Continuity estimates for porous medium type equations with measure data

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CONTINUITY ESTIMATES FOR POROUS MEDIUM TYPE EQUATIONS
WITH MEASURE DATA

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ABSTRACT. We consider parabolic equations of porous medium type of the form
\[ u_t - \text{div} \ A(x,t,u,Du) = \mu \quad \text{in} \ E_T, \]
in some space-time cylinder \( E_T \). The most prominent example covered by our assumptions
is the classical porous medium equation
\[ u_t - \Delta u^m = \mu \quad \text{in} \ E_T. \]
We establish a sufficient condition for the continuity of \( u \) in terms of a natural Riesz po-
tential of the right-hand side measure \( \mu \). As an application we come up with a borderline
condition ensuring the continuity of \( u \): more precisely, if \( \mu \in L\left( \frac{N+2}{2}, 1 \right) \), then \( u \) is
continuous in \( E_T \).

1. INTRODUCTION

In this paper we establish a characterization ensuring the continuity of solutions of non-
homogeneous porous medium type equations whose most prominent example is given by
the classical porous medium equation
\[ u_t - \text{div} \ (a(x,t)Du^m) = \mu \quad \text{in} \ E_T, \]
where the matrix \( a \) is only measurable and positive-definite in \( E_T \). Here, \( E_T \) stands for the
space-time cylinder of height \( T > 0 \) over a bounded open domain \( E \subset \mathbb{R}^N, N \geq 2 \). The
inhomogeneity \( \mu \) is a non-negative Radon-measure on \( E_T \) with finite total mass \( \mu(E_T) < \infty \).
Without loss of generality, we assume that the measure \( \mu \) is defined on \( \mathbb{R}^{N+1} \) by letting
\( \mu\left( \mathbb{R}^{N+1} \setminus E_T \right) = 0 \). More generally, we consider porous medium type equations of the
type
\[ u_t - \text{div} \ A(x,t,u,Du) = \mu \quad \text{in} \ E_T. \]
For the vector-field \( A : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) we assume that it is measurable with respect
to \( (x,t) \in E_T \) for all \( (u,\xi) \in \mathbb{R} \times \mathbb{R}^N \), and continuous with respect to \( (u,\xi) \) for a.e.
\( (x,t) \in E_T \), and moreover satisfies the following growth and ellipticity conditions:
\[ \left\{ \begin{array}{l}
A(x,t,u,\xi) \cdot \xi \geq mC_o |u|^{m-1} |\xi|^2, \\
|A(x,t,u,\xi)| \leq mC_1 |u|^{m-1} |\xi|,
\end{array} \right. \]
whenever \( (x,t) \in E_T, u \in \mathbb{R} \) and \( \xi \in \mathbb{R}^N \), for some \( 0 < C_o \leq C_1 < \infty \). Throughout the
paper we consider the case \( m \geq 1 \), i.e. we are concerned with the degenerate case in the
porous medium equation.

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In [16, 1] we established a sufficient condition ensuring the local boundedness of weak solutions to (1.2). More precisely we proved for given \( \lambda \in (0, \frac{1}{N}) \) the following pointwise bound for solutions to (1.2):

\[
(1.4) \quad u(z_o) \leq 2 \left( \frac{r^2}{\theta} \right)^{\frac{1}{m+1}} + \gamma \left[ \frac{1}{r^{n+2}} \int_{Q_r,\theta(z_o)} u^{m+\lambda} \, dx \, dt \right]^{\frac{1}{m+\lambda}} + \gamma I_2^\mu(z_o, r, \theta)
\]

which holds true whenever \( Q_r,\theta(z_o) \subset E_T \) for a.e. \( z_o \in E_T \) with a universal constant \( \gamma \) depending only on the data \( N, m, C_o, C_1 \), and on \( \lambda \). Here, the localized (or truncated) parabolic Riesz potential is defined by

\[
I_2^\mu(z_o, r, \theta) := \int_0^r \mu(Q_{r,\theta}(z_o)) \frac{dg}{g^{N+2-\beta}}, \quad \beta \in (0, N+2),
\]

for \( z_o \in E_T \) and \( r, \theta > 0 \) such that \( Q_{r,\theta}(z_o) \subset E_T \). Here, \( Q_{r,\theta}(z_o) \) stands for a general parabolic cylinder in \( E_T \), see §2.1. In the case that \( \theta = r^2 \) the potential \( I_2^\mu \) reduces to the standard localized parabolic Riesz potential. The pointwise estimate (1.4) implies the following boundedness criterion: If \( I_2^\mu(\cdot, r, r^2) \in L^\infty_{loc}(E_T) \) for some \( r > 0 \), then \( u \in L^\infty_{loc}(E_T) \). In view of this recent result, we deal with the notion of weak energy solution, of which we now give the precise definition.

**Definition 1.1** (locally bounded, weak energy solution). Consider a non-negative measurable function \( u \colon E_T \to \mathbb{R} \) satisfying

\[
(1.5) \quad u \in C^0_{loc}(0, T; L^2_{loc}(E)) \cap L^\infty_{loc}(E_T), \quad u^\frac{m+1}{m} \in L^2_{loc}(0, T; W^{1,2}_{loc}(E));
\]

it is termed locally bounded, weak energy solution of the porous medium type equation (1.2) if and only if for every subset \( U \subseteq E \) and every subinterval \([t_1, t_2] \subset (0, T]\) the following equation

\[
(1.6) \quad \int_U u \varphi \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_U \left[ -u \varphi_t + A(x, t, u, Du) \cdot D\varphi \right] \, dx \, dt = \int_{U \times (t_1, t_2)} \varphi \, d\mu\n\]

holds true for any bounded testing function

\[
\varphi \in W^{1,2}_{loc}(0, T; L^2(U)) \cap L^2_{loc}(0, T; W^{1,2}_0(U)).
\]

