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BOUNDARY VALUE PROBLEMS ON LIPSCHITZ DOMAINS IN $\mathbb{R}^n$ OR $\mathbb{C}^n$

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The purpose of this note is to bring update results on boundary value problems on Lipschitz domains in $\mathbb{R}^n$ or $\mathbb{C}^n$.

We first discuss the Dirichlet problem, the Neumann problem and the d-Neumann problem in a bounded domain in $\mathbb{R}^n$. These problems are the prototypes of coercive (or elliptic) boundary value problems when the boundary of the domain is smooth. When the domain is only Lipschitz, solutions to such problems are not necessarily smooth. The study of these boundary value problems on Lipschitz domains in $\mathbb{R}^n$ over the past thirty years has played important role in singular integrals, harmonic analysis and partial differential equations. In contrast, its counterpart in $\mathbb{C}^n$ is the $\bar\partial$-Neumann problem on pseudoconvex domains, which has not been completely understood. Our goal is to provide an overall view of these problems and review the main results with emphasis on L^2 Sobolev estimates.

We first give some basic properties of Lipschitz domains in Chapter 0. These simple and useful facts do not seem to have been systematically treated. In Chapter 1 we review the L^2 theory for boundary value problems in Lipschitz domains in $\mathbb{R}^n$. For the Dirichlet and Neumann boundary value problems, these results are mainly obtained by Dahlberg [Da], Jerison-Kenig [JK]. The reader should consult the book by Kenig [Ke] for a more detailed proof and treatment of these topics. For the d-Neumann problem, the results are due to Mitrea-Mitrea [MM] and Mitrea-Taylor [MT]. The monograph by Mitrea-Mitrea-Taylor [MMT]) gives a detailed account of the best results in this direction. Our goal here is to introduce the reader to the real d-Neumann problem, a subject of great importance in view of the Hodge theory on manifolds with boundary (see Morrey [Mo] for domains with smooth boundary). In the second part, we survey the results of the $\bar\partial$-Neumann problems on Lipschitz domains in $\mathbb{C}^n$.

Chapter 0. Preliminaries for Lipschitz domains

A bounded domain $D \subset\subset \mathbb{R}^n$ is called Lipschitz if locally the boundary bD is the graph of a Lipschitz function. Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a function which satisfies the Lipschitz condition

$$(0.1) \qquad |\psi(y') - \psi(x')| \le M|y' - x'|, \quad \text{for all } y',\ x' \in \mathbb{R}^{n-1}.$$

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A bounded domain D is called Lipschitz if near every boundary point $p \in bD$, there exists a neighborhood U of p such that after a rotation,

$$D \cap U = \{(x', x_n) \in U \mid x_n > \psi(x')\}$$

for some Lipschitz function ψ. The smallest M in which (0.1) holds will be called the bound of the Lipschitz constant. By choosing finitely many balls $\{U_i\}$ covering bD, the Lipschitz constant for a Lipschitz domain is the smallest M such that the Lipschitz constant is bounded by M in every ball U_i. A Lipschitz function is differentiable almost everywhere (See Evans-Gariepy [EG] for a proof of this fact).

Let D be a bounded Lipschitz domain. A Lipschitz function ρ is called a *global* defining function for D if $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfies $\rho < 0$ in D, $\rho > 0$ outside $\overline{D}$ and

$$C_1 < |d\rho| < C_2 \quad \text{a.e. on } bD, \tag{0.2}$$

where $C_1,\ C_2$ are positive constants.

Lemma 0.1. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$. Then D has a global Lipschitz defining function ρ.*

Proof. We cover bD by finitely many boundary coordinate patches U_i where $i = 1, \cdots, K$. Let r_i be a local defining function on U_i which is locally a Lipschitz graph. Let $\phi_i \in C_0^\infty(U_i)$ be a partition of unity such that $\sum_i \phi_i = 1$ in a neighborhood of bD. We define $\rho = \sum_i \phi_i r_i$. Then ρ is a defining function for D. To see that ρ satisfies (0.2), it is easy to see that $|\nabla\rho| \le C_2$ a.e. on bD. Since D is Lipschitz, for each $p \in bD$ such that the tangent plane of p for bD exists, one can find ξ such that $\langle \nabla r_i, \xi\rangle_p \ge C_0 > 0$ for every i such that $p \in U_i$. Also we have $\nabla\rho = \sum_i \phi_i \nabla r_i$ on bD if ∇r_i exists. Thus

$$\langle \nabla\rho, \xi\rangle_p = \langle \sum_i \phi_i \nabla r_i, \xi\rangle_p \ge \sum_i \phi_i(p) C_0 = C_0.$$

This proves that ρ satisfies (0.2).

Lemma 0.2. *If ρ is any global Lipschitz defining function for D and r is a local defining function in a neighborhood U of a boundary point $p \in bD$ such that $r(x) = x_n - \psi(x_1, \cdots, x_{n-1})$ for some Lipschitz function ψ, then there exists a positive function $h(x) \in L^\infty(\overline{D} \cap U) \cap C(D \cap U)$ such that*

$$\rho(x) = h(x) r(x), \quad x \in \overline{D} \cap U, \tag{0.3}$$

$$d\rho(x) = h(x) dr(x), \quad a.e.\ x \in bD \cap U \tag{0.4}$$

where

$$\tilde{C}_1 \le h(x) \le \tilde{C}_2, \qquad a.e.\ on\ bD$$

for some positive constants $\tilde{C}_1$ and $\tilde{C}_2$.

Proof. Since $\nabla r(x) = (-\nabla\psi, 1)$, we have

$$C_1 \le |dr(x)| \le C_2 \quad \text{a.e. on } U \cap bD.$$

Define

$$h(x) = \frac{\rho(x)}{r(x)}, \quad x \in U \cap D.$$

Then $h(x)$ is Lipschitz on D and $h(x) > 0$ in $U \cap D$. Let p be any point in $U \cap bD$ such that both $d\rho$ and dr exist at p. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma$, $-\langle \nabla r, \xi\rangle_p \ge C_0 > 0$. This implies that

$$-\langle \frac{\nabla\rho}{|\nabla\rho|}, \xi\rangle_p \ge \frac{C_0}{|\nabla r|} > 0, \quad p \in U \cap bD. \tag{0.5}$$

Since r and ρ are differentiable at p, we have for any $x \in \{\xi + \{p\}\} \cap U$,

$$\frac{\rho(x)}{r(x)} = \frac{\rho(x) - r(p)}{r(x) - \rho(p)} \to \frac{D_\xi \rho(p)}{D_\xi r(p)}. \tag{0.6}$$

Thus $h(x)$ has nontangential boundary value a.e. on $U \cap bD$. Also from (0.5) and (0.6), there exist $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ such that

$$\tilde{C}_1 \le h(x) \le \tilde{C}_2, \quad \text{a.e. for } x \in U \cap bD.$$

(0.4) follows from the definition of differentiation.

Lemma 0.3. *Let D be a bounded Lipschitz domain in $\mathbb{R}^n$. There exists an exhaustion $\{D_\nu\}$ of D such that*

(1) *$\{D_\nu\}$ is an increasing sequence of relatively compact subsets of D and*

$$\bigcup_\nu D_\nu = D.$$

(2) *Each D_ν has a C^∞ defining function η_ν, i.e., $D_\nu = \{x \in \mathbb{R}^n; \rho_\nu(x) < 0\}$.*

(3) *There exist positive constants c_1, c_2 such that $c_1 \le |\nabla \eta_\nu| \le c_2$ on ∂D_ν, where c_1, c_2 are independent of ν.*

Proof. Using Lemma 0.1, there exists a global Lipschitz defining function ρ for D satisfying (0.2) and ρ is obtained by a partition of unity of defining functions r_i which is a Lipschitz graph. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\chi \ge 0$, $\int \chi dV = 1$, $\chi(x)$ depends only on $|x|$ and vanishes when $|x| > 1$.

We define $\chi_\varepsilon(x) = \dfrac{1}{\varepsilon^n}\chi\left(\dfrac{x}{\varepsilon}\right)$ for $\varepsilon > 0$. Let $\delta_\nu \searrow 0$ and we define $D_{\delta_\nu} = \{x \in D|\ \rho(x) < -\delta_\nu\}$. Then D_{δ_ν} is a sequence of relatively compact open subsets of D with union equal to D. Each ρ_{ϵ_ν} is well defined if $0 < \epsilon_\nu < \delta_{\nu+1}$ for $x \in D_{\delta_{\nu+1}}$. Letting $c_2 = \sup_D|\nabla\rho|$, then for ϵ_ν sufficiently small, we have

$$\rho(x) < \rho_{\epsilon_\nu(x)} < \rho(x) + c_2\epsilon_\nu$$

on $D_{\delta_{\nu+1}}$. For each ν we choose

$$\epsilon_\nu = \frac{1}{2c_2}(\delta_{\nu-1} - \delta_\nu)$$

and $t_\nu \in (\delta_{\nu+1}, \delta_\nu)$. We define

$$D_\nu = \{x \in \mathbb{R}^n|\ \rho_{\varepsilon_\nu} < -t_\nu\}.$$

Since $\rho(x) < \rho_{\epsilon_\nu}(x) < -t_\nu < -\delta_{\nu+1}$, we have that $D_{\delta_{\nu+1}} \supset D_\nu$. Also if $x \in D_{\delta_{\nu-1}}$, then $\rho_{\epsilon_\nu}(x) < \rho(x) + c_2\epsilon_\nu < -\delta_\nu < -t_\nu$. Thus we have $D_{\delta_{\nu+1}} \supset D_\nu \supset D_{\delta_{\nu-1}}$ and (1) is satisfied. Each D_ν is defined by $\eta_\nu = \rho_{\varepsilon_\nu} + t_\nu$. That the subdomain D_ν has smooth boundary will follow from (3).

To prove (3), it is easy to see that $|\nabla\eta_\nu| \le c_2$ in bD_ν. To show that $|\nabla\eta_\nu|$ is uniformly bounded from below, we note bD satisfies the uniform interior cone property. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{R}^n$ such that for any unit vector $\xi \in \Gamma + \{p\}$, $-\langle\nabla\rho, \xi\rangle_p > C_0$ a.e. in $U \cap bD$, where C_0 is a positive constant independent of p if U is sufficiently small. There exist a finite covering $\{V_\mu\}_{1\le\mu\le K}$ of ∂D, a finite set of unit vectors $\{\xi_\mu\}_{1\le\mu\le K}$ and $c_1 > 0$ such that the inner product $\langle\nabla\rho, \xi_\mu\rangle \ge c_1 > 0$ a.e. for $x \in V_\mu$, $1 \le \mu \le K$. Since this is preserved by convolution, (3) is proved. This proves Lemma 0.3.

Chapter 1. Boundary value problems on Lipschitz domains in $\mathbb{R}^n$.

1.1 Green's function, the Poisson kernel and the Cauchy kernel for the model domains

In this section we recall Green's functions and the Poisson kernels for the model domains. Let Ω be either a ball or the upper half space. We are looking for the integral representation for the equation

$$(1.1.1)\qquad \begin{cases} \triangle u = f & \text{in } \Omega, \\ u = 0 & \text{on } b\Omega, \end{cases}$$

and

$$(1.1.2)\qquad \begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = f & \text{on } b\Omega, \end{cases}$$

where u, f are functions.

The starting point is the radially symmetric fundamental solution $E(x) = E(|x|)$ for $\mathbb{R}^n$. Let

$$E(x) = \begin{cases} \frac{1}{2\pi} \log |x| & n = 2, \\ \frac{-1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} = c_n \frac{1}{|x|^{n-2}} & n \geq 3, \end{cases} \tag{1.1.3}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. It is easy to check that

$$\triangle E(x) = \delta_0, \tag{1.1.4}$$

where δ_0 is the Dirac measure centered at 0. The function $E(x)$ is called a potential function centered at 0. Let f be a distribution with compact support in $\mathbb{R}^n$ and let $u = E \star f$. Then u satisfies

$$\triangle u = f \quad \text{in } \mathbb{R}^n.$$

We note that by adding to E any harmonic function in $\mathbb{R}^n$, we will get another fundamental solution. Let Ω be a bounded connected domain in $\mathbb{R}^n$. Green's function, by definition, is the fundamental solution with vanishing boundary data on $b\Omega$. We are looking for a fundamental solution $G(x, y)$ satisfying

$$\begin{cases} \triangle G(x, \cdot) = \delta_x & \text{in } \Omega, \\ G(x, \cdot) = 0 & \text{on } b\Omega, \end{cases} \tag{1.1.5}$$

where δ_x is the Dirac delta function centered at $x \in \Omega$.

