangian approach proving the regularity up to the boundary, and to [36] for the result in a bounded domain, rather than in the whole space. Useful references about the study of vortex patches can be found in the recent paper [125].

The first result for the inviscid limit problem to vortex patches is given by [29], where it is proved that the Navier-Stokes flows in $\mathbb{R}^2$ starting from a constant vortex patch converges to the constant vortex patch of the Euler flows with the same initial data. In [29] the convergence is shown in the topology of $L^\infty(0,T;L^2(\mathbb{R}^2))$, $T > 0$, for the velocity fields, with the convergence rate $(\nu T)^{\frac{1}{2}}$. The authors of [29] proved in [30] the convergence of the vorticity fields in $L^p(\mathbb{R}^2)$, $2 \le p < \infty$. The optimal rate of convergence is then achieved for constant vortex patches in $\mathbb{R}^2$ by the results in [1], as stated below.

Theorem 2.4. *Let $u_0^\nu = u_0$ be a constant C^r vortex patch in $\mathbb{R}^2$. Then there exists $C > 0$ depending only on the initial data such that*

$$\|u^\nu(t) - u(t)\|_{L^2} \le C e^{Ce^{Ct}} (\nu t)^{\frac{3}{4}} (1 + \nu t) , \qquad t > 0 .$$

The rate $(\nu t)^{\frac{3}{4}}$ is optimal in the sense that if $u_0^\nu = u_0$ is a circular vortex patch, i.e., A is a disk, then the explicit computation leads to the bound from above and below (see [1, Section 4]):

$$C(\nu t)^{\frac{3}{4}} \le \|u^\nu(t) - u(t)\|_{L^2} \le C'(\nu t)^{\frac{3}{4}} , \qquad 0 < \nu t \le 1 .$$

The proof of Theorem 2.4 in [1] is based on an energy method combined with the Littlewood-Paley decomposition, i.e., a dyadic decomposition in the Fourier variables. In [1], the authors used the bound $u^\nu \in L^\infty(0,T;\mathrm{Lip}\,(\mathbb{R}^2))$ uniformly in $\nu > 0$, which was obtained in [32] and extended in [33] for general C^r vortex patches in $\mathbb{R}^n$, to study the vanishing viscosity limit. The reader is also referred to [60] for the extension of [32], and to [61] for the analysis when the constant patch has a singularity in its boundary. Later it was pointed out in [105] that the uniform Lipschitz bound of u^ν itself is not necessary and the optimal rate is proved only under the Lipschitz bound of u. More precisely, in [105, Theorems 3.2, 3.4], the inviscid limit is verified for general C^r vortex patches in $\mathbb{R}^n$ in terms of Besov spaces, as follows.

Theorem 2.5. *Let $u_0^\nu = u_0$ be a C^r vortex patch. Assume that $\operatorname{curl} u_0 \in \dot{B}^\alpha_{2,\infty}(\mathbb{R}^n)$, $0 < \alpha < 1$, and that $u \in L^\infty(0,T;\mathrm{Lip}\,(\mathbb{R}^n))$. Then any weak solution u^ν to the Navier-Stokes equations satisfies the estimate*

$$\|u^\nu(t) - u(t)\|_{L^2} \le C(\nu t)^{\frac{1+\alpha}{2}} , \qquad 0 < t < T .$$

Here C depends only on u and T.

Remark 2.6. In Theorem 2.5 the class of weak solutions is Leray-Hopf type when $n = 3$. When $n = 2$ the solution is not always energy finite due to the structure of the vortex patch, so the class of weak solutions needs to be suitably modified at this point. This is not an essential problem here, for the global well-posedness of the Navier-Stokes equations in $\mathbb{R}^2$ is known in various functional settings.

As usual, the norms of the Besov spaces are defined in terms of the Littlewood-Paley decomposition in the Fourier variables, cf. [105, Appendix]. The convergence rate in Theorem 2.5 depends on the exponent α for the regularity of the vorticity, rather than the value r for the regularity of the vortex patch. The estimate of this type was shown also in [1, Theorem 1.1] for the two-dimensional case. It is worthwhile to note that the space $L^\infty(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$ is continuously embedded in $\dot{B}^{\frac{1}{2}}_{2,\infty}(\mathbb{R}^n)$ (see the proof of [105, Lemma 4.2]). Then, a C^r vortex patch u satisfies $\mathrm{curl}\, u \in \dot{B}^{\frac{1}{2}}_{2,\infty}(\mathbb{R}^n)$ if ω_i, ω_e in Definition 2.3 belong to $C^{\frac{1}{2}}(\mathbb{R}^n)$ in addition. For such a case Theorem 2.5 gives the convergence rate $(\nu t)^{\frac{3}{4}}$, which is known to be optimal in general. The key observation here is that if $u \in L^\infty(0,T; \mathrm{Lip}\,(\mathbb{R}^n))$ and $\mathrm{curl}\, u_0 \in \dot{B}^\alpha_{2,\infty}(\mathbb{R}^n)$, $\alpha \in (0,1)$, then the $\dot{B}^\alpha_{2,\infty}(\mathbb{R}^n)$ regularity is preserved under the convection by the flow u, and $\mathrm{curl}\, u \in L^\infty(0,T; \dot{B}^\alpha_{2,\infty}(\mathbb{R}^n))$ holds (for example, see the argument of [1, Proposition 3.1]). The formal proof of Theorem 2.5 is given below, which showcases the role of the regularity of $\mathrm{curl}\, u$ in estimating the convergence rate, while a rigirous justification is given in [105], Since the difference $w = u^\nu - u$ solves the equations

$$\partial_t w - \nu \Delta w + u \cdot \nabla w + w \cdot \nabla u + w \cdot \nabla w + \nabla(p^\nu - p) = \nu \Delta u\,, \qquad \mathrm{div}\, w = 0\,,$$

a standard energy method, using integration by parts, leads to

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= -\nu\|\nabla w\|_{L^2}^2 - \langle w \cdot \nabla u, w\rangle_{L^2} - \nu\langle \nabla u, \nabla w\rangle_{L^2} \\ &\le -\nu\|\nabla w\|_{L^2}^2 + \|\nabla u\|_{L^\infty}\|w\|_{L^2}^2 - \nu\langle \nabla u, \nabla w\rangle_{L^2}\,. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2 inner product. The duality between $\dot{B}^{-\alpha}_{2,1}(\mathbb{R}^n)$ and $\dot{B}^\alpha_{2,\infty}(\mathbb{R}^n)$ implies

$$|\langle \nabla u, \nabla w\rangle_{L^2}| \le C\|\nabla u\|_{\dot{B}^\alpha_{2,\infty}}\|\nabla w\|_{\dot{B}^{-\alpha}_{2,1}} = C\|\nabla u\|_{\dot{B}^\alpha_{2,\infty}}\|w\|_{\dot{B}^{1-\alpha}_{2,1}}\,,$$

while $\mathrm{div}\, u = 0$ and an interpolation inequality (see [105, Lemma 4.1]) yield:

$$\|\nabla u\|_{\dot{B}^\alpha_{2,\infty}} = \|\mathrm{curl}\, u\|_{\dot{B}^\alpha_{2,\infty}}\,, \qquad \|w\|_{\dot{B}^{1-\alpha}_{2,1}} \le C\|w\|_{L^2}^\alpha \|\nabla w\|_{L^2}^{1-\alpha}\,,$$

respectively. Young's inequality finally gives:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 + \frac{\nu}{2}\|\nabla w\|_{L^2}^2 \le \|\nabla u\|_{L^\infty}\|w\|_{L^2}^2 + C\nu\|\mathrm{curl}\, u\|_{\dot{B}^\alpha_{2,\infty}}^{\frac{2}{1+\alpha}}\|w\|_{L^2}^{\frac{2\alpha}{1+\alpha}}\,.$$

Hence, in virtue of the bounds $u \in L^\infty(0,T; \mathrm{Lip}\,(\mathbb{R}^n))$ and $\mathrm{curl}\, u \in L^\infty(0,T; \dot{B}^\alpha_{2,\infty}(\mathbb{R}^n))$, the estimate in Theorem 2.5 follows from the Gronwall inequality.

The results of Theorems 2.2 - 2.5 show that the Euler flow describes the first order expansion of the Navier-Stokes flow in viscosity in the energy norm for a suitable class of vortex patches. It is then natural to investigate higher-order expansions; however, this problem becomes highly nontrivial since the vorticity of a patch is discontinuous across the boundary of the patch. Due to the smoothing effect of the viscosity term $\nu \Delta u^\nu$ in the Navier-Stokes equations, one has to introduce a fast scale in the higher order expansions which represents a viscous transition at the boundary of the vortex patch. Hence, the boundary layer analysis comes to play an important role. The first result in this direction is recently obtained by [125], where a complete asymptotic expansion is provided in powers of $(\nu t)^{\frac{1}{2}}$.

(II) Singular solutions. (II-2) Vortex sheets · Vortex filaments · Point vortices: In many physical situations the vorticity of flows concentrates on a very small region. Typical examples are vorticity fields called vortex sheets and vortex filaments or tubes, and mathematically they are formulated as a class of flows the vorticities of which are Radon measures supported on hypersurfaces (vortex sheets) or curves (vortex filaments).

The velocity field associated to a vortex sheet satisfies the Euler equations on both side of the sheet and its tangential components jump across the sheet. It is a natural model of flows with small viscosity after separation from rigid walls or corners. The simplest example in the two-dimensional case is the stationary flow

$$u = (u_1, 0)\,, \qquad u_1 = \begin{cases} -\frac{1}{2}\,, & x_2 > 0\,, \\ \frac{1}{2}\,, & x_2 < 0\,. \end{cases} \tag{2.2}$$

The associated vorticity field is then given by $\omega(x) = \delta(x_2)$, where $\delta(x_2)$ is the Dirac measure supported at $x_2 = 0$. However, this simple exact solution is linearly unstable to small periodic disturbances, known as the Kelvin-Helmholtz instability. More precisely, the linearization of the Birkhoff-Rott equation, which is an equivalent formulation to the Euler equations for vortex sheets under suitable conditions ([94]), around (2.2) has a solution growing exponentially like $e^{c|k|t}$ for each kth Fourier mode of the interface corrugation (see e.g. [97, Section 9.3] for details). As a result, the solvability of the Birkhoff-Rott equation is available only for analytic initial data [126, 19, 35], and it is known to be ill posed if one goes beyond the analytic framework [35, 20]. As for the relation between the analyticity and the regularity of the solution to the Birkhoff-Rott equations, see also [82, 139] and references therein.

The Navier-Stokes equations in $\mathbb{R}^n$ with vortex sheets as initial data are locally well-posed when $n = 3$ and globally well-posed when $n = 2$ since the velocity field in this class has enough local regularity to construct a unique solution. However, due to the underlying Kelvin-Helmholtz instability it is highly nontrivial to describe the behavior of the solutions in the inviscid limit. In [21] the equations for viscous profiles of vortex sheets are presented in the two dimensional case when the radius of curvature of the sheet is much larger than the thickness of the layer, and those equations are solved within the analytic category. The equations for viscous profiles in three dimensions are announced in [125, Eq. (253)]. In both cases the equations exhibit the loss of one derivative. A rigorous justification of the asymptotic expansion described in [21, 125] is still lacking even for analytic data. At the same time, in the two dimensional case, a weak solution to the Euler equations for vortex sheet initial data can be constructed with the help of viscous solutions, if the initial vorticity has a distinguished sign (see [97, Chapter 11] for detailed discussion). However, the verification of the inviscid limit in generality as in the case of vortex patches seems to be out of reach.

For a vortex filament, the vorticity field is concentrated on a curve in $\mathbb{R}^3$. The vortex filament is more singular than the vortex sheet. The vorticity field of a vortex filament naturally belongs to a Morrey space $\mathcal{M}^{\frac{3}{2}}(\mathbb{R}^3)$, which is an invariant space under the scaling: $f_\lambda(x) = \lambda^2 f(\lambda x)$, $\lambda > 0$ (for the precise definition of the space $\mathcal{M}^{\frac{3}{2}}(\mathbb{R}^3)$, see [53]). Vortex filaments posses an infinite energy in general. Although the vortex filament can belong to the class of functional solutions to the Euler equations introduced by [24], due to the strong singularity and underlying vortex stretching mechanism in three dimensional flows, there seems to be no general existence result for vortex filaments as solutions to the Euler equations. Alternatively, the self-induction equation (localized induction approximation) and its significant generalization have been used to understand the dynamics of vortex filaments; that dynamics is out of the scope of this chapter and the interested reader is referred to [97, Chapter 7] and references therein. On the other hand, the vortex filament belongs to an invariant space for the three-dimensional Navier-Stokes equations. Hence, a general theory is available for the unique existence of solutions to the Navier-Stokes equations with vortex filaments as initial data under smallness condition on the scale-invariant norm [53].

There are two typical vorticity distributions of vortex filaments: circular vortex rings with infinitesimal cross section, and exactly parallel and straight vortex filaments with no structural variation along the axis. The first one, called vortex ring for simplicity here, corresponds to an axisymmetric flow without swirl. Then the vorticity field of the vortex ring becomes a scalar quantity and is expressed as a Dirac measure supported at a point of $(r,z) \in (0,\infty)\times\mathbb{R}$ in the cylindrical coordinates. Recent results of [39] show the global existence of solutions to the Navier-Stokes equations with vortex ring as initial data, and in fact the uniqueness can be also proved in the class of axisymmetric flows without swirl; a more detailed overview about this topic is available in another chapter of this handbook, see [42]. The inviscid limit problem for vortex rings is attempted by [102], where the initial vortex ring is slightly regularized depending on the viscosity and the radius of the initial vortex ring is assumed to tend to infinity as the viscosity goes to zero. The result of [102] is significantly extended in [16], where the radius of the initial vortex ring can be taken independently of the viscosity. A more precise description of the result in [16] is given as follows. The velocity of the axisymmetric flow without swirl is expressed as $u^\nu = (u_r^\nu, 0, u_z^\nu)$ in cylindrical coordinates (r,ϕ,z), and the vorticity field is a scalar quantity, $\omega^\nu = \partial_z u_r^\nu - \partial_r u_z^\nu$. Then the evolution of ω^ν is described in the cylindrical coordinates as

$$\partial_t \omega^\nu + (u_r^\nu \partial_r + u_z^\nu \partial_z)\omega^\nu - \frac{u_r^\nu}{r}\omega^\nu = \nu\big(\frac{1}{r}\partial_r(r\partial_r\omega^\nu) + \partial_z^2\omega^\nu - \frac{\omega^\nu}{r^2}\big)\,. \tag{2.3}$$

Set

$$\Sigma_{(r_0,z_0)}(l) = \{(r,z)\in(0,\infty)\times\mathbb{R} \mid |r-r_0|^2 + |z-z_0|^2 < l^2\}\,.$$

Then a typical case of the result stated in [16, Theorem 1.1] leads to the next theorem.

Theorem 2.7. *Assume that a sequence of initial vorticities* $\{\omega_0^\nu\}_{0<\nu<\frac14}$ *satisfies*

$$\operatorname{supp}\omega_0^\nu \subset \Sigma_{(1,0)}(\nu^{\frac12})\,, \qquad 0\le \omega_0^\nu(r,z) \le \frac{M}{\nu|\log\nu|}\,, \qquad \int_{(0,\infty)\times\mathbb{R}} \omega_0^\nu \,\mathrm{d}r\,\mathrm{d}z = \frac{2a}{|\log\nu|}\,, \tag{2.4}$$

where $M>0$ *and* $a\in\mathbb{R}$ *are constants independent of* $\nu\in(0,\frac14)$. *Then there exists a sequence* $\{(r_\nu(t), z_\nu(t))\}_{0<\nu<\frac14}$ *in* $(0,\infty)\times\mathbb{R}$ *such that for any* $T>0$ *and* $f\in BC([0,\infty)\times\mathbb{R})$,

$$\begin{gathered}\lim_{\nu\to0} r_\nu(t) = 1\,, \qquad \lim_{\nu\to0} z_\nu(t) = \frac{at}{4\pi}\,,\\ \lim_{\nu\to0}\Big(|\log\nu| \int_{\Sigma_{(r_\nu(t),z_\nu(t))}(D_\nu)} \omega^\nu(t) f \,\mathrm{d}r\,\mathrm{d}z\Big) = 2af(1,\frac{at}{4\pi})\,, \qquad 0<t\le T\,,\\ \text{where}\ \ D_\nu = C\nu^{\frac12}\exp(|\frac12\log\nu|^\gamma)\,, \qquad 0<\gamma<1\,.\end{gathered} \tag{2.5}$$

Here ω^ν *is the (unique) solution to* (2.3) *with the initial data* ω_0^ν, *and* $C>0$ *is a constant independent of* ν.