In (1.6) the symbol \( Du \) has to be understood in the sense of the following definition:

\[
Du := \frac{2}{m+1} 1_{\{u > 0\}} u^\frac{m}{m+1} D u^{m+1}.
\]

The hypothesis that the testing function \( \varphi \) must be bounded has to be imposed, in order to guarantee that the right-hand side of (1.6) is well defined. All other integrals appearing there are finite, due to the other assumptions on \( u \) and \( \varphi \). The above notion of locally bounded weak energy solution can for instance be retrieved from [7, 11] for the homogeneous, respectively inhomogeneous porous medium equation with a right-hand side \( \mu \in L^\infty(E_T) \). The notion differs from the most common one, where the regularity condition on \( u \) is replaced by the assumption \( u^m \in L^2_{loc}(0, T; W^{1,2}_{loc}(E)) \). The requirements in (1.5) allow the testing of the homogeneous equation by the solution \( u \) itself (and not by \( u^m \)) and lead to natural energy estimates for \( u \) in terms of \( Du^{m+1} \). For the homogeneous, respectively inhomogeneous equation with a bounded right-hand side \( \mu \in L^\infty(E_T) \), this notion seems to be the weakest one which allows natural energy estimates. In the following, when talking of solutions, we will omit the term locally bounded for the sake of simplicity.
As already mentioned at the beginning of the introduction, we are interested in the continuity properties of weak solutions. Our main result is a sufficient criterion guaranteeing the continuity of solutions. The precise statement is as follows:

**Theorem 1.1** (Continuity of weak energy solutions via linear potentials). Let \( u \) be a non-negative, locally bounded, weak energy solution of the porous medium equation (1.2) in the sense of Definition 1.1, where the vector-field \( A \) fulfills the growth and ellipticity conditions (1.3). Furthermore, consider \( E_\circ \subseteq E_T \) and assume that

\[
\lim_{r \to 0} \sup_{z \in E_\circ} I_2^\mu(z, r, r^2) = 0
\]

holds true. Then, \( u \) is continuous in \( E_\circ \).

The subtle point here is, that the local boundedness of \( I_2^\mu(\cdot, r, r^2) \) for some \( r > 0 \), ensures the local boundedness of \( u \), and moreover implies \( \lim_{z \to 0} I_2^\mu(z, 0, g^2) = 0 \) for a.e. \( z \in E_T \), while the information of locally uniform convergence to zero of the Riesz potential from (1.7) implies the continuity of the weak energy solution. With that respect, our results, i.e. the local boundedness and the continuity of weak solutions via Riesz potentials, are of borderline type. It is somewhat surprising that the Riesz potential plays the same role as in the linear setting. At this stage it would be interesting to consider measures \( \mu \) for which the Riesz potential \( I_2^\mu(\cdot, r, r^2) \) is locally bounded, and moreover satisfies

\[
\lim_{\epsilon \to 0} \mu(Q_{\epsilon, \rho^2}(z)) = 0
\]

locally uniformly on \( E_T \) with respect to \( z \). By our potential estimate (1.4) weak energy solutions would be locally bounded, and one might conjecture that they are also locally VMO on \( E_T \). Such a result would be between local boundedness and continuity. We will not go into this subject here.

Theorem 1.1 is stated as a result for weak energy solutions, but it also applies to very weak solutions \( u \), as introduced in [1, Definition 1.3] and then built in [1, Theorem 1.4]. Indeed, due to the boundedness of the weak energy solutions \( u_k \), making up the approximating sequence of the very weak solution \( u \), it is possible to build the starting cylinder (and consequently, the whole approximating sequence of shrinking cylinders \( Q_{r_n, \alpha_n}(z_0) \)) in a way that is independent of \( k \). A similar argument is discussed, for example, in [4, Chapter 6].

As an application of Theorem 1.1, we consider measures given by measurable functions \( \mu \in L^1(E_T) \). In Chapter 5 we establish the following important assertion:

\[
\mu \in L\left(\frac{N+2}{2}, 1\right) \implies u \text{ is locally continuous in } E_T.
\]

For the definition of the Lorentz space \( L\left(\frac{N+2}{2}, 1\right) \) we refer to (5.2). How subtle this result actually is, can be seen by the classical theory for parabolic equations of the form \( u_t - \text{div } A(x, t, Du) = \mu \in E_T \) with coefficients satisfying (1.3) with \( m = 1 \). Here, it is known that the assumption \( \mu \in L^{\frac{N+2}{2} + \epsilon}(E_T) \), for some arbitrary small \( \epsilon > 0 \), implies the continuity of \( u \). This can be retrieved for example from [5, Section IV]. For \( \mu \in L^{\frac{N+2}{2}}(E_T) \) solutions might be even unbounded. We note that \( L^{\frac{N+2}{2} + \epsilon}(E_T) \subset L\left(\frac{N+2}{2}, 1\right) \) for any \( \epsilon > 0 \). We mention, that the assumption \( \mu \in L\left(\frac{N+2}{2}, 1\right) \) is independent of \( m \geq 1 \). Finally, an assumption of the type \( L^{\frac{N+2}{2} + \epsilon}(E_T) \) falls into the range of applications covered by Corollary 5.1. Hence in the case \( m = 1 \), we recover the classical result on the continuity of weak solutions.
Before describing the method of proof, a few words concerning the history of the problem are in order. As far as the regularity for equations with the same structure considered here, with $m > 1$ and $\mu = 0$, is concerned, continuity of solutions was and is still a major issue. An important step forward was the proof that locally bounded solutions are locally Hölder continuous, due to DiBenedetto & Friedman [6]. Hölder continuity for solutions of the Cauchy problem for the prototype equations (1.1), with $a(x, t) = I_m$, was established before by Caffarelli & Friedman [3]; their approach relies on the special property of global solutions. Continuity of solutions of degenerate parabolic equations $u_t = \Delta |u|^{m-1} u$ was proved by Caffarelli & Evans [2], but the modulus of continuity implicit in their proof is essentially of logarithmic kind.

