Divisorial Extensions and the Computation of Integral Closures

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We provide a setting for analyzing the efficiency of algorithms that compute the integral closure of affine rings. It gives quadratic (cubic in the non-homogeneous case) multiplicity-based but dimension-independent bounds for the number of passes the basic construction will make. An approach that does not uses Jacobian ideals is examined in detail.

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1. Introduction

We introduce here an approach to the analysis of the construction of the integral closure $\overline{A}$ of an affine ring $A$. It will apply to an examination of the complexity of some of the existing algorithms (Vasconcelos, 1991; Brennan and Vasconcelos, 1997; de Jong, 1998). These algorithms put their trust blindly on the Noetherian condition, without any a priori numerical certificate of termination. We remedy this for any algorithm that uses a particular class of extensions termed divisorial. These are simply the extensions that satisfy the condition $S_2$ of Serre. In addition, we analyze in detail an approach to the computation of $\overline{A}$ that is theoretically distinct from the current methods.

Unlike the treatment of Ulrich and Vasconcelos (1999), that seeks to estimate the complexity of $\overline{A}$ in terms of the number of generators that will be required to give a presentation of $\overline{A}$ (i.e. its embedding dimension), here we are going to estimate the number of passes by any method that progressively builds $\overline{A}$ by taking larger integral extensions. More precisely, we will argue that while the finite generation of $\overline{A}$ as an $A$-module guarantees that chains of integral extensions,

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \overline{A},$$

are stationary, there are no bounds for $n$ if the Krull dimension of $A$ is at least two. We show that if the extensions $A_i$ are taken satisfying the $S_2$-condition of Serre, then $n$ can be bound by data essentially contained in the Jacobian ideal of $A$ (in characteristic zero at least).

The approach to the computation of the integral closure that we discuss has the following general properties:

(1) All calculations are carried out in the same ring of polynomials. If the ring $A$ is not Gorenstein, it will require one Noether normalization.

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(2) It uses the Jacobian ideal of \( A \), or of an appropriate hypersurface subring, only theoretically to control the length of the chains of the extensions.

(3) There is an explicit quadratic bound on the multiplicity for the number of passes the basic operation has to carry out (cubic in the non-homogeneous case). In particular, and surprisingly, the bound is independent of the dimension.

(4) It can make use of known properties of the ring \( A \).

It will thus differ from the algorithms proposed in either de Jong (1998) or Vasconcelos (1991), by the fact that it does not require changes in the rings for each of its basic cycles of computation. The primitive operation itself is based on elementary facts of the theory of Rees algebras. To enable the calculation we will introduce the notion of a *proper construction* and discuss instances of it.

Key to our discussion here is the elementary, but somewhat surprising, observation that the set \( S_2(A) \) of extensions with the condition \( S_2 \) of Serre (see the next section) between an equidimensional, reduced affine algebra \( A \) and its integral closure \( \tilde{A} \) satisfies not only the *ascending* chain condition inherited from the finiteness of \( \tilde{A} \) over \( A \), but the *descending* chain condition as well. Actually this is a phenomenon typical of other algebras as well.

The explanation requires the presence of a Gorenstein subalgebra \( S \subseteq A \), over which \( A \) is birational and integral. If \( A \) is not Gorenstein, \( S \) is obtained from a Noether normalization of \( A \) and the theorem of the primitive element. One shows (Theorem 2.1) that there is an inclusion reversing one–one correspondence between \( S_2(A) \) and a subset of the set \( \text{Div}(\gamma) \) of divisorial ideals of \( S \) that contain the conductor \( \gamma \) of \( \tilde{A} \) relative to \( A \). Since \( \gamma \) is not accessible one uses the Jacobian ideal as an approximation in order to determine the maximal length of chains of elements of \( S_2(A) \). In Corollary 2.5, it is shown that if \( A \) is a standard graded algebra over a field of characteristic zero, and of multiplicity \( e \), then any chain in \( S_2(A) \) has at most \( (e-1)^2 \) elements. In Section 3 we deal with non-homogeneous algebras. If \( A \) is rational over a hypersurface ring defined by an equation of degree \( e \), we show that the lengths of the chains of divisorial extensions are bounded by \( e(e-1)^2 \).