Theorem 2.7 implies that, when the vorticity is initially sharply concentrated in an annulus, then it remains concentrated during the motion even in the presence of small viscosity, and as ν goes to zero the support of the vorticity evolves via a constant motion. The support condition in (2.4) is related to the standard boundary layer thickness, i.e., the order of $\nu^{\frac12}$. On the other hand, in (2.4) the vorticity is assumed to be logarithmically smaller than the standard scale. This is related to the fact that the velocity field for the circular vortex ring with infinitesimal cross section has a logarithmic singularity in the vertical component around the location of the ring (that is $r=1$ in the setting of Theorem 2.7).

In virtue of the additional smallness of the order $\mathcal{O}(|\log\nu|^{-1})$ in (2.4) the vorticity is translated with a finite speed in the inviscid limit, as described in (2.5).

Next, the other typical class of vortex filaments, i.e., parallel straight vortex filaments, is discussed. By symmetry, these vortex filaments keep their form under the motion, and then the problem is reduced to the motion of the points which are the intersection of the vortex filaments with the hyperplane $\{x_3 = 0\}$. These are called point vortices, linear combinations of Dirac measures in $\mathbb{R}^2$. For the inviscid flow the motion of N point vortices is formally described by the following Helmholtz-Kirchhoff system

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) = \frac{1}{2\pi} \sum_{j\neq i} \alpha_j \frac{(z_i(t) - z_j(t))^{\perp}}{|z_i(t) - z_j(t)|^2}\,, \qquad z_i(0) = x_i\,. \tag{2.6}$$

Here $i, j = 1, \cdots, N$ and each x_i denotes the initial location of the ith point vortex with circulation $\alpha_i \in \mathbb{R}$. It is possible to realize the point vortices as functional solutions, introduced in [24], to the two-dimensional Euler equations. A further relation of point vortices with the Euler equations is shown by [103]. On the other hand, the two-dimensional Navier-Stokes equations are known to be globally well-posed when the initial vorticity field is given by the point vortices of the form (2.7) below. In particular, the uniqueness of solutions is also available, which is proved in [54, 72] under smallness condition on the total variation $\sum_{i=}^{N} |\alpha_i|$, and in [40] without any smallness condition on the size of α_i.

The inviscid limit problem for point vortices is rigorously analyzed in [100, 101, 41] in the time interval in which the Helmholtz-Kirchhoff system is well-posed and vortex collisions do not occur. The next result is due to [41, Theorem 2].

Theorem 2.8. *Assume that the point vortex system* (2.6) *is well-posed on the time interval* $[0, T]$. *If the initial vorticity field is given by*

$$\operatorname{curl} u_0 = \sum_{i=1}^{N} \alpha_i\, \delta(\cdot - x_i)\,, \tag{2.7}$$

then the vorticity field $\omega^\nu = \operatorname{curl} u^\nu$ *of the solution to the Navier-Stokes equations* (1.1) *converges to* $\sum_{i=1}^{N} \alpha_i \delta(\cdot - z_i(t))$ *as* $\nu \to 0$ *in the sense of measures for all* $t \in [0, T]$, *where* $z(t) = (z_1(t), \cdots, z_N(t))$ *is the solution of* (2.6).

Theorem 2.8 shows that the distribution of the vorticity field of the Navier-Stokes flow starting from (2.7) is described by the point vortex system (2.6) in the inviscid limit. A similar result as Theorem 2.8 was first proved by [100, 101], where the initial point vortices are slightly regularized in relation with the viscosity. Theorem 2.8 provides information on the location of viscous vortices, while little information is available about the shape of the viscous vortices, i.e., about the viscous profile of each point vortex. Since the vorticity field ω^ν obeys the nonlinear heat-convection equations

$$\partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu\,, \qquad \omega^\nu|_{t=0} = \sum_{i=1}^{N} \alpha_i \delta(\cdot - x_i)\,,$$

once ω^ν is constructed by solving the above system, it is then decomposed as

$$\omega^\nu = \sum_{i=1}^{N} \omega_i^\nu\,, \tag{2.8}$$

where ω_i^ν is the solution to the heat-convection equations with the initial data which is a Dirac measure supported at x_i

$$\partial_t \omega_i^\nu + u^\nu \cdot \nabla \omega_i^\nu = \nu \Delta \omega_i^\nu \,, \qquad \omega_i^\nu|_{t=0} = \alpha_i \delta(\cdot - x_i) \,. \tag{2.9}$$

An interesting and important question is then to determine a correct asymptotic profile of each ω_i^ν. It is worthwhile to recall here that if the initial vorticity field is a Dirac measure supported at the origin, $\alpha\delta(0)$, then the unique solution to (1.1) is explicitly given by the Lamb-Oseen vortex with circulation α as follows: ([43]).

$$\alpha U^\nu(t,x) = \alpha \frac{x^\perp}{2\pi |x|^2}(1 - e^{-\frac{|x|^2}{4\nu t}}) \,, \qquad \big(\mathrm{curl}\, \alpha U^\nu\big)(t,x) = \frac{\alpha}{\nu t} G(\frac{x}{\sqrt{\nu t}}) \,, \qquad G(x) = \frac{1}{4\pi} e^{-\frac{|x|^2}{4}} \,. \tag{2.10}$$

Hence, the function $\alpha_i G$, a Gaussian with total mass α_i, is a natural candidate for the viscous profile for each ω_i^ν. The next result is due to [41, Theorem 3], which established the asymptotic expansion of ω_i^ν in the limit $\nu \to 0$, and shows that the asymptotic profile of each ω_i^ν is indeed given by the Gaussian in two dimensions.

Theorem 2.9. *Assume that the point vortex system* (2.6) *is well-posed on the time interval* $[0, T]$*, and let* $\omega^\nu = \mathrm{curl}\, u^\nu$ *be the vorticity field of the solution to* (1.1) *with the initial data whose vorticity is given by* (2.7)*. If* ω^ν *is decomposed as in* (2.8)*, then the rescaled profiles* w_i^ν *defined by*

$$\omega_i^\nu(t,x) = \frac{\alpha_i}{\nu t} w_i^\nu(t, \frac{x - z_i^\nu(t)}{\sqrt{\nu t}}) \tag{2.11}$$

satisfy the estimate

$$\max_{i=1,\cdots,N} \|w_i^\nu(t) - G\|_{L^1} \le C \frac{\nu t}{d^2} \qquad\qquad t \in (0, T] \,, \tag{2.12}$$

where $d = \min\limits_{t\in[0,T]} \min\limits_{i\neq j} |z_i(t) - z_j(t)| > 0$*. Here* z_i^ν *is the solution to a regularized point vortex system (see [41, Eq. (19)]) .*

In [41], the estimate (2.12) is proved in a stronger topology, that of a weighted L^2-space that is embedding in $L^1(\mathbb{R}^2)$. The behavior of z_i^ν in (2.11) is well approximated by z_i in the limit $\nu \to 0$ with an exponential order [41, Lemma 2], and hence, Theorem 2.9 verifies the asymptotic expansion of the viscous vortices around $z_i(t)$, $t \in (0, T]$. In order to show Theorems 2.8 and 2.9 ,one has to take into account the interaction between the viscous vortices in the inviscid limit. In particular, for each viscous vortex around z_i the velocity fields produced by the other vortices play a role in the background flow. The interactions results in a deformation of each viscous vortex. The analysis of this interaction is the key to prove Theorems 2.8 and 2.9, and in fact it requires a detailed investigation of higher-order expansions due to the strong singularity of the flows. This approach is validated in a rigorous fashion in [41, Theorem 4]. One consequence is that each viscous vortex is deformed elliptically through the interaction with the other vortices.

3 Inviscid limit problem with physical boundary

This section is devoted to the analysis of a viscous fluid at very low viscosity moving in a domain Ω with physical boundaries. The behavior of the fluid is markedly influenced by the types of boundary conditions imposed. The case when rigid walls may move only parallel to itself, as in Taylor-Couette flows, will be the focus in this section. In this case, the fluid domain (assumed to be smooth) is fixed and both the viscous, inviscid flows must satisfy the *no-penetration* condition at the boundary:

$$u \cdot \mathbf{n} = 0, \tag{3.1}$$

where u is the fluid velocity and $\mathbf{n}$ is the unit outer normal to the domain, respectively. For ideal fluids, the no-penetration condition is the only one that can be imposed on the flow. It is often referred to, somewhat incorrectly, as a *slip boundary condition*, because the fluid is allowed to slip, but no slip parameters are specified. For viscous fluids, there are several possible boundary conditions that are physically consistent. The simplest, and most difficult one from the point of view of the vanishing viscosity limit, is the *no-slip* boundary condition, where (3.1) is complemented with the following condition on the tangential fluid velocity at the boundary:

$$u_{\tan} = V, \tag{3.2}$$

where V is the velocity of the boundary and $u_{\tan}$ is the tangential component of the velocity. This is the boundary condition originally proposed by Stokes. If the friction force is prescribed at the boundary, then one obtains *Navier friction* boundary condition, which allow for slip to occur:

$$u \cdot \mathbf{n} = 0, \qquad [S(u) : \mathbf{n} + \alpha u]_{\tan} = 0, \tag{3.3}$$

where $\alpha \geq 0$ is the friction coefficient and $S(u)$ is viscous stress tensor, which coincides for Newtonian fluids with a multiple of the rate of strain tensor $(\nabla u + \nabla u^T)/2$ (by $M : v$ we mean $\sum_{ij} M_{ij} v_i$). Above, given a vector field v on the boundary of Ω, $v_{\tan}$ means the component of a vector v tangent to $\partial\Omega$. These boundary conditions are the ones originally proposed by Navier and derived by Maxwell in the context of gas dynamics. In absence of friction ($\alpha = 0$), the Navier boundary condition reduces to the condition that the tangential component of the shear stress be zero at the boundary. They are also called *stress-free* boundary conditions in the literature.

The term αu can be replaced by Au, where A is a (symmetric) operator. In particular, if A is the shape operator on the boundary of the domain, then the generalized Navier boundary conditions reduce to the following [11, 51]:

$$u \cdot \mathbf{n} = 0, \qquad \operatorname{curl} u \times \mathbf{n} = 0, \tag{3.4}$$

sometimes called *slip-without-friction* boundary conditions, as they reduce to (3.3) with $\alpha = 0$ on flat portions on the boundary. In two space dimensions, they are also referred to as *free* boundary conditions, given that the second condition reduces to $\operatorname{curl} u = 0$ [83], that is, there is no vorticity production at the boundary. In higher dimensions free boundary conditions lead to an overdetermined system, but are still compatible with the time evolution of the fluid since the boundary is characteristic if the initial data satisfies the same condition. Other types of slip boundary conditions based on vorticity can be imposed [12], but they have been studied less in the literature.

Impermeability of the boundary implies that boundary conditions need to be imposed on a characteristic boundary, which complicates the analysis further. If the boundary is not characteristic, as is the case of a permeable boundary with injection and suction, where the normal velocity at the boundary is prescribed and non-zero, the zero-viscosity limit holds at least for short time [132], as boundary layers can be shown to be very weak (of exponential type). The case of non-characteristic boundary is discussed in Section 3.3.

As already mentioned, one of the main obstructions to establishing the vanishing viscosity limit in the presence of boundaries is the formation of a viscous boundary layer, where the behavior of the flow cannot be approximated by that of an inviscid flow. A formal asymptotic analysis using $\sqrt{\nu}$ as small parameter leads to a reduced set of effective equations for the leading order velocity term in the boundary layer, the so-called Prandtl equations (we refer to [122] for a historical perspective). Significant advances have been made in the analysis of these equations, which exhibit instabilities, possible blow-up, and ill-posedness. The analysis on the Prandtl equations and the connection between solutions to Prandtl equations and the validity of the vanishing viscosity limit will be reviewed in Section 3.4.

The discussion here will be confined to the classical case of the vanishing viscosity limit for incompressible, Newtonian fluids, although some partial results are available in the important cases of compressible flows [143, 138, 124], MHD [141, 142], convection in porous media [77], and for non-Newtonian (second-grade) fluids [18, 95]. In the ensuing discussion, the case of unsteady flows in bounded, simply connected domains or a half space will be the main focus. The interesting case of exterior or multiply-connected domains, such as flow outside one or more obstacles, brings in additional difficulties, for example the infinite energy in the vorticity-velocity formulation of the fluid equations in 2D (see [65, 76, 123] and references therein).

The introduction of slip makes the vanishing viscosity limit more tractable, essentially because the viscous boundary layer is weak compared to the outer Euler solution and it is possible to obtain *a priori* bounds on higher norms that are uniform in viscosity. The discussion starts, therefore, with slip-type boundary conditions in Section 3.1 and continues with the more challenging case of no-slip boundary conditions in Section 3.2.

In the remainder of this section, the solution to the Navier-Stokes equations (1.1) is denoted by u^ν, and the solution to the Euler equations (1.4) is written as u^0. Unless otherwise stated, it is also assumed that the initial data for the Navier-Stokes equations is *ill-prepared*, that is, the initial velocity $u^\nu(0)$ is divergence free and satisfies the no-penetration condition at the boundary, but not necessarily viscous-type, e.g. friction or no-slip, boundary conditions. This assumption will allow, in particular, to take the same initial data for (1.1) and (1.4):

$$u^\nu(0) = u^0(0) = u_0,$$

although this assumption can, and will at times, be relaxed. When the data is ill prepared, there is a corner-type singularity at $t = 0$, $x \in \partial\Omega$ with two types of layers for the viscous evolution, an *initial layer* and the boundary layer. The initial layer is particularly relevant when the limit Euler flow is steady, as it affects vorticity production in the limit.

3.1 Case of slip-type boundary condition

If the viscous boundary layer is of size much smaller than predicted by Prandtl asymptotic theory, one expects that the fluid will effectively slip at the boundary (see [81] for a review of experimental results). This situation is more likely the rougher the boundary is. In fact, it can be shown rigorously that homogenization of the no-slip boundary condition on a highly oscillating boundary will give rise in the limit to slip boundary conditions (see [69, 17, 113] and references therein).

With Navier friction boundary conditions, the vanishing viscosity limit holds in two and three space dimensions under additional regularity conditions on the initial data, even if vorticity is produced at the boundary and the boundary is characteristic. In two space dimensions, the problem can be studied in the vorticity-velocity formulation, as the Navier friction condition gives rise to a useful boundary condition for vorticity, namely:

$$\omega_\nu = (2\kappa - \alpha) u^\nu_{\tan}, \tag{3.5}$$

where κ is the curvature of the boundary. The 2D Navier-Stokes initial value problem in vorticity-velocity formulation then reads:

$$\begin{cases} \omega_t^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu, & \text{on } (0,T)\times\Omega, \\ u^\nu = K_\Omega[\omega^\nu], & \text{on } (0,T)\times\Omega, \\ \omega^\nu(0) = \omega_0, & \text{on } \Omega \\ \omega_\nu = (2\kappa - \alpha) u^\nu_{\text{tan}}, \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.6}$$

with K_Ω the Biot-Savart operator associated to the domain Ω. A bound on the initial vorticity in L^p, $p > 1$, ensures, in particular, a uniform-in-time bound on the L^p-norm of the vorticity and hence global existence and uniqueness of a strong solution to (3.6). In general, a solution of this problem is not a weak solution of (1.1), even if $\omega_0 \in L^p(\Omega) \cap L^1(\Omega)$ for some $p > 1$, as u^ν is not of finite energy. However, this is the case if Ω is a bounded domain.