Coming to the method of proof, the continuity of $u$ is, heuristically, the consequence of the following fact: there exists a family of nested and shrinking cylinders $Q_{r_n, \theta_n}(z_o)$, all with the same vertex, such that the oscillation of $u$ in $Q_{r_n, \theta_n}(z_o)$ tends to zero as $n \to \infty$ in a way quantitatively determined by the structure conditions (1.3), and by the measure $\mu$. In order to achieve such a kind of controlled decay, one needs to study separately two cases: Either in the cylinder $Q_{r_n, \theta_n}(z_o)$ $u$ is mostly large in a proper measure-theoretical sense (this will be our first alternative), or such a situation does not occur (this represents the second alternative). In either case, the conclusion is that the oscillation of $u$ in a smaller cylinder about $z_o$ decreases in a way that can be quantitatively measured. By the well-known intrinsic scaling technique originally introduced by DiBenedetto [5], the cylinders have to be rescaled, in order to reflect the degeneracy, that is, their height has to be suitably stretched to take into account the lack of homogeneity of the equation. If $m = 1$, the cylinders would be the standard parabolic cylinders, reflecting the natural homogeneity of the space and time variables.

There is a further aspect to be taken into account: if at a certain step $\tilde{n}$, the solution $u$ is all bounded away from zero in $Q_{r_n, \theta_n}(z_o)$ in a precisely quantified way, then, $u$ will remain bounded away from zero in all smaller cylinders $Q_{r_n, \theta_n}(z_o)$ for any $n > \tilde{n}$; correspondingly, the equation is no longer degenerate, behaves like a second order, quasilinear parabolic equation with growth of order 2, as considered in [14], and our result follows by classical methods. This last possibility is sketched at the end of the proof of Theorem 1.1.

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2. Preliminaries

2.1. Notations. For a point $z \in \mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ we shall always write $z = (x, t)$. By $B_r(x_o) \equiv \{x \in \mathbb{R}^N : |x - x_o| < r\}$ we denote the open ball in $\mathbb{R}^N$ with center $x_o \in \mathbb{R}^N$ and radius $r > 0$. Moreover, we write

$$Q_{r, \theta}(z_o) := B_r(x_o) \times (t_o - \theta, t_o),$$

where $z_o = (x_o, t_o) \in \mathbb{R}^{N+1}$ and $r, \theta > 0$. Whenever writing $2Q$ for a cylinder $Q \equiv Q_{r, \theta}(z_o)$ we mean $2Q = Q_{2r, 4\theta}(z_o)$.

Finally, by $\mathcal{M}(E_T)$ we denote the set of all non-negative Radon-measures on $E_T$.

2.2. Auxiliary lemmas. Throughout the paper we will frequently use the following parabolic Sobolev embedding; see [5, Prop. 3.7, p. 7].
Lemma 2.1. Let $Q_{\rho, \theta}(z_0)$ be a parabolic cylinder with $0 < \rho, \theta \leq 1$ and $1 < p < \infty$, $0 < r < \infty$. Then there exists a constant $\gamma$ depending only on $N, p, r$ such that for every $u \in L^\infty(t_0 - \theta, t_0; L^r(B_\rho(x_0))) \cap L^p(t_0 - \theta, t_0; W^{1,r}(B_\rho(x_0)))$

there holds

$$\int_{Q_{\rho, \theta}(z_0)} |u|^q \, dx \, dt \leq \gamma \left( \sup_{t \in (t_0 - \theta, t_0)} \int_{B_\rho(x_0) \times \{t\}} |u|^r \, dx \right)^{\frac{p}{q}} \int_{Q_{\rho, \theta}(z_0)} \left| \frac{u}{\rho} \right|^p + |Du|^p \, dx \, dt,$$

where

$$q = \frac{p(N + r)}{N}.$$

The following elementary result can be retrieved from [11, 12, 13].

Lemma 2.2. Assume that $u$ is a non-negative function in $E_T$ such that $u^\sigma \in L^2(0, T; W^{1,2}(E_T))$ for some $\sigma \geq 1$. Let $u^{(\varepsilon)} := \max\{u, \varepsilon\}$ for some $\varepsilon > 0$. Then, $u^{(\varepsilon)}$ has a weak derivative $Du^{(\varepsilon)} \in L^2(E_T; \mathbb{R}^N)$ such that

$$Du^{(\varepsilon)} = 1_{E_T \cap \{u > \varepsilon\}} Du,$$

where $Du$ is defined by

$$Du = \frac{1}{\varepsilon} 1_{E_T \cap \{u > 0\}} u^{1-\sigma} Du^\sigma.$$

Moreover, we have that

$$\lim_{\varepsilon \downarrow 0} \|u^{\sigma-1} Du^{(\varepsilon)} - \frac{1}{\sigma} Du^\sigma\|_{L^2(E_T)} = 0.$$

2.3. Auxiliary functions. For $\lambda \in (0, 1)$ and $s \geq 0$, we define the following functions which will show up in a natural way in the energy estimates:

$$G_\lambda(s) := \int_0^s 1 - (1 + \sigma)^{-\lambda} \, d\sigma \equiv s - \frac{1}{1-\lambda} \left((1 + s)^{1-\lambda} - 1\right),$$

$$V_\lambda(s) := \int_0^s \sigma^{m-1} (1 + \sigma)^{-\frac{m+\lambda}{m-\lambda}} \, d\sigma,$$

$$W_\lambda(s) := \int_0^s (1 + \sigma)^{-\frac{1+\lambda}{m-\lambda}} \, d\sigma \equiv \frac{2}{1-\lambda} \left((1 + s)^{\frac{1-\lambda}{m-\lambda}} - 1\right).$$

In the following we state some auxiliary estimates which will be used several times in the course of the proof of the main results. The proofs of Lemmas 2.3, 2.4, 2.5 can be found in [1].