We can use the reflection techniques to find Green's functions for model domains. Let Ω be the upper half space $\mathbb{R}^+_n = \{(x_1, \cdots, x_n); x_n > 0\}$. For a fixed $x = (x_1, \cdots, x_n) \in \mathbb{R}^+_n$, its reflection point x^* is defined as $x^* = (x_1, \cdots, -x_n)$. If we put a potential centered at x^*, the function $E(x^* - y)$ is harmonic in a neighborhood of $\overline{\mathbb{R}^+_n}$. Furthermore, $E(x^* - y)$ on the boundary $b\mathbb{R}^+_n$ is equal to $-E(x - y)$. Thus we set for each fixed $x \in \mathbb{R}^+_n$,

$$G(x, y) = \begin{cases} G(x, y) = E(x - y) - E(x^* - y), & y \in \mathbb{R}^+_n, \\ G(x, y) = 0, & y \in b\mathbb{R}^+_n. \end{cases} \tag{1.1.6}$$

It is easy to see that $G(x, y)$ defined by (1.1.6) satisfies (1.1.5).

To find Green's function for the ball $B_R(0)$ of radius R centered at 0, we use the same reflection idea. Let $x \in B_R(0)$ and $x^* = \frac{R^2}{|x|^2} x$ be its reflection point with respect to the boundary $bB_R(0)$. For $y \in bB_R(0)$, we have

$$|y - x^*| = \frac{R}{|x|} |(y - x)|.$$

The potential function $E\left(\frac{|x|}{R}(y-x^*)\right)$ is a harmonic function in a neighborhood of $\overline{B_R(0)}$ with a singularity at x^*. If we set for each fixed $x \in B_R(0)$,

$$G(x,y) = \begin{cases} G(x,y) = E(x-y) - E\left(\frac{|x|}{R}(x^*-y)\right), & y \in B_R(0), \\ G(x,y) = 0, & y \in bB_R(0). \end{cases} \tag{1.1.7}$$

It is easy to see that $G(x,y)$ is Green's function for $B_R(0)$. Green's function is the solution kernel for (1.1.1). Let $f \in C^\alpha(\Omega)$, where Ω is either the ball or the upper space and $0 < \alpha < 1$. Define

$$u = \int_\Omega G(x,y) f(y) dy,$$

where G is defined by (1.1.6) or (1.1.7). Then using singular integral theory, we have $u \in C^{2+\alpha}(\Omega)$ and u satisfies (1.1.1).

To derive the Poisson kernel for the upper half space or the ball, recall the Poisson-Green formula. Let $u \in C^2(\overline{\Omega})$ with compact support in $\overline{\Omega}$ and let G be Green's function for Ω. We have

$$\begin{aligned} u(x) &= \int_\Omega \delta(x-y) u(y) dy \\ &= \int_\Omega \triangle_y G(x,y) u(y) dy - \int_\Omega G(x,y) \triangle_y u(y) dy + \int_\Omega G(x,y) \triangle_y u(y) dy \\ &= \int_{b\Omega} \frac{\partial G(x,y)}{\partial \nu_y} u(y) d\sigma(y) + \int_\Omega G(x,y) \triangle_y u(y) dy, \end{aligned} \tag{1.1.8}$$

where ν_y is the outward unit normal at y. Thus if in addition, u is harmonic in Ω, (1.1.8) gives for $x \in \Omega$,

$$u(x) = \int_{b\Omega} \frac{\partial G(x,y)}{\partial \nu_y} u(y) d\sigma(y) = \int_{b\Omega} k(x,y) u(y) d\sigma(y),$$

where

$$k(x,y) = \frac{\partial G(x,y)}{\partial \nu_y}, \quad y \in b\Omega. \tag{1.1.9}$$

The function $k(x,y)$ is called the Poisson kernel for Ω and (1.1.9) gives the formula for computing the Poisson kernel.

Using (1.1.6) and (1.1.9), we have for $y \in b\mathbb{R}_n^+$,

$$
\begin{aligned}
k(x,y) &= \frac{\partial G(x,y)}{\partial \nu_y} \\
&= c_n \frac{\partial}{\partial y_n}\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^*-y|^{n-2}}\right) \\
&= c_n(-n+2)\left(\frac{y_n - x_n}{|x-y|^n} - \frac{y_n + x_n}{|x^*-y|^n}\right) \\
&= \frac{2}{n\omega_n}\frac{x_n}{|x-y|^n}.
\end{aligned} \tag{1.1.10}
$$

Let $\Omega = B_R(0)$ and $G(x,y)$ be defined by (1.1.7). Then the Poisson kernel can be computed as follows. Note that on $bB_R(0)$, $\frac{\partial}{\partial \nu_y} = \sum_{i=1}^n \frac{\partial}{\partial y_i}\frac{y_i}{R}$. Thus for $y \in bB_R(0)$,

$$
\begin{aligned}
k(x,y) =& \frac{\partial G(x,y)}{\partial \nu_y} \\
=& \frac{\partial}{\partial \nu_y}\left(E(x-y) - E\left(\frac{|x|}{R}(x^*-y)\right)\right) \\
=& \frac{R^2 - |x|^2}{n\omega_n R}\frac{1}{|x-y|^n}.
\end{aligned} \tag{1.10.11}
$$

We summarize the properties of the Poisson kernel $k(x,y)$ defined either in (1.1.10) or (1.1.11).

Theorem 1.1.1. *Let Ω be either a ball or the upper half space. The Poisson kernel $k(x,y)$ defined by (1.1.10) or (1.1.11) satisfies*

(1) $\triangle_x k(x,y) = 0$ *for every* $y \in b\Omega$.
(2) $k(x,y) > 0$ *on* $\Omega \times b\Omega$. *For each fixed* $y \in b\Omega$, $k(x,y)$ *vanishes continuously on* $b\Omega \setminus \{y\}$.
(3) $\int_{b\Omega} k(x,y)d\sigma(y) = 1, \quad x \in \Omega.$
(4) $k(x,y) \le \frac{C}{|x-y|^{n-1}}, \quad y \in b\Omega.$
(5) $k(x,y) \le C\frac{d(x,b\Omega)}{|x-y|^n}$, *where* $d(x,b\Omega)$ *is the distance of* x *to* $b\Omega$.

Let $\phi_y(x) = k(x,y)$. Theorem 1.1.1 shows that $\phi_y(x)$ is a positive harmonic function which vanishes on $b\Omega \setminus \{y\}$. It is called a kernel function at y and ϕ_y is a family of harmonic functions which form an approximation of identity as $x \to y$. In particular, we have the following theorem.

Theorem 1.1.2. *Let Ω and $k(x,y)$ be the same as in Theorem 1.1.1. For any $f \in C_0(b\Omega)$, then the Poisson integral of f*

$$
u(x) = Pf(x) = \int_{b\Omega} k(x,y)f(y)d\sigma(y), \quad x \in \Omega \tag{1.1.12}
$$

is a harmonic function in Ω and continuous up to the boundary with boundary value f.

Thus the Dirichlet problem (1.1.2) is solved explicitly for balls and upper half spaces by the Poisson integral when the given boundary function f is continuous on $b\Omega$. If the boundary function f is only in $L^\infty(b\Omega)$ or $L^p(b\Omega)$, notice that (1.1.12) is still well defined.

Theorem 1.1.3 (Fatou). *Let Ω and $k(x, y)$ be the same as in Theorem 1.1.1. Suppose that $1 \le p \le \infty$ and $f \in L^p(b\Omega)$. Let u be the Poisson integral of f defined by (1.1.12). Then u is a harmonic function in Ω and*

$$\lim_{y\in\Gamma_\alpha(x_0),\ y\to x_0} u(y) = f(x_0) \quad \text{for a.e. } x_0 \in b\Omega,$$

where $\Gamma_\alpha(x_0)$ is the nontangential approach region defined by

$$\Gamma_\alpha(x_0) = \{y \in \Omega, |y - x_0| \le (1 + \alpha)\, dist(y, b\Omega)\}.$$

The proof follows from Theorems 1.1.1 and 1.1.2. We refer the readers to the book by E. M. Stein [St] for the proof.

1.2 Green's function and the Poisson kernel for smooth domains

Let Ω be a bounded domain in $\mathbb{R}^n$. We study the L^2 theory of Dirichlet problem (1.1.1) on Ω first.

Equation (1.1.1) can be solved easily using the Hilbert space theory. Let $L^2(\Omega)$ denote the space of square-integrable functions on Ω. Let $W^s(\Omega)$ denote the Sobolev s-space. When s is a positive integer, the space of $W^s(\Omega)$ consists of functions in $L^2(\Omega)$ whose derivatives up to order s are also in $L^2(\Omega)$. We use $W_0^s(\Omega)$ to denote the subspace of $W^s(\Omega)$ such that $W_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under $W^s(\Omega)$ norm. We define $W^{-s}(\Omega)$ to be the dual of $W_0^s(\Omega)$ when $s > 0$ and the norm of $W^{-s}(\Omega)$ is defined by

$$\| f \|_{-s(\Omega)} = \sup \frac{|\langle f, g\rangle|}{\| g \|_{s(\Omega)}}, \tag{1.2.1}$$

where the supremum is taken over all functions $g \in C_0^\infty(\Omega)$ and $\langle\ ,\ \rangle$ denotes the pairing between distributions and its dual. The L^2 theory for (1.1.1) is given in the following theorem.

Theorem 1.2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Given any $f \in W^{-1}(\Omega)$, there exists a unique $u = Tf \in W_0^1(\Omega)$ satisfying*

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv, \quad \text{for every } v \in W_0^1(\Omega). \tag{1.2.2}$$

Proof. Let $v \in C_0^\infty(\Omega)$. The Poincaré inequality gives that

$$\int_\Omega |v|^2 \le 4R^2 \int_\Omega |\nabla v|^2, \quad v \in C_0^\infty(\Omega) \tag{1.2.3}$$

where R is the diameter of Ω. By completion, (1.2.3) also holds for any $v \in W_0^1(\Omega)$. Define a bilinear symmetric form on $W_0^1(\Omega)$ by

$$Q(w, v) = (\nabla w, \nabla v), \quad w, v \in W_0^1(\Omega). \tag{1.2.4}$$

From (1.2.3), the norm $\|v\|_{W_0^1(\Omega)}$ is equivalent to $Q(v,v)^{\frac{1}{2}} = \|\nabla v\|_{L^2(\Omega)}$. We have from (1.2.3),

$$|\langle f, v\rangle| \le \|f\|_{W^{-1}(\Omega)} \|v\|_{W_0^1(\Omega)} \le C\|f\|_{W^{-1}(\Omega)} Q(v,v)^{\frac{1}{2}}$$

This implies that the linear functional $l : C_0^\infty(\Omega) \to \mathbb{R}$ defined by

$$l(v) = \langle f, v\rangle$$

is a bounded on $W_0^1(\Omega)$ under the Q-norm. From the Riesz representation theorem, there exists a unique $u = Tf \in W_0^1(\Omega)$ such that

$$l(v) = Q(Tf, v), \quad v \in W_0^1(\Omega).$$

The theorem is proved.

Theorem 1.2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $T : W^{-1}(\Omega) \to W_0^1(\Omega)$ be the operator defined by Theorem 1.2.1. There exists a kernel function $G(x,y)$ in $\Omega \times \Omega$ satisfying the following:*

(1) *$G(x,y) \in C^\infty(\Omega \times \Omega \setminus \{(x,x), x \in \Omega)\})$.*
(2) *$(1 - \eta_y(x))G(x,y) \in W_0^1(\Omega)$ where $\eta(x) \in C_0^\infty(\Omega)$ is a cut-off function satisfying $\eta \ge 0$ and $\eta = 1$ in $B_\epsilon(y)$, $\epsilon > 0$.*
(3) *G(x,y)=G(y,x), for every $x \ne y$.*
(4) *$G(x,\cdot) \in L^1(\Omega)$ and*

$$T(f)(x) = \int_\Omega G(x,y) f(y) dy, \quad f \in C_0^\infty(\Omega). \tag{1.2.5}$$

Proof. From the Schwarz kernels theorem, to every continuous linear map T from $C_0^\infty(\Omega)$ to $\mathcal{D}'(\Omega)$, there is a unique distribution kernel $G(x,y)$ in $\Omega \times \Omega$ such that

$$\langle Tf, g\rangle = \langle G(x,y), g \otimes f\rangle, \quad f, g \in C_0^\infty(\Omega). \tag{1.2.5}$$

Since T is bounded from $C_0^\infty(\Omega)$ to $C^\infty(\Omega)$, its dual, T', can be extended as bounded linear operator from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$, where $\mathcal{E}'(\Omega)$ is the set of distributions with compact support in Ω. However, we have $T = T'$ from the fact that Q-form is symmetric. Thus $T\delta_y$ is defined and $G(\cdot, y) = T(\delta_y(\cdot))$. Since T is pseudolocal (i.e., sing supp$(Tf) \subset$ sing supp f), we have $G(x, \cdot) \in C^\infty(\Omega \setminus \{x\})$. In fact, $G(x, y)$ is in $C^\infty(\Omega \times \Omega \setminus \{(x, x), x \in \Omega)\})$.

