Characterization of decoherence processes in quantum computation

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Abstract: We show how the dynamics of open quantum systems can be fully characterized by using quantum tomography methods. We apply these methods to the case of an ion trap quantum computer, which does not operate under ideal conditions due to coupling to several environments. We study the performance of a fundamental two–bit quantum gate as a function of various parameters related to the interaction of the ions with external laser fields.

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1. Introduction

The time evolution of isolated quantum systems is unitary; it is generated by a self-adjoint operator called the Hamiltonian of the system. This apparently simple assessment, which follows from the basic postulates of quantum mechanics, does not correspond to realistic physical implementations in the lab. Systems are never perfectly isolated from the environment, which usually leads to a very complicated dynamic that cannot be described in terms of unitary operators. One says in this case that the system decoheres owing to its interaction with the environment. In several applications of quantum mechanics, including quantum cryptography [1] and computation [2], one must know how the physical process that takes place is affected by these interactions. For example, in the context of quantum computation it has been shown that any logical operation can be performed as a sequence of one–bit and two–bit quantum gates [2], that is, unitary processes acting on a single qubit or two. In a given experiment one is able to perform (approximately) these gates. The question now is to what extent this physical process which takes place in reality, approaches the ideal one. That is, one must characterize experimentally the quantum process. In a recent paper [3], we showed how to carry out this task by preparing the system in a series of states and then performing appropriate measurements on the system after the process. In this contribution, we apply such a technique to the ion trap quantum computer [4]. We wish to evaluate how sensitive is this experimental setup to the variation of the different physical parameters. Given the fact that one–bit qubit quantum gates can be performed much more efficiently than two–bit quantum gates, we will concentrate on this second case. To define quantitatively the quality of the computation we will use the concepts of averaged fidelity, purity quantum degree, and entanglement capability of a given process as defined in our previous work [3] (see also Ref. 5).

2. Characterizing open quantum systems

Here, we briefly discuss the general formalism to characterize the dynamics of an open quantum system. For a more detailed discussion we refer the reader to Ref. 3. Suppose we are trying to describe a given physical process in a particular system. Because of the interaction with the environment this process is generally described by

$$\hat{\rho}_{\rm in} \xrightarrow{\mathcal{E}} \hat{\rho}_{\rm out} = \mathcal{E}[\hat{\rho}_{\rm in}], \tag{1}$$

where $\mathcal{E}$ is the linear superoperator that must conserve trace, positivity, and self–adjointness. The goal is thus to characterize $\mathcal{E}$. If the system is initially prepared in a pure state $|\Psi_{\rm in}\rangle = \sum_{i=0}^{N} c_i |i\rangle$, where $|i\rangle$ are the basis vectors spanning the $(N+1)$–dimensional input space, after the evolution the state of the system is given by

$$\hat{\rho}_{\rm out} = \sum_{i,i'=0}^{N} c_i \, [c_{i'}]^* \, \hat{R}_{i'i} \,, \tag{2}$$

where $\hat{R}_{i'i}$ are *system operators* that do not depend on the initial state. In this way, through knowing the $(N+1)^2$ operators, it is possible to predict every final state related to a given input, even when we are considering a physical process in an open quantum system. Note that when the initial state is a mixed state, linearity allows us to predict the output state once we know the *system operators.*

In order to obtain the operators $\hat{R}_{i'i}$, one must perform the following series of experiments. First, one must prepare the system in a given set of initial states, and, second, one must perform state tomography in the final states. It is now interesting to know how close the physical implementation of our ideal process is to the real one. In

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order to study this, we introduce several parameters. First, we could be interested in knowing how close the final state after the process is to the ideal one. For that purpose, we introduce the Fidelity $\mathcal{F}$

$$\mathcal{F} = \overline{\langle \Psi_{\rm in}|\hat{U}^\dagger \hat{\rho}_{\rm out} \hat{U}|\Psi_{\rm in}\rangle},$$

where the overline indicates the average over all possible input states $|\Psi_{\rm in}\rangle$, and $\hat{U}$ is the unitary operator corresponding to the ideal process. One could also study the effect of the decoherence. To this aim, we introduce the Purity $\mathcal{P}$

$$\mathcal{P} = \overline{{\rm Tr}\{(\hat{\rho}_{\rm out})^2\}}, \tag{3}$$

which is averaged as before. Both parameters can be calculated once the tomography of the process has been implemented.