Lemma 2.3. For any $\varepsilon \in (0, 1]$ and $s \geq 0$ we have

$$V_\lambda(s) \leq \frac{2}{m-\lambda} s^{\frac{m-\lambda}{m}}$$

and

$$s^{m+\lambda} \leq \varepsilon^{1+\lambda} s^{-1} + \gamma_\varepsilon V_\lambda(s) \frac{2(m+\lambda)}{m},$$

where the constant $\gamma_\varepsilon$ blows up as $\varepsilon^{-(1+\lambda)} \frac{2(m+\lambda)}{m-\lambda}$ in the limit $\varepsilon \downarrow 0$. We note that $\gamma_\varepsilon$ also depends on $m$ and $\lambda$. 
Lemma 2.4. For any $\varepsilon \in (0, 1]$ and $s \geq 0$ there holds
\[
W_{\lambda}(s) \leq 2^{1-\lambda} s^{\frac{1-\lambda}{\lambda}}
\]
and
\[
s^{1+\lambda} \leq \varepsilon^{1+\lambda} + \gamma_{\varepsilon} W_{\lambda}(s)^{2(1+\lambda)\frac{1}{1-\lambda}},
\]
where the constant $\gamma_{\varepsilon}$ blows up as $\varepsilon \to 0$. We note that $\gamma_{\varepsilon}$ also depends on $\lambda$.

Lemma 2.5. For any $\varepsilon \in (0, 1]$ and $s \geq 0$ there holds
\[
s \leq \varepsilon + \gamma_{\varepsilon} G_{\lambda}(s)
\]
for a constant $\gamma_{\varepsilon} \equiv \gamma(\lambda)\varepsilon^{-1}$.

2.4. The logarithmic function. For later purposes we introduce the Logarithmic function $\psi$ as follows: For parameters $a, b, c$ with $0 < c < a$ and $b \geq 0$ we define for $s < a + b + c$ the function
\[
\psi_{(a,b,c)}(s) := \ln_{+} \left( \frac{a}{a - (s - b)_+ + c} \right)
\]
\[
:= \begin{cases} 
\ln \left( \frac{a}{a - s + b + c} \right), & \text{if } b + c < s < a + b + c, \\
0, & \text{if } s \leq b + c.
\end{cases}
\]

The first and second derivative can be computed easily. For the first derivative we have
\[
0 \leq \left( \psi_{(a,b,c)} \right)'(s) = \begin{cases} 
\frac{1}{b - s + a + c}, & \text{if } b + c < s < a + b + c, \\
0, & \text{if } s < b + c.
\end{cases}
\]

The second derivative away from $s = b + c$ is given by
\[
(\psi_{(a,b,c)})''(s) = \left[ (\psi_{(a,b,c)})'(s) \right]^2 \geq 0.
\]

3. Energy estimates

Let $k$ be any real number and for a function $v \in L^1(E)$ we consider the truncations of $v$ given by
\[
(v - k)_+ \equiv \sup \{ v - k; 0 \}, \quad (v - k)_- \equiv \sup \{ -(v - k); 0 \}.
\]

In the following we prove energy estimates for $(u - k)_-$ and $(u - k)_+$.

Proposition 3.1. There exists a positive constant $\gamma = \gamma(m, C_0, C_1)$, such that there holds: Whenever $u$ is a non-negative weak energy solution to (1.2) in $E_T$, in the sense of Definition 1.1, then for every cylinder $Q_{\varepsilon, \theta}(z_o) \subset E_T$, every $k \geq 0$, and every cutoff function $\zeta \in W^{1,\infty}_0(B_0(x_0))$, and such that $0 \leq \zeta \leq 1$, we have
\[
\begin{align*}
\text{ess sup}_{t_o-\theta < t \leq t_o} \int_{B_0(x_0) \times \{ t \}} & (u - k)^2 \zeta^2 dx - \int_{B_0(x_0) \times \{ t_o-\theta \}} (u - k)^2 \zeta^2 dx \\
& + C_0 \int_{Q_{\varepsilon, \theta}(z_o)} |u|^{m-1} |D(u - k)_-|^2 \zeta^2 dx dt \\
& \leq \gamma \int_{Q_{\varepsilon, \theta}(z_o)} (u - k)^2 \zeta dx dt + \gamma \int_{Q_{\varepsilon, \theta}(z_o)} |u|^{m-1} (u - k)^2 |D\zeta|^2 dx dt
\end{align*}
\]
and, moreover
\[
\text{ess sup}_{t_0-\theta < t \leq t_0} \int_{B_\rho(x_o) \times \{t\}} (u - k)_+^2 \zeta^2 dx - \int_{B_\rho(x_o) \times \{t_0-\theta\}} (u - k)_+^2 \zeta^2 dx + C_o \int_{Q_{\rho,\theta}(z_o)} |u|^{m-1} |D(u - k)_+^2 \zeta^2 dx dt 
\]
\[
\leq \gamma \int_{Q_{\rho,\theta}(z_o)} (u - k)_+^2 \zeta dx dt + \gamma \int_{Q_{\rho,\theta}(z_o)} |u|^{m-1}(u - k)_+^2 |D\zeta|^2 dx dt 
\]
(3.2)
\[
+ \gamma \int_{Q_{\rho,\theta}(z_o)} (u - k)_+ d\mu. 
\]