The zero-viscosity limit was first established for bounded initial vorticities and forcing in [27] and for unbounded forcing in [120]. This result was then extended to initial vorticities in L^p, $p > 2$, [93], and to initial vorticities in the Yudovich uniqueness class, [73], when the forcing is zero. To be precise the result in [93] is stated here, which applies to the largest class of 2D initial data for the Euler equations.

Theorem 3.1. *Let $\omega_0 \in L^p(\Omega)$, $p > 2$, and let $u_0 = K_\Omega[\omega_0]$. Let u^ν be the unique weak solution of* (1.1) *with $u^\nu(0) = u_0$. Then there exists a sequence $\nu_k \to 0$ and a distributional solution u^0 of the Euler equations* (1.4) *with initial data u_0, such that $u^\nu_k \to u^0$ strongly in $C([0,T]; L^2(\Omega))$ as $k \to \infty$.*

The proof relies on *a priori* bounds on higher Sobolev norms for the velocity that are uniform in viscosity. These bounds in turn allow by compactness, via an Aubin-Lions-type lemma, to pass to a limit along subsequences as $\nu \to 0$. Because compactness arguments are used, the proof does not give rates of convergence of u^ν to u^0. Uniqueness of the limit can be guaranteed *a posteriori* if ω_0 is sufficiently regular. The key ingredient in establishing the vanishing viscosity limit is an a priori bound on the vorticity in $L^\infty([0.T], L^p)$ uniform in viscosity (for e.g. $\nu \in (0,1]$). This estimate in turn is achieved through the use of the maximum principle, which requires estimating the L^∞-norm of the velocity at the boundary, except in the case of free boundary conditions. The condition $p > 2$ allow to obtain such an estimate via the Sobolev embedding theorem, but it is not expected to be sharp. A more natural condition would be $p > 1$, which would ensure passing to a limit in non-linear terms.

In three space dimensions, it is generally only possible to establish the vanishing viscosity limit for more regular Euler initial data, namely u_0 in the Sobolev space H^s, $s > \frac{5}{2}$, and only for the time of existence of the strong Euler solution. Weak *wild* solutions are generically too irregular to allow passage to the vanishing viscosity limit even in the full space (this point will be revisited in Section 3.2). For regular initial data u_0, the results in Theorem 3.1 extend to three space dimensions [66]. In fact, since the maximum principle for the vorticity no longer holds due to vortex stretching, a direct energy estimate on the velocity is employed and, consequently, the approximating sequence of Navier Stokes solutions u^{ν_k} can be taken in the Leray-Hopf class of weak solutions.

Uniform bounds in viscosity in higher Sobolev norms H^s, $s > \frac{5}{2}$, do not hold, due to the presence of a boundary layer, except in the case of free boundary conditions. Nevertheless, it is possible to show convergence of the Navier-Stokes velocity u^ν to the Euler solution u^0 uniformly in space and time, but utilizing so-called co-normal Sobolev spaces [106]. These are Sobolev spaces that, at the boundary, give control on tangential derivatives, defined as

$$H^m_{\text{co}}(\Omega) = \{f \in L^2(\Omega), Z^\alpha f \in L^2(\Omega), |\alpha| \leq m\}, \tag{3.7}$$

where Z^α is a vector-valued differential operator which is tangent to $\partial\Omega$. Such co-normal spaces will be employed in studying the no-slip case as well in Section 3.2. Because of the Navier boundary condition, it is possible to obtain a bound in $L^\infty((0,T)\times\Omega)$ uniform in viscosity on the full gradient of the Navier-Stokes velocity using co-normal spaces of high enough regularity. Then, this Lipschitz control on the Navier-Stokes solution allows to pass to the limit as viscosity vanishes by using compactness again. Set

$$V^m = \{f \in H^m_{\text{co}}(\Omega), \nabla f \in H^m_{\text{co}}(\Omega), \operatorname{div} f = 0\}$$

Then the convergence result in [106] is stated as follows.

Theorem 3.2. *Fix $m \in \mathbb{Z}_+$, $m > 6$. Let $u_0 \in V^m$. Assume in addition that $\nabla u_0 \in W^{1,\infty}_{co}$. Let $u^\nu \in C([0,T], V^m)$, be the strong solution of the Navier-Stokes equations* (1.1) *with initial data u_0 and boundary conditions* (3.3)*. Then, there exists a unique solution to the Euler equations* (1.4) *with initial data u_0 and boundary condition* (3.1)*, $u^0 \in L^\infty((0,T), V^m)$, such that $\nabla u^0 \in L^\infty((0,T), W^{1,\infty})$ and such that*

$$\|u^\nu - u^0\|_{L^\infty((0,T)\times\Omega)} \to 0, \qquad \text{as } \nu \to 0.$$

Note that in Theorem 3.2 the high regularity of the initial data is needed to control the pressure at the boundary.

One can interpret the uniform convergence of Theorem 3.2 in terms of boundary layer analysis. For simplicity the boundary is assumed to be flat, or the domain Ω is locally identified with the half space $\mathbb{R}^n_+$ by writing a point $x \in \Omega$ as $x = (x', z)$ with $x' \in \mathbb{R}^{n-1}$, where $n = 2, 3$ is the space dimension again, and $z > 0$. Thus, the boundary of Ω is identified locally with $z = 0$. Then, the following asymptotic expansion of the viscous velocity holds in the energy space $L^\infty([0,T], L^2(\Omega))$ [67]:

$$u^\nu(t,x) = u^0(t,x) + \sqrt{\nu}U(t,x',\frac{z}{\sqrt{\nu}}) + O(\nu), \tag{3.8}$$

where U is a smooth, rapidly decreasing boundary layer profile on $\mathbb{R}^n_+$. Hence, the boundary layer has the same width as predicted by the Prandtl theory for the case of no-slip boundary condition, but small amplitude. By contrast, the amplitude of the boundary layer corrector to the Euler velocity can be of order one if no-slip boundary conditions are imposed. The validity of the expansion (3.8) implies, in particular, that in general one cannot expect strong convergence in high Sobolev norm, since then, by the trace theorem, the limit Euler solution would satisfy the Navier-slip boundary condition (the so-called *strong zero-viscosity limit*). For the case of slip without friction, it was shown in [11] that the boundary condition (3.4) is not necessarily preserved under the Euler evolution in three space dimensions if the boundary is not flat. The strong zero-viscosity limit does hold for free boundary conditions, hence for arbitrary smooth domains in two space dimensions, and for slip-without friction boundary conditions on domains with flat boundary [8, 9, 140, 10]. In fact, only the initial data need to satisfy the stronger free condition at the boundary [13]. It is interesting to note that the main difficulty in dealing with non-flat boundaries comes from the non-vanishing of certain integrals in the energy estimate due to the convective term $u^\nu \cdot \nabla u^\nu$. In the no-slip case, the interaction between convection and the boundary layer is thought to be a main obstruction to the validity of the zero-viscosity limit.

The last part of this subsection is about an approach to the zero viscosity limit that yields rate of convergence in viscosity. The point of departure is the expansion (3.8). If this expansion is valid, then $u^0 + \sqrt{\nu}V$ approximates the Navier-Stokes solution. One can, therefore, define an approximate Navier-Stokes solution u^ν_{approx} in terms of an *outer solution* u^ν_{ou} valid away from the walls, and an *inner solution* u^ν_{in} valid near the walls (cf. [136, 84]). The parabolic nature of the Navier-Stokes equations suggests that the viscous effects are felt in a thin layer close to the boundary of width $\sqrt{\nu}$ (this idea goes back to the original work of Prandtl in fact, again the reader is referred to [122] for more details). For simplicity Ω is assumed to be a half-space, as in (3.8). To define the inner solution on a fixed domain independent of viscosity, it is convenient to introduce the stretched variable $Z = z/\sqrt{\nu}$. If the zero-viscosity limit holds, one then expects that a regular asymptotic expansion for u^ν_{in} is valid, that is:

$$u^\nu_{\text{in}}(t,x) = \sum_{k=0}^{\infty} \nu^{\frac{k+1}{2}}\, \theta^\nu_k(t,x',Z),$$

where θ^ν_k is the kth-order *corrector* to the outer flow. It should be stressed that the correctors are not assumed to be independent of viscosity and their amplitude is dictated by the equations of motions and by the boundary conditions. Similarly, the outer solution should have a regular expansion of the form:

$$u^\nu_{\text{ou}}(t,x) = \sum_{k=0}^{\infty} \nu^{\frac{k}{2}}\, u^\nu_k(t,x',z),$$

with u^ν_0 independent of ν. In fact, $u^\nu_0 = u^0$, the Euler solution. Consistency of the formal asymptotic expansion gives effective equations for the flow correctors, together with boundary and initial conditions, from the Navier-Stokes and Euler equations. The goal is then to derive the regularity of the correctors and their decay away from the boundary $Z = 0$ from the effective equations. The regularity and decay properties typically depend on compatibility conditions between the initial and boundary data. Using these properties, norm bounds on the error $u^\nu - u^\nu_{\text{approx}}$ can then be obtained from the Navier-Stokes equations via energy estimates.

This approach was used in [51] to establish the zero-viscosity limit and rates of convergence under generalized Navier boundary conditions on any smooth, bounded domain; see Theorem 3.3 below. Both [67] and [51] employ *linear* boundary correctors, another measure of the weakness of the boundary layer under Navier conditions, but the corrector in [51] is constructed using a covariant formulation and is coordinate independent. Under geodesic boundary normal coordinates in a tubular neighborhood of the boundary, it has an explicit form, which allows to prove uniform space-time bounds on the error $u^\nu - u^0$ even close to the boundary.

Theorem 3.3. *Denote by Γ_a a tubular neighborhood of $\partial\Omega$ interior to Ω of width $a > 0$. Fix $m > 6$ and assume $u_0 \in H^m(\Omega$. Let u^ν be the unique, strong solution of* (1.1) *with generalized Navier boundary conditions and initial data u_0. Let u^0 be the unique strong solution of* (1.4) *with the no-penetration boundary conditions and initial data u_0. Then:*

$$\|u^\nu - u^0\|_{L^\infty([0,T],L^2(\Omega))} \leq \kappa\,\nu^{\frac{3}{4}}, \qquad \|u^\nu - u^0\|_{L^\infty([0,T],H^1(\Omega))} \leq \kappa\,\nu^{\frac{1}{4}}$$
$$\|u^\nu - u^0\|_{L^\infty([0,T]\times\Gamma_a))} \leq \kappa\,\nu^{\frac{3}{8}-\frac{3}{8(m-1)}}, \quad \|u^\nu - u^0\|_{L^\infty([0,T]\times\Omega\setminus\Gamma_a))} \leq \kappa\,\nu^{\frac{3}{4}-\frac{9}{8m}},$$

where κ is a constant independent of ν.

In two space dimensions, under free boundary conditions, the use of correctors allows to study the boundary layer for the vorticity and obtain rates of convergence in Sobolev spaces [50]. Lastly, in the context of Navier-type boundary conditions, the zero-viscosity limit has been used as a mean to establish existence of solutions to the Euler equations (see e.g. (see ([145], [83, pp. 87–98], [6], and [85, pp. 129–131]). Whether it is also a selection mechanism for uniqueness of weak solutions in two space dimensions remains open.

3.2 Case of no-slip boundary condition

The classical case of no-slip boundary conditions (3.2) is perhaps the most relevant in applications, and the most challenging to study from a mathematical point of view. The main difficulty stems from the formation of a possibly strong boundary layer (of amplitude order one in viscosity) in flows at sufficiently high Reynolds numbers. It is experimentally observed (see e.g. the classical experiments of flow around a solid sphere [134]) that laminar boundary layer, where the flow lines are approximately parallel to the boundary, destabilizes and detaches from the boundary, a phenomenon known as *boundary layer separation*. This layer separation is due to the presence of an adverse pressure gradient that leads to stagnation first and then flow reversal in the layer. In the unsteady case, the connection between the vanishing viscosity limit and the stability of the boundary layer has still not been completely clarified. In particular there are no known analytical examples of unsteady flows where layer separation occurs. Nevertheless, a connection can be made in terms of vorticity production at the boundary. The mismatch between no penetration and no slip at the boundary leads potentially to the creation of large gradients of velocity in the layer, in particular normal derivatives of tangential components of the velocity at the boundary. While the creation of a boundary layer can be a purely diffusive effect, it is its interaction with strong inertial terms that is thought to lead to boundary layer separation. Therefore, one expects that in the context of the Navier-Stokes equations linearized around a non-trivial profile, i.e., Oseen-type equations, it should be possible to establish the zero-viscosity limit. This is indeed the case at least if the Oseen profile is regular enough and under some compatibility conditions between the initial and boundary data [2, 3, 128, 129, 90] (see also [49] for incompatible data for the Stokes equation).

Whether the vanishing viscosity limit holds generically even for short time under no-slip boundary conditions is largely an open problem. An asymptotic ansatz for the velocity similar to (3.8), but taking into account the amplitude of the boundary layer profile, that is, postulating an expansion for the velocity of the form:

$$u^\nu(t,x) = u^0(t,x) + \theta(t, x', \frac{z}{\sqrt{\nu}}) + O(\nu^{\frac{1}{2}})\,, \tag{3.9}$$

leads to the classical Prandtl equation, which will be discussed in more details in Section 3.4. One brief remark here is that the Prandtl equations are well posed only under strong conditions on the flow, such as when boundary and the data have some degree of analyticity [5, 121, 89, 22, 79] or the data is monotonic in the normal direction to the boundary [116, 118, 78]. The most classical result verifying (3.9) is [121] in the analytic functional framework, after the pioneering work of [5]. The result of [121] is stated here only in an intuitive manner without introducing the precise definition of function spaces.

Theorem 3.4. *Suppose that given data and the Euler flow are analytic in all variables $x = (x', z)$. Then the Prandtl asymptotic expansion* (3.9) *is valid for a short time.*

The proof of [121] is based on the analysis of the integral equations for the Navier-Stokes equations with the aid of the Cauchy-Kowalewski theorem. On the other hand, quantifying the production of vorticity by the boundary at finite viscosity and its interaction with convective terms is crucial in understanding the behavior of the viscous fluid near impermeable walls. The next result from [96] shows that the zero-viscosity limit is verified for a short time by using the vorticity formulation in the half plane, identified with $\mathbb{R}^2_+$, as long as the initial vorticity stay bounded away from the boundary.

Theorem 3.5. *Let $\omega_0 = \operatorname{curl} u_0 = \operatorname{curl} u_0^\nu$ be the initial vorticity for the Euler and Navier-Stokes flows. Assume that*

$$d_0 := \operatorname{dist}(\partial\mathbb{R}^2_+, \operatorname{supp}\omega_0) > 0.$$

Define $u^\nu_{approx} = u^0 + u^\nu_P$, where u^ν_P is the boundary layer corrector, which satisfies a modified Prandtl equation. Then,

$$\|u^\nu - u^\nu_{approx}\|_{L^\infty((0,T)\times\mathbb{R}^2)} \leq C\,\nu^{\frac{1}{2}},$$

for some constant $C > 0$ independent of ν. The time T can be estimated from below as follows:

$$T \geq c\,\min\{d_0, 1\},$$

for some positive constant c which depends only on $\|\omega_0\|_{W^{4,1}\cap W^{4,2}}$.

Note that the class of natural test functions $C^\infty_{0,\sigma}(\mathbb{R}^2_+)$ (or even $C^k_{0,\sigma}(\mathbb{R}^2_+)$ for large k) is admissible for the initial data in Theorem 3.5, due to the analyticity assumption in the entire half space. This case has been excluded in Theorem 3.4 In Theorem 3.5 the L^∞ choice for the norm in which to take the limit is more natural than the energy norm, as the vorticity-velocity formulation is used to obtain uniform bounds in viscosity and, as already remarked, the velocity obtained from the Biot-Savart law, is not in L^2 unless the integral of vorticity is zero (see e.g. [97]). If the Euler flow satisfies a sign condition at the boundary, namely, Oleinik's monotonicity condition, that on a half plane reads $u^0_1(x_1, 0, t) \geq 0$, so that no back flow occurs, then it is enough for the vorticity to be not too negative in a Kato-type boundary layer (of width ν) [28].