To prove (2), we use

$$\triangle_x((1 - \eta_y(x))G(x, y)) = -(\triangle_x \eta_y)G(x, y) \in C_0^\infty(\Omega). \tag{1.2.6}$$

The kernel G is symmetric since $G(y, x)$ is the kernel for $T' = T$. This proves (3). To see that $G(x, \cdot) \in L^1(\Omega)$, we compare note that $\triangle(G(x, \cdot) - E(x - \cdot)) = 0$. Thus $G(x, \cdot) - E(x - \cdot)$ is in $C^\infty(\Omega)$. Combining with (2), we have that $G(x, \cdot)$ is in $L^1(\Omega)$ and T is represented by the integrable kernel G in the sense defined by (1.2.5).

Thus the Green's function exists for any bounded domain in $\mathbb{R}^n$. This gives the solution to the Dirichlet problem (1.1.1) when the boundary condition $u = 0$ at $b\Omega$ is interpreted as $u \in W_0^1(\Omega)$ for any bounded domain, regardless of the smoothness of the boundary. To derive the Poisson kernel, we need to impose some smoothness on the boundary $b\Omega$.

Theorem 1.2.3. *Let Ω be a bounded domain with $C^{k,\alpha}$ boundary, $k \in \mathbb{N}$ and $0 < \alpha < 1$. Then the Green's function $G(x, \cdot)$ is in $C^{k,\alpha}(\overline{\Omega} \setminus \{x\})$, $x \in \Omega$ and $G(x, y) = 0$ for $y \in b\Omega$.*

This follows from the boundary regularity for $\triangle$ (see Morrey [Mo]). In this case Green's theorem will give us the following representation.

Theorem 1.2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary, $0 < \alpha < 1$. Then for any $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$,*

$$u(x) = \int_\Omega G(x, y)\triangle u(y)dy + \int_{b\Omega} \frac{\partial G(x, y)}{\partial \nu_y} u(y)d\sigma(y) \tag{1.2.7}$$

where ν_y is the unit out normal at $y \in b\Omega$ and $d\sigma$ is the surface element on $b\Omega$.

Proof. Apply Green's theorem to $u(y)$ and $G(x, \cdot)$ on the domain $\Omega \setminus B_\epsilon(x)$ and let $\epsilon \to 0$.

Theorem 1.2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,\alpha}$ boundary. Then the Poisson kernel $k(x, y) = \frac{\partial G(x,y)}{\partial \nu_y}$ satisfies the following conditions:*

1. *$\triangle_x k(x, y) = 0$ for every $y \in b\Omega$.*
2. *$k(x, y) > 0$ for every $x \in \Omega$, $y \in b\Omega$. For each fixed $y \in b\Omega$, $k(x, y)$ vanishes continuously on $b\Omega \setminus \{y\}$.*
3. *$\int_{b\Omega} k(x, y) = 1, \quad x \in \Omega$.*
4. *$k(x, y) \leq \frac{C}{|x-y|^{n-1}}, \quad y \in b\Omega$.*
5. *$k(x, y) \leq C\frac{d(x,b\Omega)}{|x-y|^n}$, where $d(x, \Omega)$ is the distance of x to $b\Omega$.*

Corollary 1.2.6. *Let $f \in C(b\Omega)$. Let the Poisson integral of g $u = Pf$ be defined by*

$$u(x) = \int_{b\Omega} k(x, y) f(y) d\sigma(y). \tag{1.2.8}$$

Then u is in $C(\overline{\Omega}) \cap C^\infty(\overline{\Omega})$ and u satisfies the Dirichlet problem (1.1.2). Furthermore, if $f \in L^p(b\Omega)$, $1 \leq p \leq \infty$, then u defined by (1.2.9) has nontangential limits f almost everywhere on $b\Omega$.

The proof of Theorem 1.2.5 and Corollary 1.2.6 can be found in Stein [St]. Thus the Dirichlet problem is solved with continuous or $L^p(b\Omega)$ data when the boundary has $C^{1,\alpha}$ boundary.

1.3 L^2 theory for the homogeneous Dirichlet problem on Lipschitz domains

We consider the Dirichlet problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ u = g & \text{on } b\Omega. \end{cases} \tag{1.3.1}$$

In this section we study (1.3.1) on Lipschitz domains.

Theorem 1.3.1. *Let Ω be a bounded Lipschitz domain. Given any $f \in C(b\Omega)$, there exists a $u \in C(\overline{\Omega})$ satisfying (1.3.1).*

A domain Ω is called regular if (1.3.1) is solvable with $u \in C(\overline{\Omega})$ for any $g \in C(\partial\Omega)$. From the Perron method, any domain is regular if it has a barrier function at every boundary point. When Ω is Lipschitz, Ω is regular since it satisfies the exterior cone condition. Thus (1.3.1) is solved when the boundary function g is continuous. Our goal is to prove the Fatou's theorem on Lipschitz domains.

Using the maximum principle, for any $x \in \Omega$, the map

$$f \in C(b\Omega) \to u(x)$$

is a bounded linear functional on $b\Omega$. This implies that there exists a Borel measure ω^x such that

$$u(x) = \int_{b\Omega} f(y) d\omega^x.$$

The measure ω^x is called the harmonic measure at x for Ω.

Theorem 1.3.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let ω^x, ω^{x_0} be harmonic measures with respect to the points x, $x_0 \in \Omega$. Then*

(1) $\omega^x << \omega^{x_0}$.

(2) *ω^{x_0} satisfies the doubling condition, i.e., there exists a constant $C > o$ such that*

$$\omega^{x_0}(\beta_{2r}) \leq C\omega^{x_0}(\beta_r)$$

where β_r is any ball of radius r on $b\Omega$ centered at a point $y_0 \in b\Omega$.

(3) *For any $g \in C(b\Omega)$, the solution u to the problem (1.3.1) can be expressed as*

$$u(x) = \int_{b\Omega} g(y)k_1(x,y)d\omega^{x_0}(y) \tag{1.3.2}$$

for some $k_1(x,\cdot) \in C^\alpha(b\Omega)$ for some $0 < \alpha < 1$.

(4) *For any $g \in L^p(b\Omega)$, where $1 \leq p \leq \infty$. Define*

$$u(x) = \int_{b\Omega} g(y)k_1(x,y)d\omega^{x_0}(y).$$

Then u has nontangential limits a.e. $d\omega^{x_0}$ on $b\Omega$.

Proof. (Sketch of proof) That harmonic measures are absolutely continuous with respect to each other can be deduced from the Harnack inequality. Let K be a relatively compact subset of $b\Omega$ with $\omega^{x_0}(K) = 0$. Let U be an open neighborhood of K with $\omega^{x_0}(U) < \epsilon$ for some $\epsilon > 0$. Choose a function $g \in C(b\Omega)$ such that $g \geq 0$, $g = 1$ on K and supp$g \subset U$. Since Ω is regular, there exists a unique harmonic function $u \in C(\overline{\Omega})$ such that $u = g$ on $b\Omega$. Then we have

$$\begin{aligned}
\omega^x(K) &= \int_K d\omega^x \leq \int_{b\Omega} g d\omega^x \\
&= u(x) \leq C_{x,x_0} u(x_0) \quad \text{(Harnack inequality)} \\
&\leq C_{x,x_0} \int_{b\Omega} g d\omega^{x_0} \\
&\leq C_{x,x_0} \int_U d\omega^{x_0} \leq C_{x,x_0}\omega^{x_0}(U) < C_{x,x_0}\epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$, we have that $\omega^x(K) = 0$.

From (1), there exists a Radon-Nykodym derivative

$$k_1(x,y) \in L^1(b\Omega, d\omega^{x_0})$$

such that (1.3.2) holds. To see that $k_1(x,y)$ is actually Hölder continuous in $y \in b\Omega$, we first observe that Green's function $G(x,\cdot)$ is Hölder continuous near the boundary. This can be seen by the fact that $G(x,y)$ is a harmonic function near $b\Omega$ for a fixed $x \in \Omega$. Near $y_0 \in b\Omega$, if we straighten the boundary and write the equation $\triangle G(x,\cdot) = 0$ in the divergent form, then $G(x,\cdot)$ satisfies an elliptic equation in divergent form with L^∞ coefficients. Since $G(x,\cdot) = 0$ on $b\Omega$, we can use an odd reflection to conclude that $G(x,\cdot)$ satisfies an elliptic equation with L^∞ coefficients. Thus using the theorem of De Georgi, we have that $G(x,\cdot)$ is in C^α for some $0 < \alpha < 1$ near the boundary.

Next we have an representation due to Martin for the kernel k_1 by

$$k_1(x,y) = \lim_{y'\to y} \frac{G(x,y')}{G(x_0,y')}. \tag{1.3.3}$$

The limit on the right-hand side of (1.3.3) exists from the boundary Harnack inequality. The limit is a kernel function for any fixed $y \in b\Omega$. Using the uniqueness of the kernel function, (1.3.3) holds.

Since the kernel $k_1(x,\cdot)$ is Hölder continuous on $b\Omega$, (1.3.3) is well defined for any $f \in L^1(b\Omega)$. The nontangential convergence follows from (2) and the weak L^1 bounds of the Hardy-Littlewood Maximal functions with respect the harmonic measures.

Theorem 1.3.2 gives the solution of (1.3.1) when the boundary data is L^p function with respect to the harmonic measure $d\omega^{x_0}$. Suppose for every $x \in \Omega$, we have

$$\omega^x << \sigma, \tag{1.3.4}$$

then there exists a Radon-Nykodym derivative

$$\frac{d\omega^x}{d\sigma} = k(x,y) \in L^1(b\Omega, d\sigma). \tag{1.3.5}$$

For any $g \in C(b\Omega)$, the solution u of (1.3.1) can be written as

$$u(x) = \int_{b\Omega} g(y)k(x,y)d\sigma(y). \tag{1.3.6}$$

The kernel $k(x,y)$ is the Poisson kernel. When $b\Omega$ is $C^{1,\alpha}$ for some $0 < \alpha < 1$, the kernel $k(x,y)$ is given by the normal derivative of Green's function from Theorem 1.2.5.

When $b\Omega$ is only Lipschtz, the Green's function $G(x,y)$ is only C^α up to the boundary for some $0 < \alpha < 1$. The Poisson kernel and the solution to (1.3.1) when the function $g \in L^p(b\Omega)$ are more difficult to obtain.

Theorem 1.3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let ω^{x_0} be a harmonic measure with respect to the point $x_0 \in \Omega$. Then*

(1) $\omega^{x_0} << \sigma$.
(2) $\frac{d\omega^{x_0}(y)}{d\sigma(y)} = k(x_0,y) \in L^2(b\Omega)$.
(3) $\sigma << \omega^{x_0}$.

The proof depends on the following lemma.

Lemma 1.3.4. *Let Ω be a domain with C^∞ boundary $b\Omega$ and $0 \in \Omega$. let ω be the harmonic measure for Ω at $0 \in \Omega$ and $k(y) = \frac{d\omega}{d\sigma}$. Then the following identity holds:*

$$\frac{1}{\omega_n}\int_{b\Omega} \frac{k(y)}{|y|^{n-2}} d\sigma(y) = \int_{b\Omega} k^2(y)\langle y, \nu_y\rangle d\sigma \tag{1.3.7}$$

where ν_y is the outward unit normal at $y \in b\Omega$.