In the next section we describe a process to create chains in \( S_2(A) \). Actually the focus is on processes that produce shorter chains in \( S_2(A) \) and its elements are amenable to computation. A standard construction in the cohomology of blowups provides chains of length at most \( \lceil \frac{(e-1)^2}{2} \rceil \) (correspondingly, \( \frac{e(e-1)^2}{2} \), in the non-homogeneous case). The possible computations are discussed in the last section. They take place in the ring \( S \) and deal entirely with ideals of \( S \).

2. Divisorial Extensions of Gorenstein Rings

For elementary properties of Cohen–Macaulay rings and modules we refer the reader to Bruns and Herzog (1993), while Becker, Kredel and Weispfenning (1993) and Vasconcelos (1997) will be our sources regarding the Buchberger algorithm and its capabilities.

Let \( A = k[x_1, \ldots, x_n]/I \) be a reduced equidimensional affine algebra over a field \( k \) of characteristic zero, let \( R = k[x_1, \ldots, x_d] \subseteq A \) be a Noether normalization and let \( S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \) be a hypersurface ring such that the extension \( S \subseteq A \) is birational. Denote by \( J \) the Jacobian ideal of \( S \); that is, the image in \( S \) of the ideal generated by the partial derivatives of the polynomial \( f \).
From $S \subseteq A \subseteq \mathbb{S} = \mathbb{F}$ and Noether (1950) we have that $J$ is contained in the conductor of $\mathbb{S}$. To fix the terminology, we denote the annihilator of the $S$-module $A/S$ by $\gamma(A/S)$. Note the identification $\gamma(A/S) = \text{Hom}_S(A,A)$.

We want to benefit from the fact that $S$ is a Gorenstein ring, in particular that its divisorial ideals have a rich structure. Let us recall some of those. Denote by $K$ the total ring of fractions of $S$. A finitely generated submodule $L$ of $K$ is said to be divisorial if it is faithful and the canonical mapping $L \mapsto \text{Hom}_S(\text{Hom}_S(L,S),S)$ is an isomorphism. Since $S$ is Gorenstein, for a proper ideal $L \subset S$ this simply means that all the primary components of $L$ have codimension one. We sum up some of these properties in the following proposition (see Bruns and Herzog, 1993, for general properties of Gorenstein rings and Vasconcelos, 1997, Section 6.3, for specific details on the $S_2$ condition of Serre).

**Proposition 2.1.** Let $S$ be a hypersurface ring as above (or more generally a Gorenstein ring). Then:

(a) A finitely generated faithful submodule of $K$ is divisorial if and only if it satisfies the $S_2$ condition of Serre.

(b) Let $A \subseteq B$ be finite birational extensions of $S$ such that $A$ and $B$ have the condition $S_2$ of Serre. Then $A = B$ if and only if $\gamma(A/S) = \gamma(B/S)$.

(c) If $A$ is an algebra such that $S \subseteq A \subseteq \mathbb{S}$, then

$$\text{Hom}_S(\text{Hom}_S(A,S),S) = \text{Hom}_S(\gamma(A/S),S)$$

is the $S_2$-closure of $A$.

**Remark 2.2.** We will also denote $A^{-1} = \text{Hom}_S(A,S)$, and the $S_2$-closure of an algebra $A$ by $\mathcal{C}(A) = (A^{-1})^{-1}$. Note that one has the equality of conductors $\gamma(A/S) = \gamma(\mathcal{C}(A)/S)$. Actually these same properties of the duality theory over Gorenstein rings hold if $S$ is $S_2$ and is a Gorenstein ring in codimension one.

We now define our two notions and begin exploring their relationship.