The data discussed in Theorems 3.4 and 3.5 have analytic regularity at least near the boundary. If the analyticity is totally absent, ione should not expect an asymptotic expansion of the form (3.9) to be valid in general, given the underlying strong instability mechanism at high frequencies. A classical instability result is given in [56], and recalled on Theorem 3.6 below, where the invalidity of (3.9) is shown when the initial boundary layer profile is linearly unstable for the Euler equations.

Theorem 3.6. *Let $U_s(z)$ be a smooth shear layer satisfying $U_s(0) = 0$, such that $U_s\mathbf{e}_1 = (U_s, 0)$ is a linearly unstable stationary solution to the Euler equations. Let n be an integer, arbitrarily large. Then there exists $\delta_0 > 0$ such that the following statement holds. For every large s and sufficiently small ν there exist $T_\nu > 0$ and $v^\nu_0 \in H^s(\mathbb{R}^2_+) \cap L^2_\sigma(\mathbb{R}^2_+)$ such that*

$$\lim_{\nu\to 0} T_\nu = 0\,, \qquad \|v^\nu_0\|_{H^s(\mathbb{R}^2_+)} \leq \nu^n\,,$$

and the solution u^ν to (1.1) in $\mathbb{R}^2_+$ (under the no-slip boundary condition) with the initial data $u^\nu_0(x) = U_s(\frac{x_2}{\sqrt{\nu}})\mathbf{e}_1 + v^\nu_0(x)$ satisfies the estimate

$$\lim_{\nu\to 0} \|\mathrm{curl}\, u^\nu(T_\nu) - \mathrm{curl}\,\big(u_s(T_\nu, \frac{\cdot}{\sqrt{\nu}})\mathbf{e}_1\big)\|_{L^\infty(\mathbb{R}^2_+)} = \infty\,,$$
$$\lim_{\nu\to 0} \|u^\nu(T_\nu) - u_s(T_\nu, \frac{\cdot}{\sqrt{\nu}})\mathbf{e}_1\|_{L^\infty(\mathbb{R}^2_+)} \geq \delta_0 \nu^{\frac{1}{4}}\,.$$

Here $u_s(t, z)$ is the smooth solution to the heat equations $\partial_t u_s - \partial_z^2 u_s = 0$, $u_s|_{t=0} = U_s$, and $u_s|_{z=0} = 0$.

In the proof given in [56], the time T_ν is of the order $\mathcal{O}(\nu^{\frac{1}{2}}|\log \nu|)$. In particular, Theorem 3.6 implies that the expansion (3.9) may cease to be valid in general at least in this very short time. Theorem 3.6 is proved based on the instability of the shear profile U_s for the Euler equations, but in the rescaled variables $X = \frac{x}{\sqrt{\nu}}$. On the other hand, as will be mentioned in Section 3.4.2, the invalidity of the asymptotic estimate (3.9) can be also derived from the high-frequency instability of the shear profile in the Prandtl equations, which is proved by [57]. Note that, however, the invalidity of (3.9) observed in [56] and [57] relies on the assumption that the boundary layer is formed already at the initial time (and thus, the initial data for the Naiver-Stokes flows is also assumed to depend on the viscosity coefficient, more precisely, on the fast variable $\frac{x_2}{\sqrt{\nu}}$). It is still open whether (3.9) can be disproved, or proved, in the case when the Sobolev initial data of the Navier-Stokes flows is taken independently in ν, for in this case there is no boundary layer at the initial time and the layer forms only at a positive time.

Mathematically, the main difficulty in the case of the no-slip boundary condition is the lack of *a priori* estimates on strong enough norms to pass to the limit, which in turns is due to the lack of a useful boundary condition for vorticity or pressure. Then, the other types of results found in the literature can be roughly divided into two groups:

(I) conditional convergence results for generic flows under conditions on the flow that control the growth of gradients in the layer, such as Kato's condition on the energy dissipation rate, discussed below;

(II) convergence results for specific classes of flows, where some conditions as in (I) are valid automatically, such as parallel flows in pipes and channels discussed below.

Again, for general initial conditions, the zero viscosity limit is sought to hold on the interval of existence of the Euler solutions.

Kato [71] realized that the vanishing of energy dissipation in a small layer near the boundary is equivalent to the validity of the zero-viscosity limit in the energy space. In fact, this condition is enough to pass to the limit in the non-linear term in the weak formulation of the equations.

Theorem 3.7 (Kato's criterion)**.** *Let u^ν be a Leray-Hopf weak solution of the Navier-Stokes equations* (1.1) *with initial data $u_0^\nu \in L^2(\Omega)$. Let u^0 be a strong solution of the Euler equations* (1.4) *with initial data $u_0 \in H^s$, $s > 5/2$ on the time interval $[0,T]$. Then, $u_0^\nu \to u_0$ strongly in $L^2(\Omega)$, i.e., the vanishing viscosity limit holds, if and only if, for $T' \leq T$,*

$$\lim_{\nu \to 0_+} \nu \int_0^{T'} \|\nabla u^\nu(t)\|^2_{L^2(\Gamma_{c\nu})}\, \mathrm{d}t = 0,$$

where $c > 0$ is a fixed, but arbitrary, constant and $\Gamma_{c\nu}$ is a boundary strip of width $c\nu$.

Variants of Kato's criterion have been established, involving only the partial gradient of the velocity field and allowing for non-zero boundary velocity [130, 137], or involving the vorticity [74] for instance. Unfortunately, it is not know whether flows generically satisfies Kato's criterion at least for short time. In fact, Kato's criterion cannot hold if boundary layer separation occur by a result of [75] stated below, where it is shown that the zero viscosity limit holds if and only if vorticity accumulates only at the boundary as a conormal distribution in $H^{-1}(\mathbb{R}^d)$.

Theorem 3.8. *In the hypothesis of Kato's criterion, the following are equivalent:*

(a) $u^\nu \to u^0$ *in* $L^\infty([0,T], L^2(\Omega))$*;*

(b) $\omega^\nu \to \omega^0 - \dfrac{1}{2} u^0 \times n\, \mu$ *weakly in* $L^\infty([0,T), H^1(\mathbb{R}^d)')$*.*

where n is the unit outer normal and μ is a Radon measure that agrees with surface area on $\partial\Omega$.

It should be noted that the equivalence of (a) and (b) above is purely kinematic, in the sense that is a direct consequence of the validity of the limit and results on weak convergence of gradients. If dynamics can be taken into account, for example when the initial-boundary-value problem for vorticity is solvable, the convergence in (b) can be improved to convergence in the sense of Radom measures on the closure of Ω. This will be the case for parallel pipe and channel flows discussed later in this subsection. In this situation, one can interpret the extra measure on the boundary appearing in the limit as a *vortex sheet* due to a jump in velocity across the boundary even if there is no fluid outside the domain Ω.

A consequence of the theorem is that boundary layer separation cannot occur if the zero-viscosity limit holds, even though the convergence is in a relatively weak norm, the energy norm, because then the convergence of u^ν to u^0 is in the H^1-Sobolev norm in the interior, and this strong convergence is incompatible with the layer separation.

The vanishing viscosity limit and associated boundary layer can be studied for special classes of flows that satisfy strong symmetry assumptions. In this situation, no additional assumptions are made on the flow, except assuming symmetry of the initial data, as the symmetry is preserved by the Navier-Stokes and Euler evolution, at least for strong solutions. (For a discussion of possible symmetry breaking in the context of weak solutions, the reader is referred to [7].) The classes of flows that can be studied are so-called *parallel flows* in straight infinite channels or straight infinite circular pipes. These flows can be thought of as generalization of the classical Poiseuille and Couette flows, but they are unsteady and generally non-linear. In fact, the walls of the pipe or channel are allowed to move rigidly along itself, as in the classical Taylor-Couette case, so that the no-slip boundary condition for solutions to (1.1) takes the form (3.2) with $V \neq 0$. Parallel channel and pipe flows were considered before in the context of the zero viscosity limit by [137], who lists them as cases for which Kato's criterion applies. In fact, it is easy to see that the criterion applies in the extension due to [130] if the boundary velocity V is not too rough. However, due to the symmetry in the problem it is possible to obtain a detailed analysis in the boundary layer and quantify vorticity production even in the case of impulsively started and stopped boundary motions, where V is of bounded variation in time. It should be noted that the boundary layer is not weak here and, in fact, it has the width predicted by the Prandtl theory proportional to $\sqrt{\nu}$. But, because of symmetry, the flow stays laminar and the boundary layer never detaches. A similar analysis for truly non-linear, symmetric flows, such as axisymmetric flows (without swirl) and helical flows seems out of reach at the moment.

In what follows, $\{e_r, e_\phi\}$ will denote the orthonormal frame associated to polar coordinates (r, ϕ) in the plane, and $\{e_r, e_\phi, e_x\}$ will denote the orthonormal frame associated to cylindrical coordinates (r, ϕ, x) in space. The symmetric flows that have been considered in the literature are:

(i) *Circularly Symmetric Flows (RSF)*: planar flows in a disk centered at the origin $\Omega = \{x^2+y^2 < R\}$. The velocity is of the form

$$u = V(t)e_\phi, \tag{3.10}$$

using polar coordinates, where $V(t))$ is radial function. The vorticity, which can be identified with a scalar for planar flows, is also radial.

(ii) *Plane-parallel flows*: 3D flows in a infinite channel, with periodicity imposed in the x and y-directions. The velocity takes the form:

$$u = (u_1(t, z),\ u_2(t, x, z),\ 0), \tag{3.11}$$

and is given on the domain

$$\Omega := (0, L)^2 \times (0, h)$$

where h is the width of the channel and u_1, u_2 satisfy periodic boundary conditions in x and y. The boundary is identified with the set $\partial\Omega = [0, L]^2 \times [0, h]$

(iii) *Parallel pipe flows*: 3D flows in a infinite straight, circular pipe, with periodicity imposed along the pipe axis, identified with the x-axis. The velocity is of the form

$$u = u_\phi(t, r)\mathbf{e}_\phi + u_x(t, \phi, r)\mathbf{e}_x, \tag{3.12}$$

using cylindrical coordinates on the domain

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 < R,\ 0 < x < L\},$$

where R is the radius of circular cross-section and u_ϕ, u_x satisfy periodic boundary conditions in x. The boundary is identified with the set $\partial\Omega = \{(y, z) \in \mathbb{R}^2 \mid y^2 + z^2 = R\} \times [0, h]$.

For all these flows, the divergence-free condition is automatically satisfied. In the case of circular symmetry, the Navier-Stokes equations reduces to a heat equation and the Euler flow is steady, making this more of a pedagogical example. Both for the channel and pipe geometry, symmetry and periodicity ensure uniqueness of solutions to NSE and EE, in particular by forcing the only pressure-driven flow to be the trivial flow. The well-posedness is global in time for both Euler and Navier-Stokes for sufficiently regular initial data.

As an illustration, only parallel pipe flows will be discussed here, which is the most interesting case due to the effect of curvature of the boundary. The reader is referred to [107, 15, 91, 92] for the case of circularly symmetric flows, and to [111, 109] for the case of channel flows. The velocity is independent of the variable along the pipe axis and, in any circular cross section of the pipe, it is the sum of a circularly symmetric, planar velocity field and a velocity field pointing in the direction of the axis. As in the case of plane-parallel flows, even though the flows are not planar, the Navier-Stokes and Euler equations reduce to a weakly non-linear system in only two space variables, given respectively by (for simplicity it is assumed that the boundary is stationary):

$$\begin{cases} \dfrac{\partial u_\phi^\nu}{\partial t} - \nu \Delta u_\phi^\nu + \nu \dfrac{1}{r^2} u_\phi^\nu = 0 & \text{in } (0,T)\times\Omega\,, \\ \dfrac{\partial u_x^\nu}{\partial t} - \nu \Delta u_x^\nu + \dfrac{1}{r} u_\phi^\nu \dfrac{\partial u_x^\nu}{\partial \phi} = 0 & \text{in } (0,T)\times\Omega\,, \\ -\dfrac{1}{r}(u_\phi^\nu)^2 + \dfrac{\partial p^\nu}{\partial r} = 0 & \text{in } (0,T)\times\Omega\,, \\ u_i^\nu = 0\,,\ i=\phi,x & \text{on } (0,T)\times\partial\Omega\,, \end{cases}$$

and by

$$\begin{cases} \dfrac{\partial u_\phi^0}{\partial t} = 0 & \text{in } (0,T)\times\Omega\,, \\ \dfrac{\partial u_x^0}{\partial t} + \dfrac{1}{r} u_\phi^0 \dfrac{\partial u_x^0}{\partial \phi} = 0 & \text{in } (0,T)\times\Omega\,, \\ -\dfrac{1}{r}(u_\phi^0)^2 + \dfrac{\partial p^0}{\partial r} = 0 & \text{in } (0,T)\times\Omega\,. \end{cases}$$

The Navier-Stokes system is amenable to the analysis of the vanishing viscosity limit primarily because it is diffusion dominated and because the pressure is slaved to the velocity and drops out of the momentum equation.

A detailed analysis of the boundary layer using techniques borrowed from semiclassical analysis was performed in [110], for ill-prepared data. There, in particular, convergence rates in viscosity for the L^∞ norm were derived by constructing a parametrix to a suitable associated linear problem and taking the corrector as the double layer potential associated to this problem. For well-prepared data, convergence rates in higher Sobolev norms were obtained by the use of flow correctors and effective equations in [58]. By the use of a different type of correctors, it is possible to obtain similar results for ill prepared data and quantify production of vorticity at the boundary [52]. Similarly to the case of Navier boundary conditions, one defines an approximate Navier-Stokes solution u^ν_{approx} as a sum of an outer solution u^ν_{ou} and an inner solution u^ν_{in} . At zero order in viscosity, $u^\nu_{\text{ou}} = u^0$, the Euler solution, while u^ν_{in} is given by a smooth, radial cut-off ψ supported in a collar neighborhood of the boundary times a corrector θ of the form:

$$\theta(t,x) = \theta_\phi(t,r)\mathbf{e}_\phi + \theta_x(t,\phi,r)\mathbf{e}_x\,, \tag{3.13}$$

using again cylindrical coordinates (r,ϕ,x), where θ_ϕ and θ_x satisfy weakly coupled parabolic systems. Then, the following convergence rates can be obtained [110, 58, 52].

Theorem 3.9. *Assume $u_0 \in H^k(\Omega)$, k large enough ($k > 4$ suffices), and has symmetry* (3.12)*. Then, the zero-viscosity limit hold on $(0,T)$, for all $0 < T < \infty$ and, in particular:*

$$\begin{cases} \|u^\nu - u^\nu_{approx}\|_{L^\infty(0,T;L^2(\Omega))} + \nu^{\frac{1}{2}} \|\nabla u^\nu - \nabla u^\nu_{approx}\|_{L^2(0,T;L^2(\Omega))} \le \kappa_T\, \nu^{\frac{3}{4}}, \\ \|u^\nu - u^0\|_{L^\infty(0,T;L^2(\Omega))} \le \kappa_T\, \nu^{\frac{1}{2}}, \end{cases} \tag{3.14}$$

where κ_T is a constant independent of ν. In addition,

$$\omega^\nu \to \omega^0 + (u^0 \times n)\mu \quad \text{weakly}^* \text{ in } L^\infty(0,T;\mathcal{M}(\bar{\Omega})), \tag{3.15}$$

where $\mathcal{M}(\bar{\Omega})$ is the space of Radon measures on $\bar{\Omega}$, and μ is a measure supported on $\partial\Omega$, which on the boundary agrees with the normalized surface area.