Proof. After a translation we may assume \((x_o, t_o) = (0, 0)\). We limit ourselves to the proof for \((u - k)_-\), the one for \((u - k)_+\) being completely analogous, except for the extra term coming from the measure \(\mu\). In (1.6) take the testing function
\[
\varphi_- = -\zeta^2 (u - k)_-
\]
over \(B_\rho \times (-\theta, t]\), where \(-\theta < t \leq 0\). The use of \(-(u - k)_-\) in this testing function is justified, modulus a mollification procedure with respect to \(t\), as explained in detail in [1, Chapter 2, 3]. We omit the details, since the procedure is quite standard. With this respect the following computations are done on a formal basis, when writing \(u_t\). The testing gives
\[
- \int_{B_\rho \times (-\theta, t]} u_t (u - k)_- \zeta^2 dx dt - \int_{B_\rho \times (-\theta, t]} A(x, \tau, u, Du) \cdot D(u - k)_- \zeta^2 dx d\tau 
\]
\[
- 2 \int_{B_\rho \times (-\theta, t]} (u - k)_- A(x, \tau, u, Du) \cdot D\zeta \zeta dx d\tau \leq 0, 
\]
where we have directly taken into account that
\[
- \int_{B_\rho \times (-\theta, t]} (u - k)_- \zeta^2 d\mu \leq 0. 
\]
This is precisely the term, which cannot be discarded, when working with \((u - k)_+\). The first term can be transformed as usual, to get
\[
- \int_{B_\rho \times (-\theta, t]} u_t (u - k)_- \zeta^2 dx dt = - \int_{Q_{\rho,\theta}} (u - k)_-^2 \zeta \zeta dx d\tau + \frac{1}{2} \int_{B_\rho \times \{t\}} (u - k)_-^2 \zeta^2 dx - \frac{1}{2} \int_{B_\rho \times (-\theta, t]} (u - k)_-^2 \zeta^2 dx. 
\]
From the first structure condition (1.3) it follows that
\[
- \int_{B_\rho \times (-\theta, t]} A(x, \tau, u, Du) \cdot D(u - k)_- \zeta^2 dx d\tau 
\]
\[
\geq C_o m \int_{B_\rho \times (-\theta, t]} |u|^{m-1} |D(u - k)_-^2 \zeta^2 dx d\tau, 
\]
and from the second condition in (1.3) and Young’s inequality it follows that
\[
2 \left| \int_{B_\rho \times (-\theta, t]} (u - k)_- A(x, \tau, u, Du) \cdot D\zeta \zeta dx d\tau \right| 
\]
\[
\leq 2 C_1 m \int_{B_\rho \times (-\theta, t]} |u|^{m-1}(u - k)_- |D(u - k)_- \zeta| D\zeta dx d\tau 
\]
Combining these estimates, and taking the supremum over \( t \in (0,1) \) proves the proposition.

In the sequel we need another estimate for \((u-k)_+\). Let \( \lambda \in (0,1), a,d > 0 \). We consider (1.6) on parabolic cylinders of the form \( Q^{(a)}_{(\rho)}(z_0) := B_\rho(x_0) \times \Lambda^{(a)}_{(\rho)}(t_0) = B_\rho(x_0) \times (t_0 - a^{1-m} \rho^2, t_0) \). These cylinders are natural, since they take into account the structure (scaling) of the parabolic equation which arises from the degeneracy in the \( u \)-variable. From [1, (3.2)] we recall the following energy estimate:

**Proposition 3.2.** There exists a positive constant \( \gamma = \gamma(m,C_\alpha,C_1,\lambda) \geq 1 \), such that there holds: Whenever \( u \) is a weak energy solution to (1.2), in \( E_T \), in the sense of Definition 1.1, then for every cylinder \( Q^{(a)}_{(\rho)}(z_0) \subseteq E_T \), and every \( a \geq 0 \) we have that the energy estimate

\[
\sup_{t \in \Lambda_{(\rho/2)}^{(a)}} \int_{B_\rho(x_0)} G_\lambda \left( \frac{u-a}{d} \right) dx \\
+ \iint_{Q^{(a)}_{(\rho/2)} \cap \{ u > a \}} a^{m-1} \left| D \lambda \left( \frac{u-a}{d} \right) \right|^2 + a^{m-1} \left| D W \lambda \left( \frac{u-a}{d} \right) \right|^2 dx dt
\]

\[
\leq \frac{\gamma}{\rho^2} \iint_{Q^{(a)}_{(\rho/2)} \cap \{ u > a \}} u^{m-1} \left( 1 + \frac{u-a}{d} \right)^{1+\lambda} dx dt + \frac{\gamma \mu(Q^{(a)}_{(\rho)})}{d}
\]

holds true.

**Remark 3.1.** Proposition 3.2 has been stated and proved for cylinders \( Q^{(a)}_{(\rho)}(z_0) \) in [1]. Later on we shall work with slightly different cylinders of the form

\[
Q^{(\omega/A)}_{(\rho)}(z_0) = B_\rho(x_0) \times \left( t_0 - \left( \frac{\omega}{A} \right)^{1-m} \rho^2, t_0 \right),
\]

with \( A \in (0,1) \) and \( \omega > 0 \) such that \( \frac{1}{4} \omega \leq a \leq \frac{1}{12} \omega \). Then, a careful reading of the proof shows that the energy estimate continues to hold with a constant \( \frac{\gamma}{\rho^2} \) instead of \( \frac{\gamma}{\rho^2} \) on the right-hand side.

In the following Lemma we consider a weak energy solution to (1.2) in \( E_T \) and a general cylinder \( Q_{(\rho,\theta)}(z_0) \subseteq E_T \). From our potential estimate we already know that \( \sup_{Q_{(\rho,\theta)}(z_0)} u < \infty \) and therefore also

\[
H := \sup_{Q_{(\rho,\theta)}(z_0)} (u-k)_+ < \infty
\]

for any \( k \geq 0 \). Without loss of generality we can assume that \( k < \sup_{Q_{(\rho,\theta)}(z_0)} u \). Otherwise, we would have \( H = 0 \). Finally, let \( 0 < c < \min\{1,H\} \). From (2.1) we recall the definition of the logarithmic function \( \psi_{(H,k,c)} \). With that at hand we define for \( z \in Q_{(\rho,\theta)}(z_0) \) the function

\[
\psi(u)(z) := (\psi_{(H,k,c)} \circ u)(z) = \ln_+ \left[ \frac{H}{H - (u(z) - k)_+ + c} \right],
\]