**Definition 2.3.** Let $I$ be an ideal containing regular elements of a Noetherian ring $S$. The degree of $I$ is the integer

$$\deg(I) = \sum_{\text{height } \nu = 1} \ell((S/I)_\nu).$$

**Definition 2.4.** A proper operation for the purpose of computing the integral closure of $S$ is a method such that whenever $S \subset A \subset \mathbb{S}$ are strict inclusions, the operation produces a divisorial extension $B$ such that $A \subset B \subset \mathbb{S}$.

**Theorem 2.1.** Let $A$ be a Gorenstein ring with a finite integral closure $\overline{A}$. Let $S_2(A)$ be the set of extensions $A \subseteq B \subseteq \overline{A}$ that satisfy the $S_2$ condition of Serre and denote by $\gamma$ the conductor of $\overline{A}$ over $A$. There is an inclusion reversing one-one correspondence between those elements of $S_2(A)$ and a subset of divisorial ideals of $S$ containing $\gamma$. In particular, $S_2(A)$ satisfies the descending chain condition.
Proof. Any ascending chain of divisorial extensions

\[ A \subset A_1 \subset A_2 \subset \cdots \subset A_n \subseteq \overline{A} \]

gives rise to a descending chain of divisorial ideals

\[ \gamma(A_1/A) \supset \gamma(A_2/A) \supset \cdots \supset \gamma(A_n/A) \]

of the same length by Proposition 2.1(b). But each of these divisorial ideals contain \( \gamma \), which gives the assertion. \( \square \)

We sum up these properties in the following manner. Let \( \wp_1, \ldots, \wp_n \) be the associated primes of \( \gamma \) and let \( U = \bigcup_{i=1}^n \wp_i \). There is an embedding of partially ordered sets

\[ S_2(A) \hookrightarrow \text{Ideals}(S_U/\gamma_U) \]

where \( S_2(A)' \) is \( S_2(A) \) with the order reversed. Note that \( S_U/\gamma_U \) is an Artinian ring.

Of course the conductor ideal \( \gamma \) is usually not known in advance, or the ring \( A \) is not always Gorenstein. In case \( A \) is a reduced equidimensional affine algebra over a field of large characteristic we may replace it by a hypersurface subring \( S \) with integral closure \( \overline{A} \). On the other hand, by Noether (1950) (see more general results in Lipman and Sathaye, 1981; Kunz, 1986), the Jacobian ideal \( J \) of \( S \) is contained in the conductor of \( S \). We obtain the less tight but more explicit rephrasing of Theorem 2.1.

**Theorem 2.2.** Let \( S \) be a reduced hypersurface ring

\[ S = k[x_1, \ldots, x_{d+1}]/(f) \]

over a field of characteristic zero and let \( J \) be its Jacobian ideal. Then the integral closure of \( S \) can be obtained in at most \( \deg(J) \) proper operations on \( S \).

In characteristic zero, the relationship between the Jacobian ideal \( J \) of \( S \) and the conductor \( \gamma \) of \( S \) is difficult to express in detail. In any event the two ideals have the same associated primes of codimension one, a condition we can write as the equality of radical ideals

\[ \sqrt{\gamma} = \sqrt{(J^{-1})^{-1}} \]

**Corollary 2.5.** Let \( k \) be a field of characteristic zero and let \( A \) be a standard graded domain over \( k \) of dimension \( d \) and multiplicity \( e \). Let \( S \) be a hypersurface subring of \( A \) such that \( S \subseteq A \) is finite and birational. Then the integral closure of \( A \) can be obtained after \((e - 1)^2 \) proper operations on \( S \).