A complication over plane-parallel flows is that the effect of non-vanishing curvature cannot be neglected in the analysis. Furthermore, in cylindrical coordinates the behavior of the solution near the axis cannot be controlled as well as it can be away from the axis, similarly to the case of axisymmetric flows. To overcome this difficulty, a two-step localization, one near the boundary where curvilinear coordinates are used, the other near the axis where Cartesian coordinates and energy estimates are employed, is utilized. As a consequence, however, the error estimates for the approximate solution suffer from the loss of one derivative. In particular, the estimates for the correctors are not as sharp as in the case of a pipe with annular cross section.

As seen below, there are other situations for which it is possible to pass to the limit due to the fact that the boundary layer is weak or absent. This is the case, for example, of flows outside shrinking obstacles, if the obstacle is shrinking faster than viscosity vanishes. For such flows the local Reynolds number, built by taking the size of the obstacle as characteristic length, stays of order one, as already observed in [65]. In this context, the Navier-Stokes solution in the exterior of the obstacle is expected to converge to the Euler solution in the whole space. Most of the results concern flows in the plane, as the vorticity-velocity formulation is used to obtain the inviscid solution. At the same time, the fact that the exterior of compact obstacles is not simply connected in two space dimensions adds some technical difficulties, which are overcome by assuming that the circulation around the obstacles is zero.

Let ϵ be the scale of the obstacle. The vanishing viscosity limit was shown to hold in the exterior of one obstacle diametrically shrinking to a point in [65] by assuming the condition $\epsilon \le C\,\nu$ for some positive constant C, which depends on the initial data for the Euler equations in $\mathbb{R}^2$, u_0, and the shape of the obstacle, and assuming that the initial condition for the Navier-Stokes solution, u^ν_0, extended by zero to the whole plane, converges to u_0 in $L^2(\mathbb{R}^2)$, with an optimal rate of convergence of $\sqrt{\nu}$. (See [76] for the opposite situation of an expanding domain.) This result can be extended to the exterior of a finite number of fixed obstacles. It is interesting to ask whether a similar result hold in the setting of a porous medium, that is, if the domain for the viscous flow is the exterior of an array of particles. It is known that homogenization of the Navier-Stokes equations and Euler equations gives different filtration laws (Darcy or Brinkman, for example) depending on the relative ratio the particle size ϵ and inter-particle distance d_ϵ, and the permeability of the homogenized medium is very different between the viscous and inviscid (see the discussion in [80, 114] and references therein). Therefore, it is relevant to study the joint limit of vanishing ϵ, d_ϵ, ν.

In [80], the limit was established under the condition that $d_\epsilon > \epsilon$ and $\epsilon \le A\,\nu$; see Theorem 3.10 below. In this regime, one expects that the limit Euler flow, defined in the whole plane, does not feel the presence of the porous medium. Below, for each ϵ, the domain Ω_ϵ is set as the viscous fluid domain. For simplicity the exterior of a regular array of identical particles is arranged in a square.

Theorem 3.10. *Given $\omega_0 \in C_c^\infty(\mathbb{R}^2)$, let u^0 the solution of the Euler equations in the whole plane with initial condition $u_0 := K_{\mathbb{R}^2}[\omega_0]$. For any $\epsilon, \nu > 0$, let $d^\epsilon \geq \epsilon$. Let $u^{\nu,\epsilon}$ be the solution of the Navier-Stokes equations in Ω^ϵ with initial velocity $u_0^{\nu,\epsilon}$. Then, there exists a constant A depending only on the particle shape, such that if*

$$\frac{\epsilon}{d_\epsilon} \leq \frac{A\nu}{\|\omega_0\|_{L^1\cap L^\infty(\mathbb{R}^2)}},$$

and if ω_0 is supported in Ω^ϵ, then for any $T > 0$ we have

$$\sup_{0\leq t\leq T} \|u^{\nu,\epsilon} - u^0\|_{L^2(\Omega^\epsilon)} \leq B_T \left(\frac{\sqrt{\nu}}{d_\epsilon} + \|u_0^{\nu,\epsilon} - u_0\|_{L^2(\Omega^\epsilon)} \right) \tag{3.16}$$

where B_T is a constant depending only on T, $\|\omega_0\|_{L^1\cap W^{1,\infty}(\mathbb{R}^2)}$, and the particle shape.

It is then possible to construct initial data $u_0^{\nu,\epsilon}$ such that $u_0^{\nu,\epsilon} \to u_0$ in L^2, which establishes the limit with rate $\sqrt{\nu}/d_\epsilon$. Therefore, there is a ghost of the porous medium in the convergence rate. It should be noted that in the case of the Darcy-Brinkman system, the equations for the boundary corrector are linear, and thus, the passage to the zero-viscosity limit is possible [77, 59].

3.3 Non-characteristic boundary case

One of the main difficulties in treating the vanishing viscosity limit for classical no-slip boundary conditions is the fact that the boundary is characteristic for the problem, that is it consists of streamlines for both the viscous and inviscid flows. Hence, any attempt to control the flow in the interior from the boundary seems unsuccessful unless analyticity or monotonicity of the data is imposed.

If non-characteristic boundary conditions are imposed, in particular, if the walls are permeable, then under certain conditions the boundary layer is stable, hence there is no layer separation and one expects the vanishing viscosity limit to hold. This is the case when injection and suction rates are imposed at the boundary. For simplicity we describe the set up in the geometry of a (periodized) channel $[0, L]^2 \times [0, h]$, where the boundary conditions are imposed only at the top and bottom walls. The velocity at the boundary for Navier-Stokes is given as:

$$u_i^\nu(t, x) = (0, 0, -U_i(t, x_1, x_2)), \qquad i = \text{top}, \text{bot}, \tag{3.17}$$

where $U_i \geq a_i > 0$ for some constants a_i and top, bot refer to top and bottom of the channel. For Euler, one needs to specify the entire velocity and the inlet/outlet. These conditions are also appropriate when a domain is truncated, e.g. for computational reasons, when making a Galilean coordinate transformation.

By correcting the velocity field, it was shown in [131] that the zero viscosity limit holds with sharp rates of convergence of $\nu^{1/4}$ in the uniform norm. In particular, there is only a stable boundary layer at the suction wall (the bottom) that is exponentially small. It is interesting to note that, differently than in the non-linear case, for the Oseen equations, adding injection and suction at the boundary does not seem to change the size of the boundary layer (see [90]).

3.4 Prandtl equations for boundary layer

This subsection is devoted to an overview of the study of the Prandtl equations, introduced by Prandtl in 1904 in order to describe a viscous incompressible flow near the boundary at high Reynolds number [119]. The Prandtl equations are derived from the Navier-Stokes equations with no-slip boundary condition, and the derivation is briefly recalled here in the case the fluid domain is the half plane $\mathbb{R}^2_+$:

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p^\nu = \nu \Delta u^\nu\,, & t > 0\,, \quad x \in \mathbb{R}^2_+\,, \\ \operatorname{div} u^\nu = 0\,, & t \geq 0\,, \quad x \in \mathbb{R}^2_+\,, \\ u^\nu|_{t=0} = u^\nu_0\,, & \qquad\quad x \in \mathbb{R}^2_+\,. \end{cases} \tag{3.18}$$

As is discussed in the previous sections, the equations in the limit $\nu = 0$ are the Euler equations, for which only the impermeability condition $u^\nu_2 = 0$ on $\partial\mathbb{R}^2_+$ can be prescribed. Heuristically, such an incompatibility in the boundary condition leads to a fast change of the tangential component of the velocity field, which is u^ν_1 when the fluid domain is $\mathbb{R}^2_+$. As a result, the derivative of u^ν_1 in the vertical direction tends to have a singularity near the boundary and forms a boundary layer. To study the formation of the boundary layer, Prandtl made the ansatz that u^ν near the boundary has the following asymptotic form:

$$u^\nu_1(t, x_1, x_2) \sim u^P_1(t, x_1, \frac{x_2}{\sqrt{\nu}})\,, \qquad u^\nu_2(t, x_1, x_2) \sim \sqrt{\nu} u^P_2(t, x_1, \frac{x_2}{\sqrt{\nu}})\,. \tag{3.19}$$

The thickness of the boundary layer $\mathcal{O}(\sqrt{\nu})$ is coherent with the parabolic nature of the Navier-Stokes equations. The underlying assumption here is that the velocity u^ν remains of order $\mathcal{O}(1)$ in all $\partial^k_1 u^\nu$, $k = 0, 1, \cdots$, in the limit $\nu \to 0$, and then the vertical component u^ν_2 is expected to be of the order $\mathcal{O}(\sqrt{\nu})$ since the boundary condition for the normal component is preserved in the limit, and it is also compatible with the divergence free condition. By formally substituting the ansatz (3.19) into the first equation of (3.18), the velocity profile u^P_1 and the associated pressure p^P should obey the equations

$$\partial_t u^P_1 + u^P \cdot \nabla u^P_1 + \partial_1 p^P - \partial^2_2 u^P_1 = 0\,, \qquad \partial_2 p^P = 0\,,$$

and u^P_1 must satisfy the no-slip boundary condition $u^P_1 = 0$ on $\partial\mathbb{R}^2_+$. Here the spatial derivatives are for the rescaled variables $X_1 = x_1$ and $X_2 = \frac{x_2}{\sqrt{\nu}}$, but in this subsection the same notation ∇ and $\partial_j = \frac{\partial}{\partial X_j}$ will be used for notational ease. The rescaled variables X will be also relabeled as x from now on. The vertical component u^P_2 is recovered from u^P_1 and the boundary condition $u^P_2 = 0$ on $\partial\mathbb{R}^2_+$ in virtue of the second equation (divergence-free condition) of (3.18), which yields

$$u^P_2(t, x) = -\int_0^{x_2} \partial_1 u^P_1(t, x_1, y_2)\, \mathrm{d}y_2\,.$$

The velocity in the boundary layer has to match with the outer flow which is assumed to satishy the Euler equations. This requirement leads to the following boundary condition (*matching conditions*) on u^P_1 and p^P at $x_2 = \infty$:

$$\lim_{x_2 \to \infty} u^P_1 = u^E\,, \qquad \lim_{x_2 \to \infty} p^P = p^E\,,$$

where $u^E(t, x_1) = u^0(t, x_1, 0)$ and $p^E(t, x_1) = p^0(t, x_1, 0)$, and (u^0, p^0) is the solution to the Euler equations (1.4) in $\mathbb{R}^2_+$. Since p^P must be independent of x_2, because of the equation $\partial_2 p^P = 0$ in $\mathbb{R}^2_+$, the matching condition on p^P at $x_2 = \infty$ implies that

$$p^P = p^E\,.$$

That is, the pressure field is not one of the unknowns in the Prandtl equations. Collecting the above equations gives the Prandtl equations in $\mathbb{R}^2_+$ (in the spatial variables), which are a system of equations for the scalar unknown function u_1^P:

$$\begin{cases} \partial_t u_1^P + u^P \cdot \nabla u_1^P - \partial_2^2 u_1^P = -\partial_1 p^E \,, \\ u_2^P = -\displaystyle\int_0^{x_2} \partial_1 u_1^P \, \mathrm{d}y_2 \,, \\ u_1^P|_{t=0} = u_{0,1}^P \,, \qquad u_1^P|_{x_2=0} = 0 \,, \qquad \displaystyle\lim_{x_2 \to \infty} u_1^P = u^E \,. \end{cases} \tag{3.20}$$

The reader is referred to [122, 104, 86] for more details about the formal derivation of the Prandtl equations. Noe that, by taking the boundary trace in the Euler equations, the data (u^E, p^E) coming from the Euler flows in the outer region is subject to the Bernoulli law

$$\partial_t u^E + u^E \partial_1 u^E + \partial_1 p^E = 0 \,. \tag{3.21}$$

The Prandtl equations are deceptively simpler than the original Navier-Stokes equations. In fact , due to the inherent instability of boundary layers, well-posedness of (3.20) has been proven only in some specific situations (Section 3.4.1), while strong ill-posedness results are present in the literature (Section 3.4.2).

3.4.1 Well-posedness results for the Prandtl equations

The Prandtl equations are known to be well-posed under some restricted conditions. This subsection gives a list of the categories in which the well-posedness of the Prandtl equations holds, at least for short time.

(I) Monotonic data. This category is the most classical in the theory of the Prandtl equations. The system (3.20) is considered for $0 < t < T$ and for $(x_1, x_2) \in \Omega_1 \times \mathbb{R}_+$, where Ω_1 is usually set as either $\{0 < x_1 < L\}$, $\mathbb{T}$, or $\mathbb{R}$. When $\Omega_1 = \{0 < x_1 < L\}$, an additional boundary condition has to be imposed on u_1^P at the boundary $\{x_1 = 0\}$:

$$u_1^P(t, 0, x_2) = u_{1,1}^P(t, x_2) \,.$$

The given boundary data $u_{1,1}^P$ also has to be compatible with the monotonicity. The basic assumption describing the monotonicity is

$$\partial_2 u_{0,1}^P(x_1, x_2) > 0 \,, \qquad x_1 \in \overline{\Omega_1} \,,\ x_2 \geq 0 \,, \tag{3.22}$$

$$\partial_2 u_{1,1}^P(t, x_2) > 0 \,, \qquad t > 0 \,,\ x_2 \geq 0 \,. \tag{3.23}$$

As a compatibility condition, the outer flow u^E and given data $u_{0,1}^P$, $u_{1,1}^P$ must be positive for $x_2 > 0$, and the solution u_1^P is also expected to be positive for $x_2 > 0$ together with its derivative in the x_2 variable. The solvability of (3.20) in this class has been established by Oleinik and her co-workers, especially for the case $\Omega_1 = \{0 < x_1 < L\}$.(See [115, 116, 117]. The reader is also referred to [118] for more details and references.) The steady problem is solved in [115] for small $L > 0$, and this local existence result is extended in [108], where it is shown that the solution can be continued to the separation point. For the unsteady problem, unique solvability is proved in [116] for a short time if L is given and fixed, while for an arbitrary time if L is sufficiently small. The stability of the steady solutions is shown in [117].

A natural question arises, already present in the monograph[118], namely, under which condition the solutions exist globally in time without any smallness of $L > 0$. A significant contribution to this problem is given by [144], where the global existence of weak solutions to (3.20) is proved when the pressure gradient is favorable:

$$\partial_1 p^E(t, x_1) \leq 0 \,, \qquad t > 0 \,,\quad 0 < x_1 < L \,. \tag{3.24}$$

The analysis in [116, 144] uses the classical Crocco transformation

$$\tau = t\,, \qquad \xi = x_1\,, \qquad \eta = \frac{u_1^P(t,x_1,x_2)}{u^E(t,x_1)}\,, \qquad w(\tau,\xi,\eta) = \frac{\partial_2 u_1^P(t,x_1,x_2)}{u^E(t,x_1)}\,,$$

which transforms the domain $\{(t,x_1,x_2) \mid 0<t<T\,,\ 0<x_1<L\,,\ x_2>0\}$, $T>0$, into

$$Q_T = \{(\tau,\xi,\eta) \mid 0<\tau<T\,,\ 0<\xi<L\,,\ 0<\eta<1\}\,.$$

Then the Prandtl equations for the case $\Omega_1 = \{0<x_1<L\}$ is transformed into

$$\begin{cases} \partial_\tau w^{-1} + \eta u^E \partial_\xi w^{-1} + A\partial_\eta w^{-1} - Bw^{-1} = -\partial_\eta^2 w \qquad \text{in } \ Q_T\,, \\ w|_{\tau=0} = w_0 = \dfrac{\partial_2 u_1^P}{u^E}|_{t=0}\,, \quad w|_{\xi=0} = w_1\,, \quad (w\partial_\eta w)|_{\eta=0} = \dfrac{\partial_1 p^E}{u^E}\,, \quad w|_{\eta=1} = 0\,. \end{cases} \tag{3.25}$$

Here

$$A = (1-\eta^2)\partial_1 u^E + (1-\eta)\frac{\partial_t u^E}{u^E}\,, \qquad B = \eta\partial_1 u^E + \frac{\partial_t u^E}{u^E}\,, \qquad w_1 = \frac{\partial_2 u_{1,1}^P}{u^E}|_{x_1=0}\,.$$

The following result is proved in [144, Theorem 1.1].