which will be used in the formulation of the following Lemma.
Proposition 3.3. There exists a constant $\gamma$, depending only on $N, m, C_o, C_1$, such that for any weak energy solution $u$ to (1.2), in $E_T$ in the sense of Definition 1.1, for every cylinder $Q_{\xi,\theta}(z_0) \subseteq E_T$, and for every level $k \geq 0$, there holds:

$$
\sup_{t_0 - \theta < t < t_0} \int_{B_\rho(x_\rho) \times \{t\}} \psi^2(u)\zeta^2 \, dx \\
\leq \int_{B_\rho(x_\rho) \times \{t_0 - \theta\}} \psi^2(u)\zeta^2 \, dx + \gamma \int_{Q_{\xi,\theta}(z_0)} u^{m-1}\psi(u)|D\zeta|^2 \, dx \, dt \\
+ \frac{2}{c} \left( \ln \frac{H}{\rho} \right) \int_{Q_{\xi,\theta}(z_0)} \chi_{\{u > k\}} \, d\mu.
$$

Here, $\zeta \in W^{1,\infty}(B_\rho(x_\rho))$ is a cutoff function independent of $t$.

Proof. Take $(x_\rho, t_0) = (0, 0)$ and work within the cylinder $Q^t \equiv B_\rho \times (-\theta, t)$, with $-\theta < t < 0$. In the weak formulation (1.6) take the testing function $\varphi = \zeta^2 \psi'(u) = 2\psi(u)\psi'(u)\zeta^2$. By direct calculation we infer that $|\psi^2|'' = 2(1 + \psi)\psi'^2$, and therefore

$$
|\psi^2|''(u) = \left[ 2(1 + \psi)\psi'^2 \right](u) \in L^\infty(Q_{\xi,\theta}(z_0))
$$

which implies that such a $\varphi$ is an admissible testing function, modulo a mollification procedure with respect to time. Note that $\psi(u) \neq 0$ implies that $u > k + c > 0$ and therefore $|D\varphi| \in L^2(Q_{\xi,\theta}(z_0))$. Now, since $\psi(u)$ vanishes on the set where $(u - k)_+ = 0$, we find

$$
\int_{Q^t} u_\tau [\psi^2]'(u)\zeta^2 \, dx \, d\tau = \int_{B_\rho \times \{t\}} \psi^2(u)\zeta^2 \, dx - \int_{B_\rho \times \{-\theta\}} \psi^2(u)\zeta^2 \, dx.
$$

The term involving $A(x, \tau, u, Du)$ is estimated with the help of the lower bound from (1.3) as follows:

$$
\int_{Q^t} A(x, \tau, u, Du) \cdot D\varphi \, dx \, d\tau \geq 2m C_o \int_{Q^t} \left[ (1 + \psi)\psi'^2 \right](u) u^{m-1}|Du|^2 \zeta^2 \, dx \, d\tau \\
- 4m C_1 \int_{Q^t} u^{m-1}|Du| \left| \psi \psi' \right|(u) \zeta |D\zeta| \, dx \, d\tau.
$$

By an application of Young’s inequality we obtain from this

$$
\int_{Q^t} A(x, \tau, u, Du) \cdot D\varphi \, dx \, d\tau \\
\geq m C_o \int_{Q^t} \left[ (1 + \psi)\psi'^2 \right](u) u^{m-1}|Du|^2 \zeta^2 \, dx \, d\tau - \gamma \int_{Q^t} \psi(u) u^{m-1}|D\zeta|^2 \, dx \, d\tau,
$$

with a constant $\gamma$ depending on $m, C_o$ and $C_1$. As for the remaining term, since by the definition of $\psi(u)$ we estimate

$$
\psi(u) \leq \ln \left( \frac{H}{\rho} \right) \quad \text{and} \quad \psi'(u) \leq \frac{1}{c},
$$

we have

$$
2 \int_{Q^t} \left[ \psi \psi' \right]'(u)\zeta^2 \, d\mu \leq \frac{2}{c} \left( \ln \frac{H}{\rho} \right) \int_{Q_{\xi,\theta}} \chi_{\{u > k\}} \, d\mu.
$$

Here, we also used the fact that $\zeta^2 \in [0, 1]$. Collecting these estimates, discarding the positive term in the second last inequality, and taking the supremum over $t \in (-\theta, 0)$, establishes the claim of the proposition. \qed
4. PROOF OF THEOREM 1.1

For a cylinder $Q \Subset E_T$ let

$$\mu_{2,Q}(r, \theta) := \sup_{z \in Q} I_{\omega}(z, r, \theta).$$

Theorem 1.1 will be a consequence of the following result.

**Proposition 4.1.** There exist constants $C > 1,$ and $\delta, \nu_* \in (0, 1),$ that can be quantitatively determined only in terms of the structural constants $m, C_{\nu_*} C_1,$ and the dimension $N,$ such that with $\eta := \sqrt{\frac{1}{2} \nu_* \delta^{m-1}}$ the following assertion holds true: Let $u$ be a weak energy solution to (1.2) in $E_T$ and $z_0 \in E_T.$ There exist $\rho > 0,$ and $\omega > 0$ such that with $\varrho_o := \rho,$ $\omega_o := \omega,$ and

$$\theta_n := \omega_n^{1-m}, \quad \varrho_n := \eta^m \varrho_o, \quad Q_n := Q_{\varrho_o, \varrho_n, \varrho_n}^{\theta_n}(z_o)$$

and

$$\omega_{n+1} := \max \{\delta \omega_n, C \mu_{2,Q_n}(4 \varrho, 16 \theta_n \varrho_n^2)\}$$

for $n \in \mathbb{N}_0,$ the assertions

1. $Q_n \subset Q_{n-1} \subset \cdots \subset Q_o \subset E_T,$
2. $\begin{array}{c}
\text{osc}_{Q_n} u \leq \omega_n, \quad \text{for all } n \in \mathbb{N}_0.
\end{array}$

hold true.