**Proof.** Denote \( S = k[x_1, \ldots, x_{d+1}]/(f) \), where \( f \) is a form of degree \( e = \deg(A) \). By Euler’s formula, \( f \in L = \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{d+1}} \right) \). Then let \( g, h \) be forms of degree \( e - 1 \) in \( L \) forming a regular sequence in \( T = k[x_1, \ldots, x_{d+1}] \). Clearly we have that \( \deg(g, h)S \geq \deg(J) \). On the other hand, we have the following estimation of ordinary multiplicities

\[
(e - 1)^2 = \deg(T/(g, h)) = \sum_{\text{height } \wp=2} \ell(T/(g, h)_\wp) \deg(T/\wp) \\
\geq \sum_{\text{height } \wp=2} \ell(T/(g, h)_\wp)
\]
\[ \geq \sum_{\text{height } \nu = 2} \ell(T/(f, g, h)_\nu) = \deg((g, h)_S). \]  

Note that this is a pessimistic bound, that would be already cut in half by the simple hypothesis that no minimal prime of \( J \) is monomial. One does not need the base ring to be a hypersurface ring, the Gorenstein condition will do. We illustrate this with the following corollary.

**Corollary 2.6.** Let \( A \) be a reduced equidimensional Gorenstein algebra over a field of characteristic zero. Let \( J \) be the Jacobian ideal of \( S \) and let \( L \) be the corresponding divisorial ideal, \( L = (J^{-1})^{-1} \). If \( L \) is a radical ideal, then \( \overline{A} \) is the only proper divisorial extension of \( A \).

**Proof.** Let \( A \subseteq B \subseteq C \) be divisorial extensions of \( A \), and let \( K \subseteq I \) be the conductors of the extensions \( C \) and \( B \), respectively. We will show that \( K = I \). If \( \wp \) is a prime of codimension one that is not associated to \( I \), \( B_\wp \) is integrally closed and \( B_\wp = C_\wp \). Thus \( I \) and \( K \) have the primary components since they both contain \( L \).

Let us highlight the boundedness of chains of divisorial subalgebras.

**Corollary 2.7.** Let \( A \) be a reduced equidimensional standard graded algebra over a field of characteristic zero, and set \( e = \deg(A) \). Then any sequence

\[ A = A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \overline{A} \]

of finite extensions of \( A \) with the property \( S_2 \) of Serre has length at most \((e - 1)^2\).

### 3. Non-homogeneous Algebras

We will now deal with affine algebras which are not homogeneous. Suppose \( A = k[x_1, \ldots, x_n]/I \) is a reduced equidimensional algebra over a field of characteristic zero. Let

\[ S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \hookrightarrow A \]

be a hypersurface ring over which \( A \) is finite and birational. The degree of the polynomial \( f \) will play the role of the multiplicity of \( A \). Of course, we may choose \( f \) of as small degree as possible.

Our aim is to find estimates for the length of chains of algebras

\[ S = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_q = \overline{A} \]

satisfying the condition \( S_2 \), between \( S \) and its integral closure \( \overline{A} \). The argument we used required the length estimates for the length of the total ring of fractions of \( S/\gamma \), where \( \gamma \) is the conductor ideal of \( S \), \( \text{ann}(\overline{A}/S) \). Actually, it only needs estimates for the length of the total ring of fractions of \( (S/\gamma)_M \), where \( M \) ranges over the maximal ideals of \( S \).

In the homogeneous case, we found it convenient to estimate these lengths in terms of the multiplicities of \( (S/\gamma)_M \); we will do likewise here.

The first point to be made is the observation that we may replace \( k \) by \( K \simeq S/M \) and \( M \) by a maximal ideal \( M' \) of \( A' = K \otimes_k A \) lying over it. In other words, we can
replace $R = S_M$ by a faithfully flat (local) extension $R' = S'_M$. The conditions are all preserved in that $S' = K \otimes_k S$ is reduced, $S' = K \otimes_k A$, the conductor of $S$ extends to the conductor of $S'$, and chains of extensions with the $S_2$ conditions give like to likewise extensions of $K$-algebras. Furthermore, the length of the total ring of fractions of $R/\gamma$ is bounded by the length of the total ring of fractions of $R'/\gamma'$.

What this all means is that we may assume that $M$ is a rational point of the hypersurface $f = 0$. We may change the coordinates so that $M$ corresponds to the actual origin.