We will apply this formalism to the particular example of a two–bit gate. We will then consider only nonentangled states as input states and local single measurements in the output states; otherwise we would need to apply a two–bit quantum gate in the preparation and measurement, which would mask the whole procedure.

3. The trapped ion quantum gate

We now apply the above ideas in the context of the dissipative ion trap quantum computer model. First, we explain in more detail some of the requirements previously mentioned, namely the set of initial states needed and the measurements associated with the state tomography. As was stated before, we will consider only nonentangled states as input states. As an example, the initial states needed in this case can be given by the 16 product states $|\psi_a\rangle_1|\psi_b\rangle_2$ $(a, b = 1, \ldots, 4)$, where

$$\begin{array}{ll} |\psi_1\rangle = |0\rangle, & |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \\ |\psi_2\rangle = |1\rangle, & |\psi_4\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle). \end{array} \tag{4}$$

The quantum tomography of the output states can be carried out following the lines proposed by Wooters [6]. Writing the density operator as

$$\hat{\rho}_{\rm out} = \sum_{q=0}^{15} \lambda_q \hat{A}_q, \tag{5}$$

where $\hat{A}_q = \hat{\sigma}^1_{q_1} \otimes \hat{\sigma}^2_{q_2}$ $(q = 4q_1 + q_2)$, with $\hat{\sigma}^a_{q_i} = \{\hat{1}^a, \hat{\sigma}^a_x, \hat{\sigma}^a_y, \hat{\sigma}^a_z\}$, and $a = 1, 2$ refers to the first and second qubits, respectively. By measuring the observables $\hat{A}_q$, one can determine the coefficients λ_q, given that $\lambda_q = {\rm Tr}[\hat{\rho}_{\rm out}\hat{A}_q]/4$. It is important in this context that neither for the preparation of the initial states nor for the tomographic measurement are two–bit quantum gates necessary. Since it is precisely the two–bit quantum gate that we want to characterize, it is pointless to use as tools for this characterization, other two–bit quantum gates. On the other hand, single qubit measurements and operations are considered to be error free.

We have considered two ions in a linear ion trap interacting with two lasers. Let us denote by $|g\rangle_n \equiv |0\rangle_n$ and $|e\rangle_n \equiv |1\rangle_n$ two internal states of the nth ion, and by $|e'\rangle_n$ an auxiliary internal state. As we have shown in Ref. 3, the universal two–qubit gate defined by

$$|\epsilon_1\rangle_1|\epsilon_2\rangle_2 \to (-1)^{\epsilon_1\epsilon_2}|\epsilon_1\rangle_1|\epsilon_2\rangle_2, \qquad (\epsilon_{1,2} = 0, 1)\,, \tag{6}$$

can be implemented in three steps: (i) Apply a π laser pulse to the lower motional sideband corresponding to the transition $|g\rangle_1 \to |e\rangle_1$ of the first ion; (ii) apply a 2π

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laser pulse to the lower motional sideband of the transition $|g\rangle_2 \to |e'\rangle_2$ of the second ion; (iii) apply a π laser pulse, as in (i). By lower motional sideband we mean that the laser frequency must be equal to the corresponding internal transition frequency minus the trap frequency in order to excite a center of mass phonon only. An alternative way of performing conditional gates with trapped ions beyond the requirement of cooling to zero temperature has been recently developed [7]. In our description, we will also consider the presence of dissipation in the phonon modes, the most important source of dissipation in realistic experiments. The interaction of the two ions and the laser is given by the following master equation:

$$\dot{\rho} = -i[H, \rho] + \mathcal{L}\rho \tag{7}$$

where

$$\begin{aligned} H &= -\Delta_1 |e\rangle_{11}\langle e| - \Delta_2 |e'\rangle_{22}\langle e'| + \nu a^\dagger_{\rm cm} a_{\rm cm} + \sqrt{3}\nu a^\dagger_{\rm r} a_{\rm r} \\ &\quad + \frac{\Omega_1(t)}{2}[|e\rangle_{11}\langle g| e^{-i\eta_{\rm cm}(a_{\rm cm}+a^\dagger_{\rm cm})} e^{-i\eta_{\rm r}(a_{\rm r}+a^\dagger_{\rm r})} + \text{H.c.}] \\ &\quad + \frac{\Omega_2(t)}{2}[|e'\rangle_{22}\langle g| e^{-i\eta_{\rm cm}(a_{\rm cm}+a^\dagger_{\rm cm})} e^{i\eta_{\rm r}(a_{\rm r}+a^\dagger_{\rm r})} + \text{H.c.}], \\ \mathcal{L}\rho &= \kappa_{\rm cm}(2a_{\rm cm}\rho a^\dagger_{\rm cm} - a^\dagger_{\rm cm} a_{\rm cm}\rho - \rho a^\dagger_{\rm cm} a_{\rm cm}) \\ &\quad + \kappa_{\rm r}(2a_{\rm r}\rho a^\dagger_{\rm r} - a^\dagger_{\rm r} a_{\rm r}\rho - \rho a^\dagger_{\rm r} a_{\rm r}). \end{aligned} \tag{8}$$

Here, $\Delta_{1,2}$ and $\Omega_{1,2}$ are the laser detunings and Rabi frequencies, respectively, of the laser acting on each ion. The operators a and $a^\dagger$ are annihilation and creation operators of the center of mass (cm) and relative (r) motion mode, η is the corresponding Lamb–Dicke parameter, κ is the phonon dissipation rate, and ν is the trap frequency. Dissipation has been described by means of the standard quantum optics formalism of master equations, based on the Born–Markov and rotating wave approximations, and, by considering a linear coupling between the phonon modes and the reservoir of external modes at zero temperature.

With a numerical calculation we have simulated the measurement of the operators $\hat{R}_{i'i}$. The idea is that by working only in the system space we are able to study the realistic gate that we are performing (see Fig. 1).

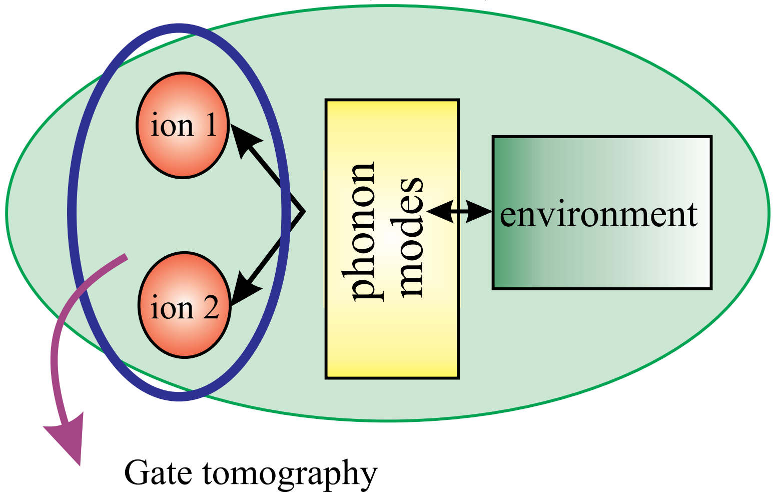

Fig. 1. System environment scheme in a realistic process.

Apart from the previously introduced Fidelity and Purity, we introduce also the so–called "Quantum Degree of the Gate" $\mathcal{Q}$, defined as the maximum value of the overlap between all possible output states that are obtained starting from an unentangled state and all the maximally entangled states, i.e.,

$$\mathcal{Q} = \max_{\tilde{\rho}_{\rm out}, |\Psi_{\rm me}\rangle} \langle \Psi_{\rm me} | \tilde{\rho}_{\rm out} | \Psi_{\rm me} \rangle,$$

where $\tilde{\rho}_{\rm out}$ denote the output states corresponding to unentangled input states $|\Psi_{\rm in}\rangle = |\psi_a\rangle_1 |\psi_b\rangle_2$, and $|\Psi_{\rm me}\rangle$ is a maximum entangled state. As has been shown, when the overlap between a density operator and a maximally entangled state is larger than $(2+3\sqrt{2})/8 \simeq 0.78$, Clauser–Horne–Shimony–Holt inequalities are violated [8]. Finally, another useful parameter is the "Entanglement Capability" $\mathcal{C}$ [9,10], given as the smallest eigenvalue of the partial transposed density matrix $\hat{\rho}_{\rm out}$, for unentangled inputs states. As recently shown[6], the negativity of this quantity is a necessary and sufficient condition for nonseparability of density operators of two spin 1/2 systems. These quantities can be calculated numerically starting from the gate operators $\hat{R}_{i'i}$ with a maximization/minimization procedure. Both parameters are important to quantify the ability to create entanglement between the two ions. In Fig. 2 we have plotted the different gate parameters as functions of the dissipation rate. As expected, the gate becomes unreliable whenever $\kappa t_g \simeq 1$, where t_g is the time required for the gate that is of the order of $4\pi/(\Omega\eta)$. We note also that the purity of the gate decreases faster compared to that of the fidelity, which in turn decays in the same way as the degree of entanglement.