Proof. Let $G(y) = G(0, y)$ be the Green function at 0. Since $b\Omega$ is smooth, from Theorem 1.2.4, Green's function $G(y) \in C^\infty(\overline{\Omega} \setminus \{0\})$ and $k(y) = \frac{\partial G(y)}{\partial \nu_y}$. Furthermore, we have $G(y) = E(y) - u(y)$ where $E(y) = \frac{-1}{(n-2)\omega_n}|y|^{n-2}$ and $u(y)$ is a harmonic function smooth up to the boundary such that u is chosen so that $G(y) = 0$ when $y \in b\Omega$. Let y be a vector in $\mathbb{R}^n$. We decompose

$$y = \tau_y + n_y$$

where $n_y = \langle y, \nu_y\rangle\nu_y$ is the normal component and $\tau_y = y - n_y$ is the tangential component. Since $G(y) = 0$ on the boundary, we have $\langle \tau_y, \nabla G(y)\rangle = 0$ on $b\Omega$. Thus

$$\begin{aligned}&\int_{b\Omega} k(y)\langle y, \nabla G(y)\rangle d\sigma(y)\\ &= \int_{b\Omega} k(y)\langle n_y, \nabla G(y)\rangle d\sigma(y) = \int_{b\Omega} k(y)^2\langle y, \nu_y\rangle d\sigma(y).\end{aligned} \tag{1.3.8}$$

On the other hand, note that the function $v(y) = \langle y, , \nabla u(y)\rangle$ is a harmonic function. We have

$$\int_{b\Omega} k(y)\langle y, \nabla u(y)\rangle d\sigma(y) = v(0) = 0.$$

Hence

$$\begin{aligned}&\int_{b\Omega} k(y)\langle y, \nabla G(y)\rangle d\sigma(y)\\ &= \int_{b\Omega} k(y)\langle y, \nabla E(y)\rangle d\sigma(y) - \int_{b\Omega} k(y)\langle y, \nabla u(y)\rangle d\sigma(y)\\ &= \frac{1}{\omega_n}\int_{b\Omega} \frac{k(y)}{|y|^{n-2}} d\sigma(y)\end{aligned} \tag{1.3.9}$$

The lemma follows from (1.3.8) and (1.3.9).

Proof of Theorem 1.3.3. The theorem will be proved by approximation. We may assume that Ω is star-shaped and $0 \in \Omega$. Otherwise one can use a a partition of unity $\{\zeta_i\}_{i=1}^N$ such that locally each Ω_i is star-shaped. Let Ω_ν be a family of smooth subdomains obtained in Lemma 0.3. For each Ω_ν, there exists a constant C_0 such that

$$\langle y, \nu_y \rangle = \langle y, \frac{\nabla \rho_\nu}{|\nabla \rho_\nu|} \rangle \geq C_0 \quad \text{a.e. on } \Omega_\nu, \tag{1.3.10}$$

where C_0 is independent of ν. Let $k_\nu(y)$ be the corresponding Poisson kernel at 0 on Ω_ν. Then k_ν is a $C^\infty(b\Omega_\nu)$ function. Using Lemma 1.3.4, k_ν satisfies

$$\frac{1}{\omega_n} \int_{b\Omega_\nu} \frac{k_\nu(y)}{|y|^{n-2}} d\sigma_\nu(y) = \int_{b\Omega_\nu} k_\nu^2(y) \langle y, \nu_y \rangle d\sigma_\nu(y). \tag{1.3.11}$$

Using (1.3.10) and (1.3.11), we have

$$\begin{aligned} & C_0 \int_{b\Omega_\nu} k_\nu^2(y) d\sigma_\nu(y) \\ & \leq \int_{b\Omega_\nu} k_\nu^2(y) \langle y, \nu_y \rangle d\sigma_\nu(y) \leq C_1 \int_{b\Omega_\nu} k_\nu(y) d\sigma_\nu(y) \\ & = C, \end{aligned} \tag{1.3.12}$$

where C_1 is a positive constant independent of ν.

Using a partition of unity, we can assume that $f \in C(b\Omega)$ and f is supported in a small coordinate patch $b\Omega \cap U$ such that $U = V \times (-\delta, \delta)$, $V \subset \mathbb{R}^{n-1}$, $\delta > 0$ and $\Omega \cap U = \{x' \in V, x_n < \phi(x')\}$. Define $\tilde{f}_\nu \in C(b\Omega_\nu)$ by $\tilde{f}_\nu(x', \phi_\nu(x')) = f(x', \phi(x'))$ and $\tilde{k}_\nu(x', \phi(x')) = k_\nu(x', \phi_\nu(x'))$. Then

$$\begin{aligned} & \int_{b\Omega} \tilde{k}_\nu(y) f(y) d\sigma(y) \\ & = \int_{y' \in V \subset \mathbb{R}^n} \tilde{k}_\nu(y', \phi(y')) f(y', \phi(y')) \sqrt{1 + |\nabla \phi(y')|^2} dy' \\ & = \int_{y' \in V \subset \mathbb{R}^n} \tilde{k}_\nu(y', \phi_\nu(y')) \tilde{f}_\nu(y', \phi_\nu(y')) \sqrt{1 + |\nabla \phi(y')|^2} dy' \\ & = \int_{b\Omega_\nu} k_\nu(y) \tilde{f}_\nu(y) \frac{\sqrt{1 + |\nabla \phi(y')|^2}}{\sqrt{1 + |\nabla \phi_\nu(y')|^2}} d\sigma_\nu(y). \end{aligned} \tag{1.3.13}$$

Using (1.3.12) and (1.3.13), we have

$$\begin{aligned} & \int_{b\Omega} \tilde{k}_\nu(y) f(y) d\sigma(y) \\ & \leq \left(\int_{b\Omega_\nu} k_\nu(y)^2 d\sigma_\nu(y) \right)^{\frac{1}{2}} \left(\int_{b\Omega_\nu} \tilde{f}_\nu(y)^2 d\sigma_\nu(y) \right)^{\frac{1}{2}} \\ & \leq C \left(\int_{b\Omega} f(y)^2 d\sigma(y) \right)^{\frac{1}{2}}, \end{aligned} \tag{1.3.14}$$

where C is independent of ν. Thus $\tilde{k}_\nu$ is bounded in $L^2(b\Omega)$. There exists a subsequence, still denoted by $\tilde{k}_\nu$, which converges weakly to some function $k \in L^2(b\Omega)$. It remains to show that $d\omega^{x_0} = kd\sigma$.

For any $f \in C((b\Omega)$, let u be the harmonic function with boundary value f. We have

$$\int_{b\Omega} f\tilde{k}_\nu d\sigma = \int_{b\Omega_\nu} \tilde{f}_\nu k_\nu d\sigma_\nu = \int_{b\Omega_\nu} (\tilde{f}_\nu - u)k_\nu d\sigma_\nu + \int_{b\Omega_\nu} uk_\nu d\sigma_\nu \to u(x_0)$$

as $\nu \to \infty$ since $\sup_{b\Omega_\nu}|\tilde{f}_\nu - u| \to 0$ and $\int_{b\Omega_\nu} uk_\nu d\sigma_\nu = u(x_0)$. We have $\tilde{\omega}_\nu$ converges to ω^{x_0} weakly in measures and $d\omega^{x_0} = kd\sigma$ and $\sigma << \omega^{x_0}$. This proves (1) and (2) in Theorem 1.3.3.

To prove (3), let E be a Borel set in $b\Omega$ and $\omega^{x_0}(E) = 0$. This implies that $\omega^x(E) = 0$ for every $x \in \Omega$. Let χ_E denote the characteristic function of E.

$$u(x) = \int_{b\Omega} \chi_E d\omega^x = \omega^x(E) = 0.$$

But from Theorem 1.3.2, u has nontangential limit χ_E a.e. $d\omega^{x_0}$. From (1), u has nontangential limit a.e. $d\sigma$. If $\sigma(E) > 0$, then u is not identically zero in Ω. Thus $\sigma(E) = 0$. The proof of Theorem 1.3.3 is complete.

Remark. The kernel $k(x, \cdot) \in L^2(b\Omega)$ is best possible for arbitrary Lipschitz domains. Let Ω be a wedge in $\mathbb{R}^2$ defined by $\Omega = \{re^{i\theta}; 0 < \theta < \alpha,\ r > 0\}$. Let Ω' denote the upper half plane. The Poisson kernel for Ω' is the real part of the Cauchy integral. For any $f \in C(b\Omega')$,

$$P_{\Omega'} f(z') = \mathrm{Re}\frac{1}{2\pi i} \int_{b\Omega'} \frac{f(\zeta')}{\zeta' - z'} d\zeta'.$$

Using the conformal map $\zeta \to \zeta' = \zeta^{\frac{\pi}{\alpha}}$, we see that the kernel

$$P_\Omega f(z) = \mathrm{Re}\frac{1}{2\alpha i} \int_{b\Omega} \frac{f(\zeta)}{\zeta^{\frac{\pi}{\alpha}} - z^{\frac{\pi}{\alpha}}} \zeta^{\frac{\pi}{\alpha}-1} d\zeta.$$

Thus the kernel P_Ω has a singularity of order $r^{\frac{\pi}{\alpha}-1}$, which is in $L^{2+\epsilon}$ for some $\epsilon \to 0$ as the angle $\alpha \to 2\pi$.

Theorem 1.3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for any $f \in L^p(b\Omega)$, $2 \le p \le \infty$, there exists a unique u such that u is harmonic in Ω and u has nontangential limits a.e. $d\sigma$ to f and*

$$\|u^*\|_{L^p(b\Omega)} \le C\|f\|_{L^p(b\Omega)}, \tag{1.3.15}$$

where u^* is the nontangential maximal function of u.

Proof. We define for any $f \in C(b\Omega)$,

$$u(x) = \int_{b\Omega} f(y)d\omega^x(y) = \int_{b\Omega} f(y)k(x,y)d\sigma(y). \tag{1.3.16}$$

Since Ω is regular, we have that u is continuous with boundary value f. When $f \in L^2(b\Omega)$, using (2) in Theorem 1.3.3, the kernel $k(x,y)$ is bounded in $L^2(b\Omega)$ and (1.3.16) is well defined. From Theorem 1.3.2, u has boundary value a.e. $d\omega^{x_0}$, thus a.e. $d\sigma$.

To prove (1.3.15), first we note that $u^*(x) \le C\mathcal{M}f(x)$, $x \in b\Omega$, where $\mathcal{M}f$ is the Hardy-Littlewood maximal function of f. Inequality (1.3.15) obviously holds for $p = \infty$. By interpolation, it suffices to prove

$$\|u^*\|_{L^2(b\Omega)} \le C\|\mathcal{M}f\|_{L^2(b\Omega)}.$$

We refer the reader to Kenig [Ke] for details.

1.4 The Dirichlet problem on Lipschitz domains in Sobolev spaces

We study the Dirichlet problem (1.1.1) and (1.1.2) in Sobolev spaces when the domain Ω is Lipschitz. From Theorem 1.2.1, for any $f \in W^{-1}(\Omega)$, there exists a $u = Tf$ in $W_0^1(\Omega)$ satisfying (1.1.1).

When Ω has C^2 boundary, for any given f in $L^2(\Omega)$, we have that $u = Tf \in W^2(\Omega)$ (see Evans [Ev]). In this section, we will further investigate the L^2 theory on Lipschitz or C^1 domains.

Lemma 1.4.1 (Rellich identity). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let u be a harmonic function in $C^1(\overline{\Omega})$. Then*

$$\begin{aligned}&\int_{b\Omega} \langle y, \nu_y\rangle |\nabla u(y)|^2 d\sigma(y)\\ &= 2\int_{b\Omega} \langle y, \nabla u(y)\rangle\langle \nu_y, \nabla u(y)\rangle d\sigma(y) + (n-2)\int_\Omega |\nabla u|^2 dV.\end{aligned} \tag{1.4.1}$$

Proof. Since

$$\sum_{i=1}^n \frac{\partial}{\partial y_i}(y_i|\nabla u|^2) = (n-2)|\nabla u|^2 + 2\sum_{i=1}^n \frac{\partial}{\partial y_i}\left(\langle y, \nabla u\rangle \frac{\partial u}{\partial y_i}\right),$$

(1.4.1) follows from integration by parts.

Theorem 1.4.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let u be a harmonic function in $C^1(\overline{\Omega})$. Then*

$$\int_{b\Omega} |\nabla u|^2 \le C \int_{b\Omega} |\nabla_n u|^2, \tag{1.4.2}$$

$$\int_{b\Omega} |\nabla u|^2 \le C \int_{b\Omega} |\nabla_t u|^2, \tag{1.4.3}$$

where $\nabla_n = \langle \nu_y, \nabla \rangle = \frac{\partial}{\partial \nu_y}$ and $\nabla_t = \nabla - \nabla_n$.