Proposition 3.1. Let $A = k[x_1, \ldots, x_d]$ be the ring of polynomials over the infinite field $k$ and let $f, g$ be polynomials in $A$ vanishing at the origin. Suppose $f, g$ is a regular sequence and $\deg f = m \leq n = \deg g$. Then the multiplicity of the local ring $(A/(f, g))(x_1, \ldots, x_n)$ is at most $nm^2$.

Proof. Write $f$ as the sum of its homogeneous components,

$$f = f_m + f_{m-1} + \cdots + f_r,$$

and similarly for $g$,

$$g = g_n + g_{n-1} + \cdots + g_s.$$

We first discuss the route the argument will take. Suppose that $g_s$ is not a multiple of $f_r$. We denote by $R$ the localization of $A$ at the origin, and its maximal ideal by $M$. We observe that $A/(f_r)$ is the associated graded ring of $R/(f)$, and the image of $g_s$ is the initial form $g$. Thus the associated graded ring of $R/(f, g)$ is a homomorphic image of $A/(f_r, g_s)$. If $f_r$ and $g_s$ are relatively prime polynomials, it will follow that the multiplicity of $R/(f, g)$ will be bounded by $r \cdot s$,

$$\deg R/(f, g) \leq r \cdot s.$$

We are going to ensure that these conditions on $f$ and $g$ are realized for $f$ and another element $h$ of the ideal $(f, g)$. After a linear, homogeneous change of variables (as $k$ is infinite), we may assume that each non-vanishing component of $f$ and of $g$ has unit coefficient in the variable $x_d$. For that end it suffices to use the usual procedure on the product of all non-zero components of $f$ and $g$. At this point we may assume that $f$ and $g$ are monic.

Now rewrite

$$f = x_d^m + a_{m-1}x_d^{m-1} + \cdots + a_0,$$

$$g = x_d^n + b_{n-1}x_d^{n-1} + \cdots + b_0$$

with the $a_i, b_j$ in $k[x_1, \ldots, x_{d-1}]$. Now consider the resultant of these two polynomials.
with respect to $x_d$:

$$\text{Res}(f, g) = \det \begin{bmatrix} 1 & a_{m-1} & a_{m-2} & \ldots & a_0 \\ 1 & a_{m-1} & a_{m-2} & \ldots & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{m-1} & a_{m-2} & \ldots & a_0 \\ 1 & b_{n-1} & b_{n-2} & \ldots & b_0 \\ 1 & b_{n-1} & b_{n-2} & \ldots & b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b_{n-1} & b_{n-2} & \ldots & a_0 \end{bmatrix}.$$  

We recall that $h = \text{Res}(f, g)$ lies in the ideal $(f, g)$. Scanning the rows of the matrix above ($n$ rows of entries of degree at most $m$, $m$ rows of entries of degree at most $n$), it follows that $\deg h \leq 2mn$. A closer examination of the distribution of the degrees shows that $\deg h \leq mn$. If $h_p$ is the initial form of $h$, then clearly $h_p$ and $f_r$ are relatively prime since the latter is monic (more precisely, has leading unit coefficient) in $x_d$, while $h_p$ lacks any term with $x_d$.

Assembling the estimates, one has

$$\deg R/(f, g) \leq \deg R/(f, h) \leq r \cdot p \leq m \cdot mn = nm^2,$$

as claimed. □

**COROLLARY 3.2.** Let $S = k[x_1, \ldots, x_d+1]/(f)$ be a reduced hypersurface ring over a field of characteristic zero, with $\deg f = e$. Then any chain of algebras between $S$ and its integral closure, satisfying the condition $S_2$, has length at most $e(e-1)^2$.

**4. A Rees Algebra Approach to the Integral Closure**

In this section we introduce and analyze in detail one proper operation which does not use the Jacobian ideals. The reason for this move lies with the multiply exponential estimates of Ulrich and Vasconcelos (1999) for the embedding dimensions of the integral closure of $A$.