Theorem 3.11. *Assume that* (3.22) *and* (3.23) *hold together with compatibility conditions. If the pressure condition* (3.24) *holds in addition, then there exists a weak solution* $w \in BV(Q_T) \cap L^\infty(Q_T)$ *to* (3.25) *such that for some* $C>0$,

$$C^{-1}(1-\eta) \le w \le C(1-\eta) \qquad \text{in } \ Q_T\,,$$

and $\partial_\eta^2 w$ *is a locally bounded measure in* Q_T.

In Theorem 3.11 the initial and boundary conditions are satisfied in the sense of trace, and (3.25) is considered in the sense of distributions. The reader is referred to [144] for more properties on the regularity of weak solutions obtained in the theorem. As a consequence of Theorem 3.11, global existence of weak solutions to (3.20) follows. However, uniqueness and smoothness of weak solutions in the Crocco variables seem to be still unsettled. In particular, when L is not small enough the global existence of smooth solutions to the Prandtl equations remains open even under the monotonicity conditions (3.22), (3.23), and the pressure condition (3.24).

The Crocco transformation has been a basic tool in the classical works [116, 117, 118, 144]. Recently, an alternative approach, found indipendantly, has been presented in [4, 99], where the crucial part of the proof is based on a direct energy method but for new dependent variables. In particular, the Crocco transformation in not needed in this new approach.. The key new unknown is $w = \partial_2(\frac{u_1^P}{\tilde\omega^P})$ in [4] and $g = \omega^P \partial_2(\frac{u_1^P}{\omega^P})$ in [99], where $\omega^P = \partial_2 u_1^P$ and $\tilde\omega^P = \partial_2 \tilde u_1^P$ represent the vorticity fields of the Prandtl flow u_1^P and of the background Prandtl flow $\tilde u_1^P$, respectively. The function w is introduced in [4] in the analysis of the linearized Prandtl equations around $\tilde u_1^P$, and the function g is analyzed in [99] for the nonlinear energy estimate. As explained in [99] for example, the crucial obstacle one meets in the energy estimate for the standard unknowns $\partial_1^j u_1^P$ or $\partial_1^j \omega^P$ is the presence of the terms $(\partial_1^j u_2^P)\omega^P$ or $(\partial_1^j u_2^P)\partial_2\omega^P$, since they contain the highest order derivatives in x_1 and do not vanish after integration by parts. The new unknown is in fact chosen so that these crucial terms cancel. The development of the Prandtl theory in [4, 99] has had a significant impact in the field, and has led to various recent progress in this ield [78, 47, 68].

Very recently, the Prandtl equations have been studied studied in the three dimensional half space in [88, 87]. These works shows that the monotonicity condition on the tangential velocities is not enough to ensure the local well-posedness, and a sharp borderline condition is found for t well-posedness/ill-posedness.

(II) Analytic data. The second classical category for the local well-posedness of the Prandtl equations is the space of analytic functions. Under the analyticity of the initial data and of the outer Euler flow, the local existence of the Prandtl equations has been proven in [121] after the pioneer work [5], where the analyticity is imposed on both variables x_1 and x_2. Later, it was realized that the condition of the analyticity in the vertical variable x_2 can be removed, and the local well-posedness is known to hold under only analyticity in the tangential variable [89, 79, 22, 23].

To be precise, a typical existence result available by now in this category is stated here. Let $\sigma \in \mathbb{R}$, $\alpha, \beta, T > 0$. The space $\mathcal{H}^{\sigma,\alpha}$ is the space of functions $f(x_1, x_2)$, 2π periodic in x_1, such that the norm

$$|f|_{\sigma,\alpha} = \sum_{j\leq 2} \sup_{x_2\in\mathbb{R}_+} \langle x_2\rangle^\alpha \sum_{k\in\mathbb{Z}} |\partial_2^j \hat{f}(k, x_2)| e^{|k|\sigma}\,, \qquad \langle x_2\rangle = (1 + x_2^2)^{\frac{1}{2}}\,,$$

is finite. Here $\hat{f}(k, x_2)$ is the kth Fourier mode of f with respect to x_1. The space $\mathcal{H}^{\sigma,\alpha}_{\beta,T}$ is the space of functions $f(t, x_1, x_2)$, 2π periodic in x_1, such that the norm

$$|f|_{\sigma,\alpha,\beta,T} = \sum_{j\leq 2} \sup_{0\leq t\leq T} |\partial_2^j f(t)|_{\sigma-\beta t,\alpha} + \sup_{0\leq t\leq T} |\partial_t f(t)|_{\sigma-\beta t,\alpha}$$

is finite. The space $\mathcal{H}^{\sigma}_{\beta,T}$ is the space of functions $f(t, x_1)$, 2π periodic in x_1, such that the norm

$$|f|_{\sigma,\beta,T} = \sum_{i=0,1} \sup_{0\leq t\leq T} |\partial_t^i f(t)|_{\sigma-\beta t}$$

is finite. The spaces $\mathcal{H}^{\sigma,\alpha}$ and $\mathcal{H}^{\sigma,\alpha}_{\beta,T}$ are used for the boundary layer profiles, while $\mathcal{H}^{\sigma}_{\beta,T}$ is used for the Euler flows. The next theorem is shown in [23], where the local solvability is obtained even for the incompatible initial data $u^P_{0,1}|_{x_2=0} \neq 0$.

Theorem 3.12. *Let $u^E \in \mathcal{H}^{\sigma_0}_{\beta_0,T_0}$ and $u^P_{0,1} - u^E|_{t=0} \in \mathcal{H}^{\sigma_0,\alpha}$ for some $\sigma_0, \beta_0, T_0 > 0$ and $\alpha > \frac{1}{2}$. Then there exist $\sigma \in (0, \sigma_0)$, $\beta \in (0, \beta_0)$, and $T \in (0, T_0)$ such that Prandtl equations* (3.20) *admit a unique solution u^P_1 in $[0, T]$ of the form*

$$u^P_1(t, x_1, x_2) = -2u^P_{0,1}(x_1, 0)\,\mathrm{erfc}\,\Big(\frac{x_2}{2\sqrt{t}}\Big) + \tilde{u}(t, x_1, x_2) + u^E(t, x_1)\,,$$

where $\tilde{u} \in \mathcal{H}^{\sigma,\alpha}_{\beta,T}$. Here $\mathrm{erfc}\,\Big(\frac{x_2}{2\sqrt{t}}\Big) = \frac{1}{\sqrt{\pi t}} \int_{x_2}^{\infty} \exp\Big(-\frac{y_2^2}{4t}\Big)\,\mathrm{d}y_2$.

As in [5, 121, 89, 22], the proof given in [23] relies on, the abstract Cauchy-Kowalewski theorem, which is applied for the integral equations associated with (3.20). The existence of solutions under a polynomial decay condition on $u^P_{0,1} - u^E|_{t=0}$ is firstly proved by [79]. The proof of [79] is based on the direct energy method, rather than the use of the abstract Cauchy-Kowalewski theorem and the integral equations.

Without monotonicity, one cannot expect the global existence of smooth solutions to the Prandtl equations in general even if the given data is analytic. Indeed, the existence of finite time blowup solutions is shown ([37]; see Section 3.4.2 below). However, the class of initial data for blowup solutions in [37] must have $\mathcal{O}(1)$ size, and hence, it is still not clear whether the global existence is valid for sufficiently small given data or not. Recently an important progress has been achieved in this direction, and the long time well-posedness is established in [147, 68] for small solutions in the analytic functional framework. In [147] the life span of local solutions is estimated from below to be of the order $\mathcal{O}(\epsilon^{-\frac{3}{4}})$ when the uniform Euler flow $u^E = \underline{u}$ is of the order $\mathcal{O}(\epsilon^{\frac{5}{3}})$ and the initial data $u^P_{0,1}$ is $\mathcal{O}(\epsilon)$ in a suitable norm measuring the tangential analyticity. In [68] the almost global existence is established, where the smallness condition on the uniform Euler flow is removed and the life span is estimated from below as $\mathcal{O}(\exp(-\frac{1}{\epsilon \log \epsilon}))$, $0 < \epsilon \ll 1$, when $u^P_{0,1}$ is $\mathcal{O}(\epsilon)$.

(III) Other categories: beyond analyticity or monotonicity. Without monotonicity or analyticity of given data the solvability of the unsteady Prandtl equations becomes a highly difficult problem even locally in time. There are a few classes of initial data that are not strictly included in the categories (I) and (II) above but for which the Prandtl equations can be solved for a short time.

(1) *Gevrey class with a nondegenerate vorticity.* As will be seen in Section 3.4.2, the Prandtl equations are ill-posed in general in Sobolev spaces. This ill-posedness is due to the instability in high frequencies for the tangential components, occurring when the monotonicity of given data is absent. A key argument for this instability is given by [46], where it is proved that the linearization around the nonmonotonic shear flow satisfying (3.29) for some $a > 0$ has a solution growing exponentially in time with the growth rate $\mathcal{O}(|n|^{\frac{1}{2}})$ for high tangential frequencies n. Although such a high frequency instability yields the ill-posedness in Sobolev spaces, there is still a hope to obtain the well-posedness for initial data whose nth Fourier mode in the x_1 variable decays in the order $\mathcal{O}(e^{-c|n|^\gamma})$ for $|n| \gg 1$ with some $\gamma > \frac{1}{2}$, that is, the Gevrey class less than 2. A crucial difference between the Gevrey class 1 ($\gamma = 1$, analytic functions) and the Gevrey classes m with $m > 1$ ($\gamma = \frac{1}{m}$) is that the latter class can contain compactly supported functions. The verification of the instability by [46] motivates the work of [47], where the local solvability is established for a set of initial data without monotonicity, but belonging to the Gevrey class $\frac{7}{4}$ in the x_1 variable. The key condition for the initial data in [47] is that the monotonicity is absent only ona single smooth curve but in a nondegenerate manner. More precisely, in [47] it is assumed that $u^E = p^E = 0$ and $u^P_{0,1}$ is periodic in x_1 with a Gevrey $\frac{7}{4}$ regularity, and that

$$\partial_2 u^P_{0,1}(x_1, x_2) = 0 \quad \text{iff} \quad x_2 = a_0(x_1) > 0 \quad \text{with} \quad \partial_2^2 u^P_{0,1}(x_1, a_0(x_1)) > 0 \qquad \text{for all} \quad x_1 \in \mathbb{T}\,. \tag{3.26}$$

Note that the condition (3.26) is a natural counterpart of (3.29). The crucial observation for the proof in [47] is that in the region where the monotonicity is absent the flow is expected to be convex in virtue of the nondegenerate condition, while away from the curve of the critical points one can use the monotonicity of the flow. However, due to the nonlocal nature of the Prandtl equations, taking each advantage in a different region requires an intricate analysis, and the difficulty is overcome in [47] by introducing various kinds of energy.

(2) *Data with multiple monotonicity/analyticity regions.* The flows in this class are introduced by [78], where the local existence and uniqueness of the Prandtl equations are proved for initial data with multiple monotonicity regions, by assuming that the initial data is tangentially real analytic on the complement of the monotonicity regions. A typical example of the initial data in this category is

$$\partial_2 u^P_{0,1} < 0 \quad \text{for } x_1 < 0\,, \quad \partial_2 u^P_{0,1} > 0 \quad \text{for } x_1 > 0\,, \qquad u^P_{0,1} \text{ is real analytic in } x_1 \text{ around } x_1 = 0\,.$$

That is, the monotonicity of the initial flow is lost around $x_1 = 0$, but instead, the analyticity in the tangential variable is imposed there. Because of this complementary distributions of two totally different structures and the nonlocal nature of the problem, the methods developed in the categories (I) and (II) are not enough in constructing local solutions in this class. Indeed, there are several difficulties in this problem; the norms for the class of the analyticity and the monotonicity are not compatible, and moreover, nontrivial analytic functions cannot have a compact support, which indicates the breakdown of the standard localizing argument. The key observation in [78] is that one can in fact construct the analytic solution around $x_1 = 0$ in a decoupled manner without using the lateral boundary conditions. On the other hand, by introducing a suitable extension of the data in the tangential direction one can construct a monotone solution in the entire half plane. Finally, in virtue of the finite propagation property in the tangential direction the analytic solution and the monotonic solution actually coincides with each other on some strip regions, which implies the existence of solutions to the original Prandtl equations. The construction of [78] reveals in some sense a possibility of localizing the Prandtl equations in the tangential direction. Moreover, the result of [78] indicates that, even at the point of separation, the flow can be stable at least locally in time and space, if the flow is analytic in the tangential variable around the separation point.

3.4.2 Ill-posedness results for the Prandtl equations

Although the local solvability of the unsteady Prandtl equations still remains open for general initial data in Sobolev spaces, several ill-posedness results have been reported by now. This subsection is devoted to give an overview on a recent progress in this direction.

(I) Ill-posedness of the Prandtl equations in Sobolev spaces. When given data are not monotonic the unsteady Prandtl equations are known to be ill-posed in the sense of Hadamard. The ill-posedness is triggered by the instability of nonmonotonic shear flows at high frequencies. The first rigorous result for this instability is given by [46], where the linearization around a shear flow possessing a nondegenerate critical point is studied in details. To be precise, let $u^P = (u_1^P, u_2^P)$ be the solution to the Prandtl equations (3.20) for a constant data $u^E = \underline{u} \in \mathbb{R}$ and $\partial_1 p^E = 0$ (that is, the Euler flow u^0 is a stationary shear flow $u^0 = (u_1^0(x_2), 0)$ and $\underline{u} = u_1^0(0)$). In this case u^P is also a shear flow $u^P(t,x) = (u_s(t,x_2), 0)$, and u_s is the solution to the heat equations

$$\begin{cases} \partial_t u_s - \partial_2^2 u_s = 0\,, \\ u_s|_{t=0} = U_s\,, \qquad u_s|_{x_2=0} = 0\,, \qquad \lim_{x_2\to\infty} u_s = \underline{u}\,. \end{cases} \tag{3.27}$$

Here U_s is a given initial shear profile satisfying the compatibility conditions. Then, a natural question is whether or not one can construct a solution to the Prandtl equations around this shear flow. The key step to tackle this problem is to analyze the linearization around u_s:

$$\begin{cases} \partial_t v_1^P + u_s \partial_1 v_1^P + v_2^P \partial_2 u_s - \partial_2^2 v_1^P = 0\,, \\ v_2^P = -\displaystyle\int_0^{x_2} \partial_1 v_1^P \, \mathrm{d}y_2\,, \\ v_1^P|_{t=s} = v_{0,1}^P\,, \qquad v_1^P|_{x_2=0} = 0\,, \qquad \lim_{x_2\to\infty} v_1^P = 0\,. \end{cases} \tag{3.28}$$

Here, $t > s$, $x_1 \in \mathbb{T}$, and $x_2 \in \mathbb{R}_+$. The equations (3.28) is uniquely solvable at least locally in time if the initial data $v_{0,1}^P$ is analytic in the x_1 variable, and then the evolution operator $T(t,s)$, $T(t,s)v_{0,1}^P := v_1^P(t)$, is shown to be locally well-defined in the analytic functional framework; see [46, Proposition 1]. With this observation one can define the operator norm of $T(t,s)$ from H^{m_1} to H^{m_2}, where H^{m_1} and H^{m_2} are suitable Sobolev space in $\mathbb{T} \times \mathbb{R}_+$, and the exponents m_1, m_2 denote the order of the Sobolev regularity;

see [46] for the precise definition of H^m. In [46] the given initial data U_s in (3.27) is assumed to have a nondegenerate critical point: there is $a > 0$ such that

$$U_s'(a) = 0\,, \qquad U_s''(a) \neq 0\,. \tag{3.29}$$

Then the following ill-posedness in the Sobolev class is given by [46, Theorem 1].