Assuming Proposition 4.1 to be true, we proceed with the proof of Theorem 1.1. The idea of the proof is to obtain a quantitative decay of the oscillation in terms of the radius from the discrete decay on the cylinders $Q_n.$

**Proof.** By the definitions of $\theta_n$ and $\omega_n,$ we have

$$\theta_{n+1} = \omega_{n+1}^{1-m} = \left[\max \left\{\delta \omega_n, C \mu_{2,Q_n}(4 \varrho, 16 \theta_n \varrho_n^2)\right\}\right]^{1-m}$$

$$= \min \left\{ (\delta \omega_n)^{1-m}, (C \mu_{2,Q_n}(4 \varrho, 16 \theta_n \varrho_n^2))^{1-m} \right\} \leq (\delta \omega_n)^{1-m}.$$

By iteration, we can then conclude that

$$\theta_n \leq (\delta^{n-k} \omega_k)^{1-m}, \quad \text{for all } k = 0, 1, \ldots, n - 1.$$

This implies that

$$\omega_{n+1} \leq \delta \omega_n + C \mu_{2,Q_n}(4 \varrho, 16 \omega_n^{1-m} \delta^{(1-m)n} \eta^{2n} \varrho^2)$$

$$\leq \delta \omega_n + C \mu_{2,Q_n}(4 \varrho, 16 \omega_o^{1-m} \varrho^2),$$

where we used in the last line the inclusion $Q_n \subset Q_o$ and the fact that $\delta^{1-m} \eta^2 < 1$ which is a consequence of the definition of $\eta$ and $\nu_* < 1.$ We iterate the preceding inequality for $j = 0, \ldots, n$ in order to obtain a first rough bound for $\omega_n.$ Abbreviating

$$I_o := \mu_{2,Q_o}(4 \varrho, 16 \omega_o^{1-m} \varrho^2),$$

we have

$$\omega_n \leq \delta^n \omega_o + C I_o \sum_{j=0}^{n-1} \delta^{n-1-j} \leq \omega_o + \frac{C}{1-\delta} I_o.$$
for all \( n \in \mathbb{N} \). Now, we utilize the inequalities \( \eta \leq (\frac{1}{2} \nu_*)^{\frac{1}{2}} \) and \( \delta^{1-m} \eta^2 \leq \frac{1}{8} \nu_* \), which also follows from the definition of \( \eta \). Instead of (4.4) we now get

\[
(4.6) \quad \omega_{n+1} \leq \delta \omega_n + C \| \nu \|_{\mathcal{W}_2 \mathcal{Q}_o} \left( 4 \left( \frac{\nu_*}{8} \right)^{\frac{1}{2}} \varrho, 16 \omega_1^{1-m} \left( \frac{\nu_*}{8} \right)^{\delta^2} \right),
\]

for any \( n \in \mathbb{N} \). Now, for \( \bar{\varrho} \in (0, \varrho) \) there exists \( k \in \mathbb{N} \), such that

\[
\left( \frac{\nu_*}{8} \right)^{\frac{1}{2}} \varrho \leq \bar{\varrho} < \left( \frac{\nu_*}{8} \right)^{\frac{k-1}{2}} \varrho.
\]

The number \( k \) is uniquely determined by the requirement

\[
(4.7) \quad k - 1 < \frac{\ln \bar{\varrho}}{\ln \left( \frac{\nu_*}{8} \right)} < k.
\]

Iterating (4.6) yields

\[
\omega_n \leq \delta^{n-k} \omega_k + C \sum_{j=k}^{n-1} \| \nu \|_{\mathcal{W}_2 \mathcal{Q}_o} \left( 4 \left( \frac{\nu_*}{8} \right)^{\frac{1}{2}} \varrho, 16 \omega_1^{1-m} \left( \frac{\nu_*}{8} \right)^{\delta^2} \right) \delta^{n-1-j},
\]

\[
\leq \delta^{n-k} \omega_k + C \| \nu \|_{\mathcal{W}_2 \mathcal{Q}_o} \left( 4 \left( \frac{\nu_*}{8} \right)^{\frac{1}{2}} \varrho, 16 \omega_1^{1-m} \left( \frac{\nu_*}{8} \right)^{\delta^2} \right) \sum_{j=k}^{n-1} \delta^{n-1-j},
\]

\[
\leq \delta^{n-k} \omega_k + \frac{C}{1-\delta} \| \nu \|_{\mathcal{W}_2 \mathcal{Q}_o} \left( 4 \bar{\varrho}, 16 \omega_1^{1-m} \bar{\varrho}^2 \right).
\]

Now, let \( \bar{\rho} \in (0, \bar{\varrho}) \) be fixed. At this stage we can argue as in [5, Chapter III, § 3]. First, we fix a number \( 0 < b < \delta \) and choose \( \ell \in \mathbb{N} \) such that \( b^\ell \bar{\varrho} < \bar{\rho} \leq b^{\ell-1} \bar{\varrho} \). The number \( \ell \) is uniquely determined by

\[
(4.8) \quad \ell - 1 \leq \frac{\ln \bar{\rho}}{\ln b} < \ell.
\]

Then, with \( n = k + \ell \) we estimate

\[
\delta^{n-k} = \delta^\ell = \exp(\ell \log \delta) = b^\alpha < \left( \frac{\bar{\varrho}}{\bar{\rho}} \right)^\alpha,
\]

where \( \alpha \) is defined by

\[
\alpha := \frac{\ln \delta}{\ln b}.
\]