**PROPOSITION 4.1.** There are proper operations whose iteration leads to the integral closure of $S$.

**PROOF.** For point of reference, the operations used in de Jong (1998) and Vasconcelos (1991) are

$$A_1 = \begin{cases} \text{Hom}_S(\sqrt{J}, \sqrt{J}) \\ \text{Hom}_S(\text{Hom}_S(J, S), \text{Hom}_S(J, S)) \end{cases},$$

respectively. If $S$ occurs as above, $S \subset A$, one considers the ideal $I = \gamma(A/S)$ and takes $A_1 = \text{Hom}_S(I, I)$. Both processes are guaranteed to produce proper extensions of $S$ (and of $A$ in the first case).

We assume that we have $S \subset A \subseteq \overline{S}$ and now we introduce a construction of a divisorial algebra $B$ that enlarges $A$ whenever $A \neq \overline{S}$. Denote $I = \gamma(A/S)$. More strictly we shall first find a finite, proper extension and then take its divisorial closure.
A natural choice is:

\[ B = \bigcup_{n \geq 1} \text{Hom}_S(I^n, I^n). \]

In other words, if \( R[I] \) is the Rees algebra of the ideal \( I \), \( X = \text{Proj}(R[I]) \), then \( B = H^0(X, O_X) \). To show that \( B \) is properly larger than \( A \), it suffices to show this at one of the associated prime ideals of \( I \). Let \( \wp \) be a minimal prime of \( I \) for which \( S_{\wp} \neq A_{\wp} \). If \( I_{\wp} \) is principal we have that \( I_{\wp} = tS_{\wp} \) and therefore \( S_{\wp} = A_{\wp} \), which is a contradiction. This means that the ideal \( I_{\wp} \) has a minimal reduction of reduction number at least one; that is, \( I_{\wp}^{r+1} = uI_{\wp}^{r} \) for some \( r \geq 1 \). This equation shows that

\[ I_{\wp}^{r}u^{-r} \subseteq B_{\wp}, \]

which gives the desired contradiction since \( I_{\wp}^{r}u^{-r} \) contains \( A_{\wp} \) properly as \( I_{\wp} \) is not principal.

We observe that a bound for the integer \( r \) can be found as well: let \( g \in J \) be a form of degree \( e - 1 \) which together with \( f \) form a regular sequence of \( R = k[x_1, \ldots, x_{d+1}] \). According to Vasconcelos (1997, Corollary 9.5.3), the reduction number of the ideal \( I_{\wp} \) is bounded by the Hilbert–Samuel multiplicity \( e(I_{\wp}) \) of \( I_{\wp} \) minus one. In a manner similar to that which we argued earlier,

\[ e(I_{\wp}) \leq e(e - 1). \]

By taking divisorial closures, we have

\[ C(B) = C(\text{Hom}_S(I^r, I^r)) \]

for \( r < e(e - 1) \), as desired. \( \Box \)

The proper operations above lead to a “fast” approach (i.e. beats the theoretical limit by a factor of at least 2) to the integral closure according to the following observation.

**Proposition 4.2.** Let \( S \subseteq A \) be a divisorial extension and let \( I = \gamma(A/S) \). If \( B \) is the divisorial closure of \( \bigcup_{n \geq 1} \text{Hom}_S(I^n, I^n) \), then

\[ \deg(\gamma(A/S)) \geq \deg(\gamma(B/S)) + 2. \]

In particular, for any standard graded algebra of multiplicity \( e \), the number of terms in any chain of algebras obtained in this manner will have at most \( \lceil \frac{(e-1)^2}{2} \rceil \) divisorial extensions.

**Proof.** Suppose that the degrees of the conductors of \( A \) and \( B \) differ by 1. This means that the two algebras agree at all localizations of \( S \) at height 1 primes, except at \( R = S_{\wp} \), and that \( \ell((B/A)_{\wp}) = 1 \). We are going to show that for the given choice of how \( B \) is built that is a contradiction.