Theorem 3.13. *If* (3.29) *holds then there exists* $\sigma > 0$ *such that for all* $\delta > 0$,

$$\sup_{0\le s\le t\le \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_1|}} T(t,s)\|_{\mathcal{L}(H^m, H^{m-\mu})} = \infty \qquad \text{for all} \quad m \ge 0\,, \quad \mu \in [0, \frac{1}{2})\,.$$

Moreover, there is a solution u_s *to* (3.27) *and* $\sigma > 0$ *such tat for all* $\delta > 0$,

$$\sup_{0\le s\le t\le \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_1|}} T(t,s)\|_{\mathcal{L}(H^{m_1}, H^{m_2})} = \infty \qquad \text{for all} \quad m_1, m_2 \ge 0\,.$$

Th key of the proof in [46] is to construct an approximate solution to (3.28), which grows in time with the order $e^{\delta |n|^{\frac{1}{2}}(t-s)}$, $\delta > 0$, for a tangential frequency $|n| \gg 1$. This growth rate $\mathcal{O}(|n|^{\frac{1}{2}})$ is responsible for the weight $e^{-\sigma(t-s)|\sqrt{|\partial_1|}}$ in the statement of Theorem 3.13. The approximate solution is constructed as a singular perturbation from an explicit solution to the inviscid linearized Prandtl equations (dropping the viscous term $\partial_2^2 v_1^P$ in (3.28), and replacing u_s by U_s), for which the spectral problem has been studied in details by [62]. This instability mechanism, bifurcating from the inviscid solution and resulting the growth rate $\mathcal{O}(|n|^{\frac{1}{2}})$, was first reported in a formal level by [31]. Due to the nature of the singular perturbation, however, the rigorous justification requires a highly delicate asymptotic analysis, and it is successfully completed by [46].

The result of [46] on the ill-posedness for the linearized Prandtl equations is strengthened by [57, 48], where it is shown that the solutions to the nonlinear Prandtl equations cannot be Lipschitz continuous with respect to the initial data in Sobolev spaces. Moreover, it is proved by [57] that in the Sobolev framework one cannot expect a natural estimate of the asymptotic boundary layer expansion for the Navier-Stokes flows, if the leading term of the boundary layer is a nonmonotonic shear layer flow as in [46].

Recently the ill-posedness of the three dimensional Prandtl equations is studied in [88], and it is revealed that there is a stronger instability mechanism in the three dimensional case even in the linear level. In particular, it is shown in [88] that, in contrast to the two dimensional case, the monotonicity condition on tangential velocity fields is not sufficient for the well-posedness of the three-dimensional Prandtl equations.

(II) Blowup solutions to the Prandtl equations. In the absence of the monotonicity it is known that the solution to the Prandtl equations can blow up in a finite time. The formation of such singularity was observed numerically in [133] for data corresponding to an impulsively started flow past a cylinder. The existence of blowup solutions was also reported by [62] through numerical and asymptotic analysis for the inviscid Prandtl equations. The rigorous existence of finite time blowup solutions is first given by [37], which is stated as follows.

Theorem 3.14. *Let* $u^E = p^P = 0$. *Assume that the initial data* $u_{0,1}^P$ *is of the form* $u_{0,1}^P(x_1,x_2) = -x_1 b_0(x_1,x_2)$ *for some smooth* b_0, *and that* $a_0(x_2) = -\partial_1 u_{0,1}^P(0,x_2)$ *is nonnegative, smooth, and compactly supported. Assume in addition that*

$$E(a_0) = \frac{1}{2}\|\partial_2 a_0\|_{L^2(\mathbb{R}_+)}^2 - \frac{1}{4}\|a_0\|_{L^3(\mathbb{R}_+)}^3 < 0 \tag{3.30}$$

holds. Then there exist no global smooth solutions to (3.20).

Remark 3.15. In Theorem 3.14 the condition of the compact support of a_0 is not essential, and it can be replaced by a decay condition which ensures the boundedness of $\partial_1 u_1^P(t,0,x_2)$ in $L^1_{x_2}(\mathbb{R}_+)$ as long as the solution exists.

A simple example of the initial data satisfying the conditions of Theorem 3.14 is

$$u^p_{0,1}(x_1,x_2) = -x_1 e^{-x_1^2} f(\frac{x_2}{R}), \tag{3.31}$$

where f is a (nontrivial) nonnegative smooth function with compact support, and $R > 0$ is a sufficiently large number. Indeed, in this case the function a_0 is given by $a_0(x_2) = -f(\frac{x_2}{R})$, and the quantity $E(a_0)$ is computed as

$$E(a_0) = \frac{1}{2R}\|\partial_2 f\|^2_{L^2(\mathbb{R}_+)} - \frac{R}{4}\|f\|^3_{L^3(\mathbb{R}_+)},$$

which is negative when $R > 0$ is large enough. Since the initial data defined by (3.31) is analytic in x_1, in virtue of Theorem 3.12 there exists a unique solution to (3.20) at least for a short time. Theorem 3.14 shows that this local solution cannot be extended as a global solution, and the proof in [37] implies that the blowup occurs for the quantity $\|\partial_1 u_1^P(t)\|_{L^\infty(\mathbb{R}^2_+)}$. The singularity formation observed in [133] is studied in details by [45], in which numerical evidence is reported for the strong ill-posedness of the Prandtl equations in the Sobolev space $H^1(\mathbb{R}_+)$.

4 Conclusion

The inviscid limit problem of the Navier-Stokes flows is one of the most fundamental and classical issues in fluid mechanics, particularly in understanding the flows at high Reynolds numbers. Mathematically, the fundamental question here is whether or not the Navier-Stokes flows converge to the Euler flows in the zero viscosity limit, by taking the effect of the boundary into account if necessary. Even when there is no physical boundary the analysis of the inviscid limit is a challenging problem if one works with singular flows such as vortex sheets or filaments (Section 2). The rigorous understanding of these structures, in terms of the analysis of the Navier-Stokes equations at high Reynolds numbers, is still out of reach except for the case when the distribution of the possible singularities can be well specified in advance due to some additional prescribed symmetry. In the presence of physical boundary the verification of the inviscid limit is far from trivial in general even when given data have enough regularity, e.g., in higher order Sobolev spaces (Section 3). The main obstacle is the formation of the viscous boundary layer, whose size and stability are crucially influenced by the type of the boundary conditions. If the boundary condition allows the flow to slip on the boundary or the boundary is non-characteristic, then the effect of the boundary layer is moderate and the mathematical theory has been well developed by now in these categories (Sections 3.1, 3.3). However, if the no-slip boundary condition is imposed the size of the boundary layer is at least $\mathcal{O}(1)$ even in the formal level, and the underlying instability mechanism of the boundary layer leads to a serious difficulty in the analysis of the inviscid limit problem (Section 3.2). The general result in this research area is Kato's criterion, which describes the condition for the converges of the Navier-Stokes flows to the Euler flows in the energy space. This criterion can be confirmed under some symmetry conditions on both fluid domain and given data without assuming strong regularity of data such as analyticity. Without symmetry the known results verifying the inviscid limit (the convergence in the energy space as well as the Prandtl boundary layer expansion) require, so far, the analyticity at least near the boundary, and the boundary is also assumed to be flat there. In understanding the formation of the boundary layer the analysis of the Prandtl equations is a central issue (Section 3.4). However, the solvability of the Prandtl equations is available only for some restricted classes of given data including the monotonicity or the analyticity (Section 3.4.1). In the general Sobolev framework the strong instability and resulting ill-posedness have been observed (Section 3.4.2). Moreover, without monotonicity the large solutions to the Prandtl equations may blow up in a finite time, while the existence of smooth global solutions for small data but in a wide fluid domain remains open even under the monotonicity condition in general. Finally, even when the Prandtl equations are solved, it does not mean the validity of the inviscid limit, and in fact, there is a significant discrepancy between these two problems. This discrepancy actually indicates the limitation of the Prandtl equations for tunderstanding real boundary layer separation, although the latter has been often discussed within the framework of the Prandtl equations. The mathematical understanding of the stability of the boundary layer and the analysis of the boundary layer separation need a significant development of the present theory for the inviscid limit problem of the Navier-Stokes flows.

Cross references

- Existence and stability of viscous vortices

References

[1] H. Abidi and R. Danchin. Optimal bounds for the inviscid limit of Navier-Stokes equations. *Asymptot. Anal.*, 38:35–46, 2004.

[2] S. N. Alekseenko. Solution of a degenerate linearized Navier-Stokes system with a homogeneous boundary condition. In *Studies in integro-differential equations, No. 16*, pages 243–257. "Ilim", Frunze, 1983.

[3] S. N. Alekseenko. On vanishing viscosity in a linearized problem of the flow of an incompressible fluid. In *Studies in integro-differential equations, No. 19 (Russian)*, pages 288–304, 320. "Ilim", Frunze, 1986.

[4] R. Alexandre, Y.-G. Wang, C.-J. Xu, and T. Yang. Well-posedness of the prandtl equation in sobolev spaces. *J. Am. Math. Soc.*, 28(3):745–784, 2015.

[5] K. Asano. A note on the abstract Cauchy-Kowalewski theorem. *Proc. Japan Acad. Ser. A Math. Sci.*, 64(4):102–105, 1988.

[6] C. Bardos. Existence et unicité de la solution de l'équation d'Euler en dimension deux. *J. Math. Anal. Appl.*, 40:769–790, 1972.

[7] C. Bardos, M. C. Lopes Filho, D. Niu, H. J. Nussenzveig Lopes, and E. S. Titi. Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking. *SIAM J. Math. Anal.*, 45(3):1871–1885, 2013.

[8] H. Beirão da Veiga and F. Crispo. Sharp inviscid limit results under Navier type boundary conditions. An L^p theory. *J. Math. Fluid Mech.*, 12(3):397–411, 2010.

[9] H. Beirão da Veiga and F. Crispo. Concerning the $W^{k,p}$-inviscid limit for 3-D flows under a slip boundary condition. *J. Math. Fluid Mech.*, 13(1):117–135, 2011.

[10] H. Beirão da Veiga and F. Crispo. The 3-D inviscid limit result under slip boundary conditions. A negative answer. *J. Math. Fluid Mech.*, 14(1):55–59, 2012.

[11] H. Beirão da Veiga and F. Crispo. A missed persistence property for the Euler equations and its effect on inviscid limits. *Nonlinearity*, 25(6):1661–1669, 2012.

[12] H. Bellout, J. Neustupa, and P. Penel. On the Navier-Stokes equation with boundary conditions based on vorticity. *Math. Nachr.*, 269/270:59–72, 2004.

[13] L. C. Berselli and S. Spirito. On the vanishing viscosity limit of 3D Navier-Stokes equations under slip boundary conditions in general domains. *Comm. Math. Phys.*, 316(1):171–198, 2012.

[14] A. Bertozzi and P. Constantin. Global regularity for vortex patches. *Commun. Math. Phys.*, 152(1):19–28, 1993.

[15] J. L. Bona and J. Wu. The zero-viscosity limit of the 2D Navier-Stokes equations. *Stud. Appl. Math.*, 109(4):265–278, 2002.

[16] E. Brunelli and C. Marchioro. Vanishing viscosity limit for a smoke ring with concentrated vorticity. *J. Math. Fluid Mech.*, 13:421–428, 2011.

[17] D. Bucur, E. Feireisl, and Š. Nečasová. Boundary behavior of viscous fluids: influence of wall roughness and friction-driven boundary conditions. *Arch. Ration. Mech. Anal.*, 197(1):117–138, 2010.

[18] A. V. Busuioc, D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. Incompressible Euler as a limit of complex fluid models with Navier boundary conditions. *J. Differential Equations*, 252(1):624–640, 2012.

[19] R. E. Caflisch and O. F. Orellana. Long time existence for a slightly perturbed vortex sheet. *Comm. Pure Appl. Math.*, 39(6):807–838, 1986.

[20] R. E. Caflisch and O. F. Orellana. Singular solutions and ill-posedness for the evolution of vortex sheets. *SIAM J. Math. Anal.*, 20(2):293–307, 1989.

[21] R. E. Caflisch and M. Sammartino. Vortex layers in the small viscosity limit. In *WASCOM 2005-13th Conference on Waves and Stability in Continuous Media*, pages 59–70. World Scientific Publishing Company, Hackensack, 2006.

[22] M. Cannone, M. C. Lombardo, and M. Sammartino. Well-posedness of Prandtl equations with non-compatible data. *Nonlinearity*, 26(12):3077–3100, 2013.

[23] M. Cannone, M. C. Lombardo, and M. Sammartino. On the Prandtl boundary layer equations in presence of corner singularities. *Acta Appl. Math.*, 132:139–149, 2014.

[24] D. Chae and P. Dubovskii. Functional and measure-valued solutions to the Euler equations for flows of incompressible fluids. *Arch. Rational Mech. Anal.*, 129:385–396, 1995.

[25] J.-Y. Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. Ecole Norm. Sup. (4)*, 26(4):517–542, 1993.

[26] J.-Y. Chemin. A remark on the inviscid limit for two-dimensional incompressible fluids. *Comm. Partial Differential Equations*, 21:1771–1779, 1996.

[27] T. Clopeau, A. Mikelić, and R. Robert. On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions. *Nonlinearity*, 11(6):1625–1636, 1998.

[28] P. Constantin, I. Kukavica, and V. Vicol. On the inviscid limit of the Navier-Stokes equations. *Proc. Amer. Math. Soc.*, 143(7):3075–3090, 2015.

[29] P. Constantin and J. Wu. Inviscid limit for vortex patches. *Nonlinearity*, 8:735–742, 1995.

[30] P. Constantin and J. Wu. The inviscid limit for non-smooth vorticity. *Indiana Univ. Math. J.*, 45:67–81, 1996.

[31] S. J. Cowley, L. M. Hocking, and O. R. Tutty. The stability of solutions of the classical unsteady boundary-layer equation. *Phys. Fluids*, 28:441–443, 1985.

[32] R. Danchin. Poches de tourbillon visqueuses. *J. Math. Pures Appl.*, 76:609–647, 1997.

[33] R. Danchin. Persistance de structures géométriques et limite non visqueuse pour les fluides incompressibles en dimension quelconque. *Bull. Soc. Math. France*, 127:179–227, 1999.

[34] N. Depauw. Poche de tourbillon pour Euler 2D dans un ouvert à bord. *J. Math. Pures Appl. (9)*, 78(3):313–351, 1999.

[35] J. Duchon and R. Robert. Global vortex sheet solutions of Euler equations in the plane. *J. Differential Equations*, 73(2):215–224, 1988.

[36] A. Dutrifoy. On 3-D vortex patches in bounded domains. *Comm. Part. Differ. Eqs.*, 28(7-8):1237–1263, 2003.

[37] W. E and B. Engquist. Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.*, 50(12):1287–1293, 1997.

[38] D. Ebin and J. Marsden. Groups of diffeomorphisms and the notion of an incompressible fluid. *Ann. Math.*, 92:102–163, 1970.

[39] H. Feng and V. Šverák. On the Cauchy problem for axi-symmetric vortex rings. *Arch. Rational Mech. Anal.*, 215:89–123, 2015.

[40] I. Gallagher and T. Gallay. Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity. *Math. Ann.*, 332:287–327, 2005.

[41] T. Gallay. Interaction of vortices in weakly viscous planar flows. *Arch. Rational Mech. Anal.*, 200:445–490, 2011.

[42] T. Gallay and Y. Maekawa. Existence and stability of viscous vortices. In *Handbook of Mathematical Analysis of Mechanics in Viscous Fluids*. Springer.

[43] T. Gallay and C. E. Wayne. Global stability of vortex solutions of the two-dimensional Navier-Stokes equation. *Comm. Math. Phys.*, 255:97–129, 2005.

[44] P. Gamblin and X. Saint Raymond. On three-dimensional vortex patches. *Bull. Soc. Math. France*, 123(3):375–424, 1995.

[45] F. Gargano, M. Sammartino, and V. Sciacca. Singularity formation for Prandtl's equations. *Phys. D*, 238(19):1975–1991, 2009.

[46] D. Gérard-Varet and E. Dormy. On the ill-posedness of the Prandtl equation. *J. Amer. Math. Soc.*, 23(2):591–609, 2010.