Proof. We first assume that Ω is star-shaped and $0 \in \Omega$. From (1.4.1), we have for some $C_0 > 0$,

$$\begin{aligned}
&C_0 \int_{b\Omega} |\nabla u(y)|^2 d\sigma(y) \\
&\le 2 \int_{b\Omega} \langle y, \nabla u(y) \rangle \langle \nu_y, \nabla u(y) \rangle d\sigma(y) + (n-2) \int_{\Omega} |\nabla u|^2 dV \\
&\le C \int_{b\Omega} |\nabla u(y)| |\nabla_n u(y)| d\sigma(y) + (n-2) \int_{\Omega} |\nabla u|^2 dV \\
&\le \epsilon \int_{b\Omega} |\nabla u(y)|^2 d\sigma(y) + C_\epsilon \int_{b\Omega} |\nabla_n u(y)|^2 d\sigma(y) \\
&\quad + (n-2) \int_{\Omega} |\nabla u|^2 dV.
\end{aligned} \tag{1.4.4}$$

Using Green's theorem, we have

$$\begin{aligned}
&\int_{\Omega} |\nabla u|^2 dV = \frac{1}{2} \int_{\Omega} \triangle(u^2) dV = \int_{b\Omega} u \frac{\partial u}{\partial \nu_y} d\sigma \\
&\le C \left(\int_{b\Omega} |u|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{b\Omega} |\nabla_n u|^2 d\sigma \right)^{\frac{1}{2}}.
\end{aligned}$$

Substituting u by $u - c$, where $c = \int_{b\Omega} u d\sigma$, Poincaré's inequality gives

$$\int_{b\Omega} |u|^2 d\sigma \le C \int_{b\Omega} |\nabla_t u|^2 d\sigma.$$

Thus

$$\int_{\Omega} |\nabla u|^2 dV \le\le \epsilon \int_{b\Omega} |\nabla_t u|^2 d\sigma + C_\epsilon \int_{b\Omega} |\nabla_n u|^2 d\sigma \tag{1.4.5}$$

for $\epsilon > 0$ sufficiently small. Substituting (1.4.5) into (1.4.4), we have proved (1.4.2).

To prove (1.4.3), write $|\nabla u|^2 = |\nabla_t u|^2 + |\nabla_n u|^2$ and $\langle y, \nabla u\rangle = \langle y, \nu_y\rangle \frac{\partial u}{\partial \nu_y} + \langle y, \nabla_\tau u\rangle$. From (1.4.1),

$$\begin{aligned}
&\int_{b\Omega} \langle y, \nu_y\rangle (|\nabla_t u|^2 + |\nabla_n u(y)|^2) d\sigma(y) \\
&= 2\int_{b\Omega} \langle y, \nu_y\rangle \mid \nabla_n u \mid^2 d\sigma(y) + 2\int_{b\Omega} \langle \tau_y, \nabla u\rangle \frac{\partial u}{\partial \nu_y} d\sigma(y) \\
&+ (n-2)\int_{\Omega} |\nabla u|^2 dV.
\end{aligned} \tag{1.4.6}$$

For any small $\epsilon > 0$, there exists some large $C_\epsilon > 0$ such that

$$\left| \int_{b\Omega} \langle \tau_y, \nabla u\rangle \frac{\partial u}{\partial \nu_y} d\sigma(y) \right| \le \epsilon \int_{b\Omega} |\nabla_n u|^2 + C_\epsilon \int_{b\Omega} |\nabla_t u|^2 d\sigma(y). \tag{1.4.7}$$

Choosing $\epsilon > 0$ sufficiently small, substituting (1.4.7) into (1.4.6), we have there exists some $C_0 > 0$ such that

$$\begin{aligned}
&C_0 \int_{b\Omega} |\nabla_n u|^2 d\sigma(y) + (n-2)\int_{\Omega} |\nabla u|^2 dV \\
&\le \epsilon \int_{b\Omega} |\nabla_n u|^2 + C'_\epsilon \int_{b\Omega} |\nabla_t u|^2 d\sigma(y) \\
&\le C \int_{b\Omega} |\nabla_t u|^2
\end{aligned} \tag{1.4.8}$$

Theorem 1.4.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $f \in W^1(b\Omega)$. Then the solution $u = Pf$ is in $W^{\frac{3}{2}}(\Omega)$ and*

$$\|(\nabla u)^*\|_{L^2(b\Omega)} \le C\|f\|_{W^1(b\Omega)}.$$

Proof. From Theorem 1.3.3, Pf has nontangential limits f a.e. on $b\Omega$. Note that ∇u is also harmonic. Using an approximation argument, we have $\nabla_t u \in L^2(b\Omega)$. Using (1.4.2), $\nabla_n u$ is also in $L^2(b\Omega)$. This implies that ∇u has L^2 boundary values. Harmonic function with boundary values in $L^2(b\Omega)$ is in $W^{\frac{1}{2}}(\Omega)$. The theorem is proved.

Returning to the inhomogeneous boundary value problem (1.1.1), we have the following results.

Theorem 1.4.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $f \in W^{s-2}(\Omega)$, where $\frac{1}{2} < s < \frac{3}{2}$. Then the solution $u = Tf$ to the Dirichlet problem (1.1.1) is in $W^s(\Omega)$ and*

$$\|Tf\|_{W^s(\Omega)} \le C\|f\|_{W^{s-2}(\Omega)}. \tag{1.4.9}$$

Proof. Let $\tilde{f}$ be a bounded extension from $W^{s-2}(\Omega)$ to $W^{s-\frac{3}{2}}(\mathbb{R}^n)$ and $w = E \star f$ be the solution defined by convolution with the fundamental solution. Then $E \star f$ is in $W^s(\mathbb{R}^n)$. Then restriction operator

$$R : W^s(\mathbb{R}^n) \to W^{s-\frac{1}{2}}(b\Omega) \tag{1.4.10}$$

is bounded if and only if $\frac{1}{2} < s < \frac{3}{2}$. The restriction Rw of w to $b\Omega$ is denoted by g.

If $g \in W^{s-\frac{1}{2}}(b\Omega)$, from Theorem (1.4.3), the solution v to the homogeneous solution $v \in W^s(\Omega)$. Since $u = Ef - v$, we have that u is in $W^s(\Omega)$.

Remark. Note that (1.4.9) and (1.4.10) fail for $s = \frac{1}{2}$ or $s = \frac{3}{2}$ even for domains with C^1 boundaries. This is due to the fact that the trace operator R defined by (1.4.10) is not bounded for $s = \frac{3}{2}$ for Lipschitz or even C^1 domains. In fact, for some Ω with C^1 boundary, the trace operator $R(W^{\frac{3}{2}}(\mathbb{R}^n))$ gives a set larger than $W^1(b\Omega)$ (see Jerison-Kenig[JK]) for an example). But we do have the following result (see [JK]).

Theorem 1.4.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $f \in L^2(\Omega)$. Then the solution Tf to the Dirichlet problem (1.1.1) is in $W^{\frac{3}{2}}(\Omega)$ and*

$$\|u\|_{W^{\frac{3}{2}}(\Omega)} \le C\|f\|_{L^2(\Omega)}.$$

1.5 The Neumann problem on Lipschitz domains

Consider the homogeneous Neumann boundary problem

$$\begin{cases} \triangle u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g & \text{on } b\Omega. \end{cases} \tag{1.5.1}$$

Notice that in order that (1.5.1) is solvable, f must satisfy

$$\int_{b\Omega} g d\sigma = \int_{\Omega} \triangle u dV = 0. \tag{1.5.2)}$$

We first discuss the L^2 theory for (1.5.1).

Theorem 1.5.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $g \in W^{-\frac{1}{2}}(b\Omega)$ and $\langle g, 1\rangle = 0$. Then there exists a unique u modulo constants such that u solves the Neumann problem (1.5.1) in the sense that $u \in W^1(\Omega)$, $\triangle u = 0$ in Ω and*

$$\langle \frac{\partial u}{\partial \nu}, v\rangle_{b\Omega} = \langle g, v\rangle_{b\Omega},$$

where $v \in W^{\frac{1}{2}}(b\Omega)$.

Proof. Let D_n denote the subset of $W^1(\Omega)$ such that $D_n = \{u \in W^1(\Omega);\ \int_\Omega u dV = 0\}$. We define a semi-norm

$$Q(u,u) = |\nabla u|^2, \quad u \in D_n.$$

Using the Poincaré inequality, Q is actually a norm equivalent to $H^1(\Omega)$ in D_n since

$$\int_\Omega |u|^2 dV \le C \int_\Omega |\nabla u|^2 dV, \quad u \in D_n.$$

Let $g \in W^{-\frac{1}{2}}(b\Omega)$. For any $v \in D_n$, the restriction map $Rv = v|_{b\Omega}$ is bounded from $W^1(\Omega)$ to $W^{\frac{1}{2}}(b\Omega)$. We have

$$\begin{aligned}\langle g, v\rangle &\le \|g\|_{W^{-\frac{1}{2}}(b\Omega)} \|v\|_{W^{\frac{1}{2}}(b\Omega)} \\ &\le C\|g\|_{W^{-\frac{1}{2}}(b\Omega)} \|v\|_{W^1(\Omega)} \\ &\le C\|g\|_{W^{-\frac{1}{2}}(b\Omega)} Q(v,v)^{\frac{1}{2}}.\end{aligned}$$

Thus the map $l(v) = \langle g, v\rangle_{b\Omega}$ is a bounded linear functional on D_n under Q-norm. Riesz representation theorem gives that there exists a unique $u \in D_n$ such that

$$\langle g, v\rangle_{b\Omega} = Q(u,v) = (\nabla u, \nabla v), \quad v \in D_n. \tag{1.5.3}$$

Let v be any function in $W^1(\Omega)$ and let $\overline{v} = \int_\Omega v$. From (1.5.3) and integration by parts, we have that $\triangle u = 0$ in Ω in the distribution sense. If $u \in C^1(\overline{\Omega})$, we have

$$\int_{b\Omega} \frac{\partial u}{\partial \nu} v d\sigma = \int_{b\Omega} \frac{\partial u}{\partial \nu}(v - \overline{v}) d\sigma = \langle g, v\rangle_{b\Omega}, \quad v \in W^1(\Omega).$$

The theorem follows from approximation.

Theorem 1.5.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $g \in L^2(b\Omega)$ and $\int_{b\Omega} g d\sigma = 0$. Then there exists a unique u modulo constants such that u solves the Neumann problem (1.5.1) in the sense that $u \in W^{\frac{3}{2}}(\Omega)$, $\frac{\partial u}{\partial \nu} = g$ and*

$$\|(\nabla u)^*\|_{L^2(b\Omega)} \le C\|g\|_{L^2(b\Omega)}.$$

Proof. Let u be the solution in Theorem 1.5.1 such that $\tilde{u} \in W^1(\Omega)$ and $\frac{\partial u}{\partial \nu} = g$ on $b\Omega$. From assumption, we already have $g \in L^2(b\Omega)$. From the Rellich identity, we have that

$$\|\frac{\partial u}{\partial \nu}\|_{b\Omega} = \|\nabla_t u\|_{b\Omega}.$$

This gives that ∇u has L^2 boundary values. Since harmonic functions with L^2 boundary values is in $W^{\frac{1}{2}}(\Omega)$, we have $\nabla u \in W^{\frac{1}{2}}(\Omega)$, or equivalently, $u \in W^{\frac{3}{2}}(\Omega)$.

Next we consider the inhomogeneous Neumann boundary problem

$$\begin{cases} \triangle u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } b\Omega. \end{cases} \tag{1.5.4}$$

It is easy to see that in order that (1.5.4) is solvable, f must satisfy

$$\int_\Omega f dV = \int_\Omega \triangle u dV = \int_{b\Omega} \frac{\partial u}{\partial \nu} d\sigma = 0. \tag{1.5.5)}$$

Theorem 1.5.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $f \in W^{-1}(\Omega)$ and $\langle f, 1\rangle_\Omega = 0$. Then there exists a unique u modulo constants such that u solves the Neumann problem (1.5.4) in the sense that $u \in W^1(\Omega)$ and*

$$\int_\Omega \nabla u \nabla v = \langle f, v\rangle_\Omega, \tag{1.5.6}$$

where $v \in W^1(\Omega)$.