We localize \( S, A, B, I \) at \( \wp \) but keep a simpler notation \( S = S_{\wp} \), etc. Let \( (u) \) be a minimal reduction of \( I \), \( I^{r+1} = uI^r \). We know that \( r \geq 1 \) since \( I \) is not a principal ideal, as \( A \neq B \). We note that \( A \subseteq IU^{-1} \subseteq B \). Since \( B/A \) is a simple \( S \)-module, we have that either \( A = IU^{-1} \) or \( B = IU^{-1} \). We can readily rule out the first possibility. The other leads to the equality

\[ IU^{-1} \cdot IU^{-1} = IU^{-1}, \]
since $B$ is an algebra. But this implies that

$$Iu^{-1} \cdot I \subseteq I \subseteq S,$$

in other words, that $Iu^{-1} \subseteq \text{Hom}_S(I, S) = A$. $\Box$

**Remark 4.3.** In case $A$ is a non-homogeneous algebra, the cubic estimate of Corollary 3.2 must be considered.

### 5. Computation Issues

There remains to deal with the issues related to various approaches to the actual computation.

1. To begin, what is a nice way to carry out the calculation of $\gamma(A/S)$,

   $$S = k[x_1, \ldots, x_d, x_{d+1}] / (f) \hookrightarrow k[x_1, \ldots, x_n] / I = A?$$

   One approach is the following. Let $g$ be a non-zero element in the Jacobian ideal of $S$ and set $L = gA \subseteq S$. Note that $\gamma(A/S) = g \cdot \text{Hom}_S(gA, S) = Sg : S$.

   For this we need $gA$ expressed as an ideal of $S$. This is an elimination question, for example $gA$ is the image in $S$ of the ideal $(g, I) \cap k[x_1, \ldots, x_{d+1}]$.

   If we want to track the computation closely, pick an elimination term order for the $x_i$ such that $x_i > x_{i+1}$ (such ordering as might have been required earlier to achieve the Noether normalization). Let $G_\gamma(I)$ be a Gröbner basis of $I$ and let $NF(\cdot)$ be the corresponding normal form function. Then $gA \subseteq S$ is generated by all $NF(gx^\alpha), \quad x^\alpha = x_{d+2}^{\alpha_{d+2}} \cdots x_n^{\alpha_n}, \quad \text{where } \alpha_i < c$.

2. From the point we have the ideal $L$, the computations are always with ideals of $S$, until we reach the integral closure. That is, given an extension $S \subseteq A$, we assume that $A$ is represented as a fractionary ideal, $A = Lx^{-1}$, where $L \subseteq S$ and $x$ is a regular element of $S$. The closure of $A$ is

   $$\mathcal{C}(A) = (L^{-1})^{-1}x^{-1}$$

   while its conductor is

   $$\gamma = Sx : S L.$$

   Note that the conductors of $A$ and of $\mathcal{C}(A)$ are the same. This means that the closure operation is being used to control the chain of extension but it is only required at the last step of the computation.

3. The hard part is the calculation of $B = H^0(X, O_X)$, as indicated above. We do not know an elegant way of doing it.

4. Note that the test of termination is given by the equality $A = B$. If reached, say $S = Lx^{-1} = (a_1, \ldots, a_n)x^{-1}$, a set up for a presentation of $S$ can go as follows. Consider the homomorphism

   $$k[x_1, \ldots, x_{d+1}, T_1, \ldots, T_n] \xrightarrow{\varphi} Lx^{-1} \subseteq Sx^{-1} \subseteq S[x^{-1}],$$

   where $T_i$ is mapped to $a_i x^{-1}, i = 1, \ldots, n$. If we use the presentation

   $$S[x^{-1}] = k[x_1, \ldots, x_d, x_{d+1}, u] / (f, u\tilde{x} - 1),$$
where \( u \) is a new indeterminate and \( \tilde{x} \) is a lift of \( x \), the defining ideal of \( S \) is then

\[
\ker(\varphi) = (f, u\tilde{x} - 1, T_i - \tilde{a}_i u, i = 1 \ldots n) \cap k[x_1, \ldots, x_{d+1}, T_1, \ldots, T_n].
\]

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