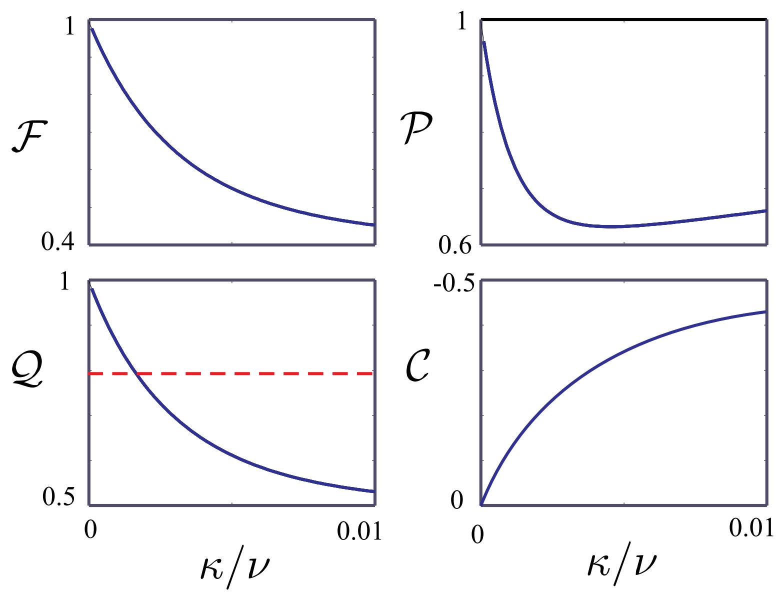

Fig. 2. Fidelity, Purity Quantum degree of a Gate and Entanglement Capability as functions of the dissipation rate. Here $\kappa_{\rm cm}/\nu$=$\kappa_{\rm r}/\nu$=κ/ν, η=1 and Ω/ν=.1 and $\Delta = -\nu$.

In Fig. 3 we have plotted the fidelity of the gate as a function of the laser detuning. Under ideal circumstances, the gate should work for a detuning $\Delta = -\nu$. As we see, when the effective Rabi frequency (proportional to $\Omega\eta/\nu$) increases, the Fidelity decreases. The maximum Fidelity occurs for a different value of Δ. This is due to the AC–Stark shift, which is a consequence of the laser radiation.

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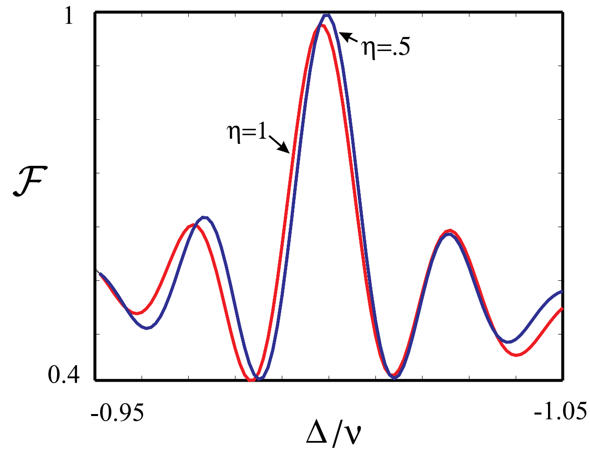

Fig. 3. Fidelity as a function of the detuning. Here we have chosen Ω/ν=.1, κ/ν=0.

4. Conclusions

In this paper, we have shown how physical processes in open quantum systems can be fully characterized. Applying recently developed techniques related to the tomography of quantum processes we have shown how to study decoherence in the context of quantum computation. In particular, we have studied the realistic performance of the ion trap quantum computer model by applying such ideas.

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