Proof. Let D_n be the same as in the proof of Theorem 1.5.1. Let $f \in W^{-1}(\Omega)$ and $v \in D_n$, we have

$$\begin{aligned} \langle f, v\rangle_\Omega &\le \|f\|_{W^{-1}(\Omega)} \|v\|_{W^1(\Omega)} \\ &\le C\|f\|_{W^{-1}(\Omega)} Q(v,v)^{\frac{1}{2}}. \end{aligned}$$

Thus there exists a unique $u \in D_n$ such that

$$\langle f, v\rangle_\Omega = Q(u,v) = (\nabla u, \nabla v), \quad v \in D_n. \tag{1.5.7}$$

Let v be any function in $W^1(\Omega)$ and let $\overline{v} = \int_\Omega v$. We have

$$(\nabla u, \nabla(v - \overline{v})) = \langle f, v - \overline{v}\rangle_\Omega = \langle f, v\rangle_\Omega.$$

Thus $\triangle u = f$ in the distribution sense and $\frac{\partial u}{\partial \nu} = 0$ as a distribution in $b\Omega$.

Theorem 1.5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $f \in L^2(\Omega)$ and $\int_\Omega f dV = 0$. Then there exists a unique u modulo constants such that $u \in W^{\frac{3}{2}}(\Omega)$ solves the Neumann problem (1.5.4).*

1.6 The d-Neumann problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. There is a natural boundary value problem associated with the de-Rham complex

$$d : C_q^\infty(\Omega) \to C_{q+1}^\infty(\Omega), \quad 0 \le q \le n$$

where $C_q^\infty(\Omega)$ denote the smooth q-forms in Ω.

Let $f = \sum_I{}' f_I dx^I$. The operator

$$d : C_q^\infty(\Omega) \to C_{q+1}^\infty(\Omega)$$

is defined by

$$df = \sum_I{}' \sum_{k=1}^n \frac{\partial f_I}{\partial \bar{x}_k} d\bar{x}_k \wedge dx^I. \tag{1.6.1}$$

Let $C_q^\infty(\overline{\Omega})$ denote q-forms which are smooth up to the boundary, i.e., forms which are the restriction of smooth q-forms in $\mathbb{R}^n$ to $\overline{\Omega}$. Let $L_q^2(\Omega)$ denote the space of q-forms with square integrable functions on Ω as coefficients. If $f = \sum'_I f_I dx^I$, $g = \sum'_I g_I dx^I$ are two q-forms in $L_q^2(\Omega)$, we define

$$\langle f, g\rangle = \sum_I{}' \langle f_I, g_I\rangle, \; |f|^2 = \langle f, f\rangle = \sum_I{}' |f_I|^2,$$

$$\| f \|^2 = \int_\Omega \langle f, f\rangle dV = \sum_I{}' \int_\Omega |f_I|^2 dV.$$

The formal adjoint of $d : C_{q-1}^\infty(\Omega) \to C_q^\infty(\Omega)$, $1 \le q \le n$, under the L^2 norm is denoted by δ, where

$$\delta : C_q^\infty(\Omega) \to C_{q-1}^\infty(\Omega).$$

The operator δ is defined by the requirement that

$$(\delta f, g) = (f, dg) \tag{1.6.2}$$

for all smooth $g \in C_{q-1}^\infty(\overline{\Omega})$ with compact support in Ω. If $g = \sum'_{|K|=q-1} g_K dx^K$, we have

$$\begin{aligned}(f,\ dg) &= \sum_K{}' \sum_{k=1}^n \left(f_{kK},\ \frac{\partial g_K}{\partial x^k}\right) \\ &= \sum_K{}' \sum_{k=1}^n \left(-\frac{\partial f_{kK}}{\partial x^k},\ g_K\right) \\ &= (\delta f,\ g).\end{aligned}$$

Therefore, δ can be expressed explicitly by

$$\delta f = -\sum_K{}' \sum_{j=1}^{n} \frac{\partial f_{jK}}{\partial x_j} \wedge dx^K, \tag{1.6.3}$$

where K are multiindices with $|K| = q-1$. It is easy to check that $d^2 = \delta^2 = 0$. Thus d and δ are systems of first order differential operators with constant coefficients.

We take the (weak) L^2 closure of the unbounded differential operator d, still denoted by d. Let

$$d : L^2_{q-1}(\Omega) \to L^2_q(\Omega)$$

be the maximal closure of the exterior differential operator defined as follows: an element $u \in L^2_{q-1}(\Omega)$ is in the domain of d if du, defined in the distribution sense, belongs to $L^2_q(\Omega)$. Then d defines a linear, closed, densely defined operator. d is closed since differentiation is a continuous operation in distribution theory. It is densely defined since $\mathrm{Dom}(d)$ contains all the compactly supported smooth $(q-1)$-forms. If D is bounded, any $f \in C^\infty_{q-1}(\overline{\Omega})$ is in $\mathrm{Dom}(d)$.

The Hilbert space adjoint of d, denoted by d^*, is a linear, closed, densely defined operator and

$$d^* : L^2_q(D) \to L^2_{q-1}(D).$$

An element f belongs to $\mathrm{Dom}(d^*)$ if there exists a $g \in L^2_{q-1}(\Omega)$ such that for every $\psi \in \mathrm{Dom}(\Omega) \cap L^2_{q-1}(\Omega)$, we have

$$(f, d\psi) = (g, \psi).$$

We then define $d^* f = g$. Note that if $f \in \mathrm{Dom}(d^*)$, then it follows that $d^* f = \delta f$ where δ is defined in the distribution sense in Ω.

If Ω is bounded, we have $C^\infty_{q-1}(\overline{\Omega}) \subset \mathrm{Dom}(d)$. However, not every element in $C^\infty_q(\overline{\Omega})$ is in $\mathrm{Dom}(d^*)$. Any element in $\mathrm{Dom}(d^*)$ must satisfy certain boundary conditions in the weak sense. If Ω has C^1 boundary $b\Omega$, any $f \in \mathrm{Dom}(d^*) \cap C^1_{(q)}(\overline{\Omega})$ must satisfy the following:

Lemma 1.6.1. *Let Ω be a bounded domain with C^1 boundary $b\Omega$ and ρ be a C^1 defining function for Ω. For any $f \in Dom(d^*) \cap C^1_q(\overline{\Omega})$, f must satisfy the boundary condition*

$$f(x) \lrcorner \nabla\rho = 0, \qquad z \in b\Omega, \tag{1.6.4}$$

where $\lrcorner$ denotes the contraction operator. More explicitly, f must satisfy

$$\sum_k f_{kK} \frac{\partial \rho}{\partial x_k} = 0 \quad \text{on } b\Omega \quad \text{for all } K, \tag{1.6.4$'$}$$

where $|K| = q-1$.

Proof. Note that (1.6.4) and (1.6.4′) are independent of the defining function ρ. We normalize ρ such that $|d\rho| = 1$ on $b\Omega$.

Let f be a 1-form and $f = \sum_{i=1}^{n} f_i dx_i$. Using integration by parts, we have for any $\psi \in C^\infty(\overline{\Omega}) \subset \text{Dom}(d)$,

$$\begin{aligned}(\delta f, \psi) &= \sum_{i=1}^{n} \left(-\frac{\partial f_i}{\partial x_i}, \psi \right) \\ &= \sum_{i=1}^{n} \left(f_i, \frac{\partial \psi}{\partial x_i} \right) - \sum_{i=1}^{n} \int_{b\Omega} f_i \frac{\partial \rho}{\partial x_i} \psi dS \\ &= (f, d\psi) + \int_{b\Omega} \langle f \lrcorner \nabla\rho, \psi \rangle dS,\end{aligned}$$

where dS is the surface measure of $b\Omega$. Similarly, for a q-form f and $\psi \in C^\infty_{q-1}(\overline{\Omega}) \subset \text{Dom}(d)$, using integration by parts, we obtain

$$(\delta f, \psi) = (f, d\psi) + \int_{b\Omega} \langle f \lrcorner \nabla\rho, \psi \rangle dS. \tag{1.6.5}$$

If, in addition, ψ has compact support in Ω, we have

$$(d^* f, \psi) = (\delta f, \psi) = (f, d\psi),$$

where the first equality follows from $f \in \text{Dom}(d^*) \cap C^1_q(\overline{\Omega})$. Since compactly supported smooth $(q-1)$-forms are dense in $L^2_{q-1}(\Omega)$, we must have

$$\int_{b\Omega} \langle f \lrcorner \nabla\rho, \psi \rangle dS = 0, \qquad \text{for any } \psi \in C^\infty_{q-1}(\overline{\Omega}).$$

This implies that $f \lrcorner \nabla\rho(x) = 0$ for $x \in b\Omega$.

For a fixed $0 \le q \le n$, we define the Laplacian of the d complex

$$L^2_{q-1}(\Omega) \underset{d_q^*}{\overset{d_{q-1}}{\rightleftarrows}} L^2_{(q)}(\Omega) \underset{d_{q+1}^*}{\overset{d_q}{\rightleftarrows}} L^2_{q+1}(\Omega).$$

Definition 1.6.2. *Let* $\triangle_q = d_{q-1}d_q^* + d_{q+1}^* d_q$ *be the operator from* $L^2_q(\Omega)$ *to* $L^2_q(\Omega)$ *such that* $Dom(\triangle_q) = \{f \in L^2_q(\Omega) \mid f \in Dom(d_q) \cap Dom(d_q^*);\ d_q f \in Dom(d_{q+1}^*)$ *and* $d_q^* f \in Dom(d_{q-1})\}$.

Proposition 1.6.3. $\triangle_q$ *is a linear, closed, densely defined self-adjoint operator.*

The following proposition shows that smooth forms in $\text{Dom}(\triangle_q)$ must satisfy two sets of boundary conditions, namely, the d-Neumann boundary conditions.

Proposition 1.6.4. *Let Ω be a bounded domain with C^1 boundary and ρ be a C^1 defining function. If $f \in C^2_q(\overline{\Omega})$, then*

$$f \in Dom(\triangle_q)$$

if and only if

$$\sigma(\delta, d\rho)f = 0 \quad and \quad \sigma(\delta, d\rho)df = 0 \quad on\ b\Omega.$$

If $f = \sum'_I f_I dx^I \in C^\infty_q(\overline{\Omega}) \cap Dom(\triangle_q)$, we have

$$\triangle_q f = -\sum_I{}' \triangle f_I dx^I, \tag{1.6.6}$$

where $\triangle = \sum_{k=1}^n \partial^2/\partial x_k \partial x_k$ is the usual Laplacian on functions.

Remark. *Let Ω be a smooth bounded domain.*

(1) *For $q = n$, $f \in C^\infty_n(\overline{\Omega})$, $f = \phi dx_1 \wedge \cdots \wedge dx_n \in Dom(\triangle_n)$ if and only if $f \in Dom(d^*_n)$ on $b\Omega$. This is the Dirichlet boundary condition, i.e.,*

$$\phi = 0 \quad on\ b\Omega.$$

(2) *For $q = 0$, $f \in C^\infty(\overline{\Omega})$, $f \in Dom(\triangle_0)$ if and only if $df \in Dom(d^*_1)$ on $b\Omega$. This is equivalent to the Neumann boundary condition*

$$\frac{\partial f}{\partial \nu} = 0 \quad on\ b\Omega.$$

(3) *When $q = 1$, the d-Neumann problem is a mixture of the Dirichlet and the Neumann problem. We assume that for some neighborhood U of $0 \in b\Omega$,*

$$\Omega \cap U = \{Im\ x_n < 0\} \cap U.$$

Let $f = \sum_k f_k dx_k \in C^\infty_1(\overline{\Omega})$ and the support of f lies in $U \cap \overline{\Omega}$. Then f is in $Dom(\triangle_1)$ if and only if f satisfies

$$f_n = 0 \qquad on\ b\Omega \cap U, \tag{a}$$

$$\frac{\partial f_i}{\partial x_n} = 0 \quad on\ b\Omega \cap U, \quad i = 1, \cdots, n-1. \tag{b}$$

Proof. For (1), we have $f \lrcorner d\rho = 0$ on $b\Omega$. This implies that $\phi = 0$ on $b\Omega$. To prove (2), that $df \in \mathrm{Dom}(d^*_1)$ implies that $df \lrcorner d\rho = 0$ on $b\Omega$. Thus

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial \rho}{\partial x_i} = \frac{\partial f}{\partial \nu} = 0 \quad \text{on} b\Omega.$$

To prove (3), (a) follows from the condition that $f \in \text{Dom}(d^*)$. To see that (b) holds, we note that $df \in \text{Dom}(d^*)$, implying

$$\frac{\partial f_i}{\partial x_n} - \frac{\partial f_n}{\partial x_i} = 0 \qquad \text{on } bD \cap U.$$

From (a), we have $\partial f_n/\partial x_i = 0$ on $bD \cap U$ for $i = 1, \cdots, n-1$ since each $\partial/\partial x_i$ is tangential. This proves (b).