[47] D. Gérard-Varet and N. Masmoudi. Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Scient. Éc. Norm. Sup.*, 48(4):1273–1325, 2015.

[48] D. Gérard-Varet and T. Nguyen. Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.*, 77(1-2):71–88, 2012.

[49] G.-M. Gie. Asymptotic expansion of the Stokes solutions at small viscosity: the case of non-compatible initial data. *Commun. Math. Sci.*, 12(2):383–400, 2014.

[50] G.-M. Gie and C.-Y. Jung. Vorticity layers of the 2D Navier-Stokes equations with a slip type boundary condition. *Asymptot. Anal.*, 84(1-2):17–33, 2013.

[51] G.-M. Gie and J. P. Kelliher. Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions. *J. Differential Equations*, 253(6):1862–1892, 2012.

[52] G.-M. Gie, J. P. Kelliher, M. C. Lopes, A. L. Mazzucato, and H. J. Nussenzveig Lopes. The vanishing viscosity limit for some symmetric flows. In final preparation.

[53] Y. Giga and T. Miyakawa. Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces. *Commnun. Partial Differential Equations*, 14(5):577–618, 1989.

[54] Y. Giga, T. Miyakawa, and H. Osada. Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.*, 104:223–250, 1988.

[55] K. K. Golovkin. Vanishing viscosity in the cauchy problem for equations of hydrodynamics. *Trudy Mat. Inst. Steklov.*, 92:31–49, 1966.

[56] E. Grenier. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.*, 53(9):1067–1091, 2000.

[57] Y. Guo and T. Nguyen. A note on Prandtl boundary layers. *Comm. Pure Appl. Math.*, 64(10):1416–1438, 2011.

[58] D. Han, A. L. Mazzucato, D. Niu, and X. Wang. Boundary layer for a class of nonlinear pipe flow. *J. Differential Equations*, 252(12):6387–6413, 2012.

[59] D. Han and X. Wang. Initial-boundary layer associated with the nonlinear Darcy-Brinkman system. *J. Differential Equations*, 256(2):609–639, 2014.

[60] T. Hmidi. Régularité h'oldérienne des poches de tourbillon visqueuses. *J. Math. Pures Appl.*, 84:1455–1495, 2005.

[61] T. Hmidi. Poches de tourbillon singulières dans un fluide faiblement visqueux. *Rev. Mat. Iberoamericana*, 22:489–543, 2006.

[62] L. Hong and J. Hunter. Singularity formation and instability in the unsteady inviscid and viscous Prandtl equations. *Comm. Math. Sci.*, 1:293–316, 2003.

[63] C. Huang. Remarks on regularity of non-constant vortex patches. *Commun. Appl. Anal.*, 3(4):449–459, 1999.

[64] C. Huang. Singular integral system approach to regularity of 3d vortex patches. *Indiana Univ. Math. J.*, 50(1):509–552, 2001.

[65] D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes. Incompressible flow around a small obstacle and the vanishing viscosity limit. *Comm. Math. Phys.*, 287(1):99–115, 2009.

[66] D. Iftimie and G. Planas. Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions. *Nonlinearity*, 19(4):899–918, 2006.

[67] D. Iftimie and F. Sueur. Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. *Arch. Ration. Mech. Anal.*, 199(1):145–175, 2011.

[68] M. Ignatova and V. Vicol. Almost global existence for the Prandtl boundary layer equations. *Arch. Rational Mech. Anal.*, 220:809–848, 2016.

[69] W. Jäger and A. Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

[70] T. Kato. Nonstationary flows of viscous and ideal fluids in $\mathbf{R}^3$. *J. Functional Analysis*, 9:296–305, 1972.

[71] T. Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 85–98. Springer, New York, 1984.

[72] T. Kato. The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity. *Differ. Integral Equ.*, 7:949–966, 1994.

[73] J. P. Kelliher. Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane. *SIAM J. Math. Anal.*, 38(1):210–232 (electronic), 2006.

[74] J. P. Kelliher. On Kato's conditions for vanishing viscosity. *Indiana Univ. Math. J.*, 56(4):1711–1721, 2007.

[75] J. P. Kelliher. Vanishing viscosity and the accumulation of vorticity on the boundary. *Commun. Math. Sci.*, 6(4):869–880, 2008.

[76] J. P. Kelliher, M. C. L. Filho, and H. J. N. Lopes. Vanishing viscosity limit for an expanding domain in space. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(6):2521–2537, 2009.

[77] J. P. Kelliher, R. Temam, and X. Wang. Boundary layer associated with the Darcy-Brinkman-Boussinesq model for convection in porous media. *Phys. D*, 240(7):619–628, 2011.

[78] I. Kukavica, N. Masmoudi, V. Vicol, and T. K. Wong. On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.*, 46(6):3865–3890, 2014.

[79] I. Kukavica and V. Vicol. On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.*, 11(1):269–292, 2013.

[80] C. Lacave and A. L. Mazzucato. The vanishing viscosity limit in the presence of a porous medium. *Math. Ann.*, 2015.

[81] E. Lauga, M. Brenner, and H. Stone. Microfluidics: The no-slip boundary condition. In C. Tropea, A. Yarin, and J. F. Foss, editors, *Springer Handbook of Experimental Fluid Mechanics*, pages 1219–1240. Springer, 2007.

[82] G. Lebeau. Régularité du problème de Kelvin-Helmholtz pour l'équation d'euler 2D. In *Séminaire: Equations aux Dérivées Partielles, 2000-2001, Exp. No. II, Séminaire: Équations aux Dérivées Partielles*, page 12 pp. École Polytechnique, Palaiseau, France, 2001.

[83] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, 1969.

[84] J.-L. Lions. *Perturbations singulières dans les problèmes aux limites et en contrôle optimal.* Lecture Notes in Mathematics, Vol. 323. Springer-Verlag, Berlin-New York, 1973.

[85] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press Oxford University Press, New York, 1996.

[86] C.-J. Liu and Y.-G. Wang. Derivation of Prandtl boundary layer equations for the incompressible Navier-Stokes equations in a curved domain. *Appl. Math. Lett.*, 34:81–85, 2014.

[87] C. J. Liu, Y.-G. Wang, and T. Yang. A well-posedness theory for the Prandtl equations in three space variables. *ArXiv e-prints*, May 2014.

[88] C.-J. Liu, Y.-G. Wang, and T. Yang. On the ill-posedness of the Prandtl equations in three-dimensional space. *Arch. Rational Mech. Anal.*, 220:83–108, 2016.

[89] M. C. Lombardo, M. Cannone, and M. Sammartino. Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.*, 35(4):987–1004 (electronic), 2003.

[90] M. C. Lombardo and M. Sammartino. Zero viscosity limit of the Oseen equations in a channel. *SIAM J. Math. Anal.*, 33(2):390–410 (electronic), 2001.

[91] M. C. Lopes Filho, A. L. Mazzucato, and H. J. Nussenzveig Lopes. Vanishing viscosity limit for incompressible flow inside a rotating circle. *Phys. D*, 237(10-12):1324–1333, 2008.

[92] M. C. Lopes Filho, A. L. Mazzucato, H. J. Nussenzveig Lopes, and M. Taylor. Vanishing viscosity limits and boundary layers for circularly symmetric 2D flows. *Bull. Braz. Math. Soc. (N.S.)*, 39(4):471–513, 2008.

[93] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and G. Planas. On the inviscid limit for two-dimensional incompressible flow with Navier friction condition. *SIAM J. Math. Anal.*, 36(4):1130–1141 (electronic), 2005.

[94] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and S. Schochet. A criterion for the equivalence of the Birkhoff-Rott and Euler descriptions of vortex sheet evolution. *Trans. Amer. Math. Soc.*, 359(9):4125–4142, 2007.

[95] M. C. Lopes Filho, H. J. Nussenzveig Lopes, E. S. Titi, and A. Zang. Approximation of 2D Euler equations by the second-grade fluid equations with Dirichlet boundary conditions. *J. Math. Fluid Mech.*, 17(2):327–340, 2015.

[96] Y. Maekawa. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. *Comm. Pure Appl. Math.*, 67(7):1045–1128, 2014.

[97] A. J. Majda and A. L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[98] M. Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.*, 39:S187–S220, 1986.

[99] N. Mamoudi and T. Wong. Local-in-time existence and uniqueness of solutions to the prandtl equations by energy methods. *Commun. Pure Appl. Math.*, 68:1683–1741, 2015.

[100] C. Marchioro. On the vanishing viscosity limit for two-dimensional Navier-Stokes equations with singular initial data. *Math. Methods Appl. Sci.*, 12:463–470, 1990.

[101] C. Marchioro. On the inviscid limit for a fluid with a concentrated vorticity. *Comm. Math. Phys.*, 196:53–65, 1998.

[102] C. Marchioro. Vanishing viscosity limit for an incompressible fluid with concentrated vorticity. *J. Math. Phys.*, 48:065302, 2007.

[103] C. Marchioro and M. Pulvirenti. Vortices and localization in Euler flows. *Comm. Math. Phys.*, 154:49–61, 1993.

[104] N. Masmoudi. Examples of singular limits in hydrodynamics. In *Handbook of differential equations: evolutionary equations. Vol III*, pages 195–275. Elsevier/North-Holland, Amsterdam, 2007.

[105] N. Masmoudi. Remarks about the inviscid limit of the Navier-Stokes system. *Comm. Math. Phys.*, 270(3):777–788, 2007.

[106] N. Masmoudi and F. Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.*, 203(2):529–575, 2012.

[107] S. Matsui. Example of zero viscosity limit for two-dimensional nonstationary Navier-Stokes flows with boundary. *Japan J. Indust. Appl. Math.*, 11(1):155–170, 1994.

[108] S. Matusi and T. Shirota. On separation points of solutions to Prandtl boundary layer problem. *Hokkaido Math. J.*, 13:92–108, 1984.

[109] A. Mazzucato, D. Niu, and X. Wang. Boundary layer associated with a class of 3D nonlinear plane parallel channel flows. *Indiana Univ. Math. J.*, 60(4):1113–1136, 2011.

[110] A. Mazzucato and M. Taylor. Vanishing viscosity limits for a class of circular pipe flows. *Comm. Partial Differential Equations*, 36(2):328–361, 2011.

[111] A. L. Mazzucato and M. E. Taylor. Vanishing viscosity plane parallel channel flow and related singular perturbation problems. *Analysis & PDE*, 1(1):35–93, 2008.

[112] F. J. McGrath. Nonstationary plane flow of viscous and ideal fluids. *Arch. Rational Mech. Anal.*, 27:329–348, 1968.

[113] A. Mikelić, Š. Nečasová, and M. Neuss-Radu. Effective slip law for general viscous flows over an oscillating surface. *Math. Methods Appl. Sci.*, 36(15):2086–2100, 2013.

[114] A. Mikelić and L. Paoli. Homogenization of the inviscid incompressible fluid flow through a 2D porous medium. *Proc. Amer. Math. Soc.*, 127(7):2019–2028, 1999.

[115] O. A. Oleĭnik. On the system of equations of the boundary layer theory. *Zh. bychisl. matem. i matem. fiz.*, 3(3):489–507, 1963.

[116] O. A. Oleĭnik. On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid. *J. Appl. Math. Mech.*, 30:951–974 (1967), 1966.

[117] O. A. Oleĭnik. On the stability of solutions of the system of boundary layer equations for a nonstationary flow of an incompressible fluid. *PMM*, 30(3):417–423, 1966.

[118] O. A. Oleĭnik and V. N. Samokhin. *Mathematical models in boundary layer theory*, volume 15 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[119] L. Prandtl. Über Flüssigkeitsbewegung bei sehr kleiner Reibung. *Verh. III Intern. Math. Kongr. Heidelberg*, pages 485–491, 1904.

[120] W. M. Rusin. On the inviscid limit for the solutions of two-dimensional incompressible Navier-Stokes equations with slip-type boundary conditions. *Nonlinearity*, 19(6):1349–1363, 2006.

[121] M. Sammartino and R. E. Caflisch. Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I, II. *Comm. Math. Phys.*, 192(2):433–491, 1998.

[122] H. Schlichting and K. Gersten. *Boundary-layer theory*. Springer-Verlag, Berlin, enlarged edition, 2000. With contributions by Egon Krause and Herbert Oertel, Jr., Translated from the ninth German edition by Katherine Mayes.

[123] F. Sueur. A Kato type theorem for the inviscid limit of the Navier-Stokes equations with a moving rigid body. *Comm. Math. Phys.*, 316(3):783–808, 2012.

[124] F. Sueur. On the inviscid limit for the compressible Navier-Stokes system in an impermeable bounded domain. *J. Math. Fluid Mech.*, 16(1):163–178, 2014.

[125] F. Sueur. Viscous profiles of vortex patches. *J. Inst. Math. Jussieu*, 14(1):1–68, 2015.

[126] C. Sulem, P.-L. Sulem, C. Bardos, and U. Frisch. Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability. *Comm. Math. Phys.*, 80(4):485–516, 1981.

[127] H. S. G. Swann. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in R_3. *Trans. Amer. Math. Soc.*, 157:373–397, 1971.

[128] R. Temam and X. Wang. Asymptotic analysis of Oseen type equations in a channel at small viscosity. *Indiana Univ. Math. J.*, 45(3):863–916, 1996.

[129] R. Temam and X. Wang. Boundary layers for Oseen's type equation in space dimension three. *Russian J. Math. Phys.*, 5(2):227–246 (1998), 1997.

[130] R. Temam and X. Wang. On the behavior of the solutions of the Navier-Stokes equations at vanishing viscosity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(3-4):807–828 (1998), 1997. Dedicated to Ennio De Giorgi.

[131] R. Temam and X. Wang. Boundary layers in channel flow with injection and suction. *Appl. Math. Lett.*, 14(1):87–91, 2001.

[132] R. Temam and X. Wang. Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case. *J. Differential Equations*, 179(2):647–686, 2002.

[133] L. L. Van Dommelen and S. F. Shen. The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comput. Phys.*, 38:125–140, 1980.

[134] M. Van Dyke. *An album of fluid motion.* Parabolic Press, 1982.

[135] M. Vishik. Incompressible flows of an ideal fluid with vorticity in borderline spaces of besov type. *Ann. Sci. Ecole Norm. Sup.*, 32:769–812, 1999.

[136] M. I. Vishik and L. A. Lyusternik. Regular degeneration and boundary layer for linear differential equations with small parameter. *Uspekki Mat. Nauk*, 12:3–122, 1957.

[137] X. Wang. A Kato type theorem on zero viscosity limit of Navier-Stokes flows. *Indiana Univ. Math. J.*, 50(Special Issue):223–241, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[138] Y.-G. Wang and Z. Xin. Zero-viscosity limit of the linearized compressible Navier-Stokes equations with highly oscillatory forces in the half-plane. *SIAM J. Math. Anal.*, 37(4):1256–1298, 2005.

[139] S. Wu. Mathematical analysis of vortex sheets. *Comm. Pure Appl. Math.*, 59(8):1065–1206, 2006.

[140] Y. Xiao and Z. Xin. On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition. *Comm. Pure Appl. Math.*, 60(7):1027–1055, 2007.

[141] Y. Xiao, Z. Xin, and J. Wu. Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition. *J. Funct. Anal.*, 257(11):3375–3394, 2009.

[142] X. Xie and C. Li. Vanishing viscosity limit for viscous magnetohydrodynamic equations with a slip boundary condition. *Appl. Math. Sci. (Ruse)*, 5(41-44):1999–2011, 2011.

[143] Z. Xin and T. Yanagisawa. Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane. *Comm. Pure Appl. Math.*, 52(4):479–541, 1999.

[144] Z. Xin and L. Zhang. On the global existence of solutions to the Prandtl's system. *Adv. Math.*, 181:88–133, 2004.

[145] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066 (Russian), 1963.

[146] V. I. Yudovisch. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2:27–38, 1995.

[147] P. Zhang and Z. Zhang. Long time well-posedness of Prandtl system with small and analytic initial data. *Journal of Functional Analysis*, 270:2591–2615, 2016.