Let Q be the form defined by

$$Q^\phi(f,f) = \|df\|^2 + \|d^* f\|^2.$$

Let $\mathcal{D}_q^1 = \text{Dom}(d^*) \cap C_q^1(\overline{\Omega})$.

Proposition 1.6.7. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with C^2 boundary $b\Omega$ and ρ be a C^2 defining function for Ω such that $|d\rho| = 1$ on $b\Omega$. For any $f = \sum'_{|J|=q} f_J dx^J \in \mathcal{D}_q^1$,*

$$\begin{aligned} Q(f,f) &= \| \, df \, \|^2 + \| \, \delta f \, \|^2 \\ &= \sum_{|J|=q}{}' \sum_k \int_\Omega \left| \frac{\partial f_J}{\partial x_k} \right|^2 dV \\ &\quad + \sum_{|K|=q-1}{}' \sum_{i,j} \int_{b\Omega} \frac{\partial^2 \rho}{\partial x_i \partial x_j} f_{iK} f_{jK} d\sigma. \end{aligned} \tag{1.6.7}$$

Proposition 1.6.8. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with C^2 boundary $b\Omega$. For any $f = \sum'_{|J|=q} f_J dx^J \in \mathcal{D}_q^1$,*

$$Q(f,f) + \| \, f \, \|^2 = \| \, df \, \|^2 + \| \, \delta f \, \|^2 + \| \, f \, \|^2 \geq C \, \| \, \nabla f \, \|^2, \tag{1.6.8}$$

where C is a positive constant.

Proof. From (1.6.7), we have

$$\begin{aligned} Q(f,f) &= \| \, df \, \|^2 + \| \, \delta f \, \|^2 \\ &\geq \| \, \nabla f \, \|^2 - C \int_{b\Omega} |f|^2 d\sigma. \end{aligned}$$

Let ρ be a C^2 defining function for $b\Omega$ such that $|d\rho| = 1$ on $b\Omega$. Then the surface element $d\sigma = \star d\rho$ where $\star$ is the Hodge star operator. It follows that $d\sigma$ is the restriction of a one form in the standard basis with C^1 coefficients. Using Stokes' theorem, we have

$$\int_{b\Omega} |f|^2 d\sigma \leq \int_\Omega d(|f|^2 \star d\rho) \leq \epsilon \, \| \, \nabla f \, \|^2 + C_\epsilon \, \| \, f \, \|^2 .$$

The inequality (1.6.8) follows by choosing $\epsilon > 0$ small.

Thus the Gårding inequality (1.6.9) holds for the d-Neumann problem on domains with C^2 boundary.

Proposition 1.6.9. *Let $\Omega \subset\subset \mathbb{R}^n$ be a convex domain with Lipschitz boundary $b\Omega$. For any $f = \sum'_{|J|=q} f_J dx^J \in \mathcal{D}^1_q$,*

$$Q(f,f) \geq \| \nabla f \|^2 . \tag{1.6.9}$$

Proof. If Ω has C^2 boundary, (1.6.9) follows from (1.6.7) directly. If Ω has only Lipschitz boundary, then we approximate Ω by smooth convex domains $\Omega_j \subset\subset \Omega$ with $\Omega_j \subset \Omega_{j+1}$ and $\cup_j \Omega_j = \Omega$.

We note that the Gårding inequality (1.6.9) only holds for the d-Neumann problem on convex Lipschitz domains. In general, we can have only the following subelliptic estimates for the d-Neumann problem on Lipschitz domains.

Theorem 1.6.10. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with Lipschitz boundary $b\Omega$. There exists a constant $C > 0$ such that for any $f = \sum'_{|J|=q} f_J dx^J \in \mathcal{D}^1_q$,*

$$Q(f,f) + \| f \|^2 \geq C \| f \|^2_{\frac{1}{2}} . \tag{1.6.10}$$

The proof of Theorem 1.6.10 uses the layer potential method, which depends on the L^2 theory for the Cauchy integral on Lipschitz curves. We refer the reader to the memoir by Mitrea-Mitrea-Taylor [MMT]. Theorem 1.6.10 implies that $\mathcal{R}(\triangle_q)$ is closed and the L^2 existence theorem holds for the d-Neumann operator.

Theorem 1.6.11. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with Lipschitz boundary $b\Omega$. For any $1 \leq q \leq n$, the space of harmonic forms $\mathcal{H}_q = ker(\triangle_q)$ is finite dimensional. Furthermore, the following estimate holds: for any $f \in Dom(d) \cap Dom(d^*) \cap \mathcal{H}_q^{\perp}$,*

$$\| f \|^2_\Omega \leq C(\| df \|^2_\Omega + \| d^* f \|^2_\Omega). \tag{1.6.11}$$

Proof. We have from (1.6.11),

$$\| f \|^2_{\frac{1}{2}(\Omega)} \leq C \| f \|^2_\Omega, \qquad f \in \mathcal{H}_q. \tag{1.6.12}$$

Since $W^{1/2}$ is compact in $L^2(\Omega)$ by the Rellich lemma (see Theorem A.8 in the Appendix), we have that the unit sphere in $\mathcal{H}_q$ is compact. Thus, $\mathcal{H}_q$ is finite dimensional.

If (1.6.12) does not hold, there exists a sequence f_n such that $f_n \in \text{Dom}(d) \cap \text{Dom}(d^*) \cap \mathcal{H}_q^{\perp}$,

$$\| f_n \|^2_\Omega \geq n(\| df_n \|^2_\Omega + \| d^* f_n \|^2_\Omega).$$

Let $\theta_n = f_n / \| f_n \|_\Omega$. Then $\| d\theta_n \|_\Omega + \| d^*\theta_n \|_\Omega \to 0$ and $\| \theta_n \|_\Omega = 1$. From (1.6.11) we have $\| \theta_n \|_{\frac{1}{2}(\Omega)} \leq C$. Using the Rellich lemma, there exists a subsequence of θ_n which converges to some element $\theta \in L^2_q \cap \mathcal{H}_q^{\perp}$. However, $d\theta = d^*\theta = 0$ and $\| \theta \|_\Omega = 1$, giving a contradiction. This proves (1.6.11).

Let H_q denote the projection onto the subspace $\mathcal{H}_q$, $1 \leq q \leq n$.

Theorem 1.6.12. *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with Lipschitz boundary $b\Omega$. For each $1 \le q \le n$, there exists a bounded operator $N_q : L^2_q(\Omega) \to L^2_q(\Omega)$ such that*

(1) $\mathcal{R}(N_q) \subset Dom(\triangle_q)$,
$N_q\triangle_q = \triangle_q N_q = I - H_q$ *on* $Dom(\triangle_q)$.
(2) *For any* $f \in L^2_q(\Omega)$, $f = dd^*N_qf \oplus d^*dN_qf \oplus H_qf$.
(3) $dN_q = N_{q+1}d$ *on* $Dom(d)$, $1 \le q \le n-1$.
(4) $d^*N_q = N_{q-1}d^*$ *on* $Dom(d^*)$, $2 \le q \le n$.
(5) *The following estimates hold for any* $f \in L^2_q(\Omega)$:

$$\| N_qf \|_{\frac{1}{2}} \le C \| f \|,$$

$$\| dN_qf \|_{\frac{1}{2}} + \| d^*N_qf \|_{\frac{1}{2}} \le C \| f \| .$$

Remark. The subelliptic $\frac{1}{2}$-estimates in Theorems 1.6.10 and 1.6.12 are best possible for general Lipschitz domains. This can be proved by examining the wedges in $\mathbb{R}^2$. Let Ω be a wedge defined by

$$\Omega = \{re^{i\theta}; 0 < \theta < \alpha,\ 0 < r < 2\} \subset\subset \mathbb{R}^2.$$

Consider the following one-form

$$\begin{cases} f &= d^*v \\ v &= \zeta(r)r^{\frac{\pi}{\alpha}} \sin \frac{\phi}{\alpha}\theta dxdy \end{cases}$$

where $\zeta \in C_0^\infty(\mathbb{R})$ with $\zeta(r) = 1$ on $r \le \frac{1}{2}$ and $\zeta = 0$ on $r \ge 1$. Then

$$\begin{cases} df = dd^*v = \triangle(\zeta(r)r^{\frac{\pi}{\alpha}} \sin \frac{\phi}{\alpha}\theta)dxdy \\ d^*f = 0. \end{cases}$$

Thus $f \in \mathrm{Dom}(d) \cap \mathrm{Dom}(d^*)$. It is easy to see that $f \in W^{\frac{1}{2}+\epsilon}(\Omega)$ with $\epsilon \to 0$ as $\alpha \to 2\pi$.

Chapter 2. The $\bar\partial$-Neumann operator on pseudoconvex Lipschitz domains

In this chapter we discuss the Sobolev estimates for the $\bar\partial$-Neumann operator N on pseudoconvex Lipschitz domains.

2.1 Strictly pseudoconvex Lipschitz domains

We first recall the well-known Levi pseudoconvexity for domains in $\mathbb{C}^n$ with C^2 boundaries.

Definition. *Let D be a bounded domain in $\mathbb{C}^n$ with C^2 boundary, $n \geq 2$, and let r be a C^2 defining function for D. D is called pseudoconvex, or Levi pseudoconvex if the Levi form*

$$\mathcal{L}_p(r; a) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p) a_j \bar{a}_k \geq 0, \ p \in \partial D,$$

for all $a \in \mathbb{C}^n, \sum_{i=1}^m a_i \frac{\partial r}{\partial z_i}(p) = 0$. The domain D is said to be strictly (or strongly) pseudoconvex if the Levi form is strictly positive for all such $a \neq 0$.

When D is not necessarily smooth, D is pseudoconvex if it can be exhausted by strongly pseudoconvex domains. To be more precise, we have the following definition.

Definition 2.1.0. *Let D be a bounded domain in $\mathbb{C}^n$, $n \geq 2$. D is called pseudoconvex if there exists an exhaustion $\{D_\nu\}$ of D such that*

(1) *the sequence $\{D_\nu\}$ is an increasing sequence of relatively compact subsets of D and $D = \underset{\nu}{\cup} D_\nu$.*

(2) *Each D_ν has a C^∞ plurisubharmonic defining function η_ν such that*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \eta_\nu}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geq c_\nu |a|^2 \quad \text{for } z \in \partial D_\nu \text{ and } a \in \mathbb{C}^n,$$

where $c_\nu > 0$ is a constant.

Thus a pseudoconvex domain can be exhausted by strongly pseudoconvex domains from inside. It is well known that if D is pseudoconvex with C^2 boundary, then the definition above agrees with the Levi pseudoconvexity (see Hörmander [Hö] or Range [Ra]).

A bounded domain D is called pseudoconvex with Lipschitz boundary if D is pseudoconvex in the sense of Definition 2.1.0 and D is Lipschitz. A bounded pseudoconvex Lipschitz domain D in $\mathbb{C}^n$ is said to have a plurisubharmonic Lipschitz defining function ρ if ρ is a global defining function and ρ is plurisubharmonic in D. Recall that a continuous function is plurisubharmonic if it is subharmonic in every complex line. Plurisubharmonicity is well-defined for continuous functions or even upper semicontinuous functions. We next define strictly (or strongly) pseudoconvex domains with Lipschitz boundaries.

Definition 2.1.1. *A bounded Lipschitz domain D in $\mathbb{C}^n$ is called strictly pseudoconvex (with Lipschitz boundary bD) if there exists an exhaustion $\{D_\nu\}$ of D such that*

(1) *The sequence $\{D_\nu\}$ is an increasing sequence of relatively compact subsets of D and $D = \underset{\nu}{\cup} D_\nu$.*

(2) *Each* D_ν *has a* C^∞ *plurisubharmonic defining function* η_ν *such that* η_ν *is uniformly bounded in* $\overline{D}$ *and*

$$\sum_{i,j=1}^{n} \frac{\partial^2 \eta_\nu}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geq c_0 |a|^2 \quad \textit{for } z \in D_\nu \cap U \textit{ and } a \in \mathbb{C}^n,$$

where U *is a neighborhood of* bD *and* $c_0 > 0$ *is a constant independent of* ν.

(3) *There exist positive constants* c_1, c_2 *such that* $c_1 \leq |\nabla \eta_\nu| \leq c_2$ *on* $D_\nu \cap U$, *where* c_1, c_2 *are independent of* ν.

We also have the following definition.

Definition 2.1.2. *A bounded Lipschitz domain* D *in* $\mathbb{C}^n$ *is called strictly pseudoconvex if it has a strictly plurisubharmonic Lipschitz defining function, i.e., there exists a Lipschitz function* $\rho : \mathbb{C}^n \to \mathbb{R}$ *such that* ρ *is locally a Lipschitz graph and*

(1) $\rho < 0$ *in* D, $\rho > 0$ *outside* $\overline{D}$ *and* $C_1 < |d\rho| < C_2$ *on* bD *almost everywhere, where* C_1, C_2 *are positive constants,*

(2) $\rho - C|z|^2$ *is plurisubharmonic for some* $C > 0$ *in* $U \cap D$ *where* U *is a neighborhood of* bD.

Proposition 2.1.3. *Let* D *be a bounded Lipschitz domain in* $\mathbb{C}^n$. *Then* D *is strictly pseudoconvex in the sense of Definition 2.1.1 and Definition 2.1.2 are equivalent.*

Proof. We first assume that D is pseudoconvex in the sense of Definition 2.1.2. We will construct a sequence of subdomains D_ν and ρ_ν. The proof is similar to the proof of Lemma 0.3. Let D_ν and η_ν be similarly defined as in Lemma 0.3. It remains to prove that η_ν satisfies (2) in Definition 2.1.1. But this follows easily from the the fact that suitable regularizations of a plurisubharmonic function are plurisubharmonic.

Let D be strictly pseudoconvex in the sense of Definition 2.1.1. Since the sequence η_ν is uniformly bounded in Λ^1 on each compact subset of D, we have from the Ascoli Theorem that there is a subsequence convergent to some limit function $\eta \in \Lambda^1(D)$. Then η is a Lipschitz defining function which satisfies (1) and (2) in Definition 2.1.2. Thus the two definitions are equivalent.

Some examples of strictly pseudoconvex domains with Lipschitz boundaries are given below.

1. Piecewise smooth strictly pseudoconvex domains. A bounded domain D in $\mathbb{C}^n$ is said to have a piecewise smooth strictly pseudoconvex boundary ∂D defined by C^2-differentiable functions if there exists a finite open covering $U_1, \ldots, U_k$ of an open neighborhood U of ∂D and C^2 strictly plurisubharmonic functions $\rho_j : U_j \to \mathbb{R}$, $j = 1, \ldots, k$ such that

(i) $D \cap U = \{x \in U |$ for each $1 \leq i \leq k$, either $x \notin U_i$ or $\rho_i(x) < 0\}$,

(ii) for $1 \leq i_1 < i_2 < \ldots \cdot < i_\ell \leq k$, the 1-forms $d\rho_{i_1}, \ldots, d\rho_{i_l}$ are linearly independent over $\mathbb{R}$ at every point of $\bigcap_{v=1}^{\ell} U_{i_\nu} \cap \partial D$.

Then D is a strongly pseudoconvex domain with Lipschitz boundary in the sense of Definition 2.1.2. To see this, first note that D is a Lipschitz domain from the assumption of transversal intersection. Let ρ_i be a C^2 strictly plurisubharmonic defining function for D_i and $D = \{z \in \mathbb{C}^n | \rho_i(z) < 0,\ i = 1, \ldots, k\}$. Set $\rho(z) = \max\{\rho_1(z), \ldots, \rho_k(z)\}$. If ρ_i is only locally defined, we define for $z \in U$,

$$\tilde{\rho}(z) = \max_{i, z \in U_i} \{\rho_i(z)\}$$

and extend $\tilde{\rho}$ to D by $\rho(z) = \max\{\tilde{\rho}(z), -\delta_0\}$ where $\delta_0 > 0$ is sufficiently small. Then ρ is a plurisubharmonic function. Let $W = \{z \in D | -\delta_0 < \rho(z) < 0\}$.

Since each ρ_i is strictly plurisubharmonic, there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} \frac{\partial^2 \rho_\ell}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j \geq c_0 |a|^2 \text{ for } z \in \overline{D} \cap U_\ell \text{ and } a \in \mathbb{C}^n,\ 1 \leq \ell \leq k.$$

It follows that $r_0(z) = \rho(z) - c_0|z|^2$ is a plurisubharmonic function on W since for each $1 \leq \ell \leq k$, $\rho_\ell - c_0|z|^2$ is a plurisubharmonic function on U_ℓ. Thus $\tilde{\rho}(z) - c_0|z|^2 = \max\limits_{i, z \in U_i} \{\rho_i(z) - c_0|z|^2\} = r_0(z)$ is plurisubharmonic on W.

Since $|\nabla \rho| = |\nabla \rho_i|$ for some i on the smooth part of ∂D, we have $|\nabla \rho| \leq C_2$ a.e. on ∂D. To show that $|\nabla \rho|$ is bounded from below, we note that ∂D is Lipschitz and satisfies the exterior cone condition. There exist a finite covering $\{V_\mu\}_{1 \leq \mu \leq k}$ of ∂D, a finite set of unit vectors $\{\xi_\mu\}_{1 \leq \mu \leq k}$ and $c_1 > 0$ such that the inner product $\langle \nabla \rho, \xi \rangle_\mu \geq c_1 > 0$ a.e. for $z \in V_\mu$, $1 \leq \mu \leq k$. $t_\nu > 0$ are pseudoconvex Definition 1.3. Thus D has a strictly plurisubharmonic Lipschitz defining function ρ and D is a strongly pseudoconvex Lipschitz domain.

2. Strictly convex domains. Let D be a strictly convex domain. By this we mean that there exists a Lipschitz defining function ρ such that $\rho - C|z|^2$ is convex for some $C > 0$ and $C_1 \leq |\nabla \rho| \leq C_2$ a.e. on bD. Since D is convex, D has Lipschitz boundary. It is easy to see that D is strongly pseudoconvex with Lipschitz boundary.

2.2 Sobolev estimates for the $\bar{\partial}$-Neumann operator on pseudoconvex Lipschitz domains

Theorem 2.2.1. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with Lipschitz boundary. There exists a constant $C > 0$ such that for any $\alpha \in L^2_{(p,q)}(D) \cap Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*)$, where $0 \leq p \leq n$ and $1 \leq q \leq n-1$,*

$$\| \alpha \|^2_{\frac{1}{2}(D)} \leq C(\| \bar{\partial}\alpha \|^2_D + \| \bar{\partial}^*\alpha \|^2_D). \tag{2.2.1}$$

Furthermore, there exists a constant $C > 0$ such that for any $\alpha \in W^{-\frac{1}{2}}_{(p,q)}(D)$, where $0 \le p \le n$ and $1 \le q \le n-1$,

$$\| N\alpha \|_{\frac{1}{2}(D)} \le C \| \alpha \|_{-\frac{1}{2}(D)}, \tag{2.2.2}$$

and

$$\| \bar\partial^* N\alpha \|_{\frac{1}{2}(D)} + \| \bar\partial N\alpha \|_{\frac{1}{2}(D)} \le C \| \alpha \|_D . \tag{2.2.3}$$

The constant C depends only on the Lipschitz constant and the diameter of D in all the inequalities.

Proof. From the assumption of strong pseudoconvexity for D, there exists a sequence of strongly pseudoconvex smooth subdomains D_ν satisfying conditions (1)-(3) in Definition (2.1.1). We have for any $f \in C^1_{(p,q)}(\overline{D}_\nu) \cap \mathrm{Dom}(\bar\partial^*_\nu)$,

$$\int_{bD_\nu} |f|^2 dS \le C(\| \bar\partial f \|^2_{D_\nu} + \| \vartheta f \|^2_{D_\nu}), \tag{2.2.4}$$

where C is independent of ν. From a theorem of Dahlberg [Da], this gives

$$\| f \|^2_{\frac{1}{2}(D_\nu)} \le C(\| \bar\partial f \|^2_{D_\nu} + \| \vartheta f \|^2_{D_\nu}), \tag{2.2.5}$$

where C is independent of ν. Thus, (2.2.5) holds for all forms $f \in C^1_{(p,q)}(\overline{D_\nu}) \cap \mathrm{Dom}(\bar\partial^*_\nu) \equiv \mathcal{D}^1_{(p,q)}(D_\nu)$. Since $\mathcal{D}^1_{(p,q)}(D_\nu)$ is dense in $\mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$ in the graph norm $\| \bar\partial f \|_{D_\nu} + \| \bar\partial^* f \|_{D_\nu}$ (See Lemma 4.3.2 in [CS]). The estimate (2.2.1) is proved by an approximation argument. Details of the proof of Theorem 2.2.1 can be found in Michel-Shaw [MS1].

We also have the following results on pseudoconvex Lipschitz domains.

Theorem 2.2.2. *Let D be a pseudoconvex domain in a $\mathbb{C}^n$ with Lipschitz boundary. Suppose that there exists a Lipschitz defining function ρ for D such that there exists some $0 < \eta \le 1$ with*

$$i\partial\bar\partial(-(-\rho^\eta) \ge 0 \text{ on } D. \tag{0.7}$$

Then the $\bar\partial$-Neumann operator $N_{(p,q)}$ exists on $L^2_{(p,q)}(D)$ where $0 \le p, q \le n$. Furthermore, $N, \bar\partial N, \bar\partial^* N$ and the Bergman projection P are exact regular operators on $W^s_{(p,q)}(D)$ for $0 < s < \frac{1}{2}\eta$ with respect to the $W^s(D)$-Sobolev norms.

Theorem 2.2.2 can be proved by the same methods used in the recent paper by Cao-Shaw-Wang [CSW] by approximation. For the regularity of $\bar\partial^* N$ and the Bergman projection P on Lipschitz domains, Theorem 2.2.2 was proved in

Berndtsson-Charpentier [BC]. When $\eta = 1$, the condition implies that there exists a plurisubharmonic defining function on D. Theorem 2.2.2 was proved in Bonami-Charpentier [BC] and Michel-Shaw [MS2]. For a general smooth pseudoconvex domain in $\mathbb{C}^n$, such a plurisubharmonic defining function does not necessarily exist (cf. [DF2]). However, Diederich-Fornaess [DF1] showed that there exists a $0 < \eta \le 1$ for any pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary.

The Bergman projection P is the orthogonal projection from $L^2_{(p,q)}(\Omega)$ to $ker(\bar\partial)$. In general, P or the $\bar\partial$-Neumann operator N is not necessarily a bounded operator from $W^s_{(p,q)}(\Omega)$ to $W^s_{(p,q)}(\Omega)$ for smooth pseudoconvex domains Ω (cf. [Ba]). In fact, Barrett [Ba] showed that, for any given β with $0 < \beta < 1$, there is a pseudoconvex domain (the so-called Diederich-Fornaess' worm domain [DF2]) with C^∞-smooth boundary such that the Bergman projection P and the $\bar\partial$-Neumann operator N are *not bounded* from $W^\beta_{(0,1)}$ to itself. One can only expect the boundary regularity of the $\bar\partial$-Neumann operator N to be regular in $W^s_{(p,q)}(\Omega)$ with small s for general pseudoconvex domains even with smooth boundary.

When D has C^∞-smooth boundary and D has a plurisubharmonic defining function on bD, Boas and Straube [BS] prove that the conclusion of Theorem 2.2.2 holds for any $s > 0$. This corresponds to the case when $\eta = 1$. For the case $\eta < 1$ and $\Omega \subset \mathbb{C}^n$, Kohn [Ko] and Berndtsson-Charpentier [BC] obtain the Sobolev regularity for the operators $\bar\partial^* N$ and the Bergman projection P.

There is also the notion of finite type piecewise smooth pseudoconvex domains (see Straube [St]). The finite type pseudoconvex domains has been extended by Harrington [Ha] to general pseudoconvex Lipschitz domains. There are also recent results on the compactness of the $\bar\partial$-Neumann operator on pseudoconvex domains (see [HI], [FS] and [Ha]). We note that Barrett-Vassiliadou [BS] has obtained the asymptotic expansion of the Bergman kernel on the intersections of two balls.

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