Note that \( \alpha \in (0, 1) \), since \( 0 < b < \delta < 1 \). Thus, we have shown that there exist \( \alpha \in (0, 1) \), and \( \gamma > 1 \), that depend only on the data, such that

\[
\omega_n \leq \left( \frac{\bar{\rho}}{\bar{\varrho}} \right)^\alpha \omega_k + \frac{C}{1-\delta} \| \nu \|_{\mathcal{W}_2 \mathcal{Q}_o} \left( 4 \bar{\varrho}, 16 \omega_1^{1-m} \bar{\varrho}^2 \right)
\]

holds true for any \( 0 < \bar{\rho} < \bar{\varrho} \leq \varrho \), where \( n = k + \ell \) and \( k, \ell \) are defined by (4.7) and (4.8). The strategy now is as follows: We fix \( \beta \in (0, 1) \) and consider radii \( 0 < \bar{\rho} \leq \varrho \). We choose \( \bar{\varrho} \) according to \( \bar{\rho} = \varrho^{1-\beta} \bar{\rho}^\beta \) and determine \( k, \ell \) according to (4.7) and (4.8) which by the choice of \( \bar{\varrho} \) is equivalent to

\[
(4.9) \quad k - 1 < \frac{\beta \ln \frac{\bar{\varrho}}{\bar{\rho}}}{\ln \left( \frac{\nu_*}{8} \right)} \leq k, \quad \text{and} \quad \ell - 1 < \frac{(1 - \beta) \ln \frac{\bar{\rho}}{\bar{\rho}}}{\ln b} \leq \ell.
\]
With these choices and letting \( n = k + \ell \) we infer from the last inequality that
\[
\text{osc}_{Q_n} u \leq \omega_n \leq \left( \frac{\bar{r}}{\varrho} \right)^{(1-\beta)} \omega_k + \frac{C}{1-\beta} \eta_{2,Q_n} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\omega_1^{-m} \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2.
\]
Plugging in the boundedness of \( \omega_k \) from (4.5) for any \( k \in \mathbb{N} \) we obtain that there holds:
\[
\text{osc}_{Q_n} u \leq \omega_n \leq \left( \frac{\bar{r}}{\varrho} \right)^{(1-\beta)} \left[ \omega_o + \frac{C}{1-\beta} I_o \right] + \frac{C}{1-\beta} \eta_{2,Q_n} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\omega_1^{-m} \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2
\]
\[=: I(\bar{r}) + II(\bar{r}).
\]
We note that both terms of the right-hand side vanish in the limit \( \bar{r} \downarrow 0 \). Therefore we can choose \( g_0 \in (0,\varrho) \) such that \( I(\bar{r}) + II(\bar{r}) \leq \omega_o \) for any \( 0 < \bar{r} \leq g_0 \). Via (4.9) we determine \( n_o \in \mathbb{N} \) such that \( \omega_n \leq \omega_o \) holds true for \( n \geq n_o \). Actually, we can take
\[
n_o = \left[ \frac{\beta \ln \frac{g_0}{\varrho}}{\ln \frac{1}{\delta} \nu_*} \right] + \left[ \frac{(1-\beta) \ln \frac{\varrho}{\varrho}}{\ln b} \right] .
\]
Then, for \( n \geq n_o \), we have
\[
\tilde{Q}_n := B_{\bar{r} \cdot \omega_1^{-m}} \times (-\omega_1^{1-m} \varrho^2, 0] \subset B_{\bar{r} \cdot \omega_1^{-m}} \times (-\varrho \cdot \varrho^2, 0] = Q_n,
\]
and therefore
\[
\text{osc}_{\tilde{Q}_n} u \leq \left( \frac{\bar{r}}{\varrho} \right)^{(1-\beta)} \left[ \omega_o + \frac{C}{1-\beta} I_o \right] + \frac{C}{1-\beta} \eta_{2,Q_o} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\omega_1^{-m} \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2.
\]
From (4.9) we obtain that
\[
n - 2 < \left( \frac{\beta}{\ln \frac{1}{\delta} \nu_*} \right) + \left( \frac{(1-\beta) \ln \varrho}{\ln b} \right) \ln \frac{\bar{r}}{\varrho} \leq n.
\]
We define
\[
\sigma := \left( \frac{\beta}{\ln \frac{1}{\delta} \nu_*} \right) + \left( \frac{(1-\beta) \ln \varrho}{\ln b} \right) \ln \eta,
\]
and obtain
\[
\varrho_n = \eta^n \varrho = \varrho e^{n \ln \eta} > \varrho \exp \left( \sigma \ln \frac{\bar{r}}{\varrho} + 2 \ln \eta \right) = \eta^2 \varrho \left( \frac{\bar{r}}{\varrho} \right)^{\sigma}.
\]
If we finally let
\[
r = \eta^2 \varrho \left( \frac{\bar{r}}{\varrho} \right)^{\sigma},
\]
since \( \tilde{Q}_n \supset Q_{r \cdot \omega_1^{-m} \varrho^2} \), we have
\[
\text{osc}_{Q_{r \cdot \omega_1^{-m} \varrho^2}} u \leq \frac{1}{\eta^{2(1-\beta)}} \left( \frac{\bar{r}}{\varrho} \right)^{(1-\beta)} \left[ \omega_o + \frac{C}{1-\beta} I_o \right]
\[
+ \frac{C}{1-\beta} \eta \eta_{2,Q_o} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\varrho^2 \omega_1^{-m} \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2.
\]
Moreover, if \( \omega_o \geq 1 \), then
\[
\eta \eta_{2,Q_o} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\varrho^2 \omega_1^{-m} \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2 \leq \eta \eta_{2,Q_o} \left( \frac{4\varrho}{\eta} \right)^{\beta}, \ 16\varrho^2 \left( \frac{\bar{r}}{\varrho} \right)^{2\beta} \varrho^2.
\]
On the other hand, if $\omega_o < 1$, then
\[
I^{\mu_2,\omega_o} \left( \frac{4 \rho}{\eta^{m/2}}, \frac{r}{\rho} \right)^{\frac{\alpha}{m}} \frac{16 \rho^2}{\eta^{m/2}} \omega_o^{1-m} \left( \frac{r}{\rho} \right)^{\frac{\alpha}{m}} \leq I^{\mu_2,\omega_o} \left( \frac{4 \rho}{\eta^{m/2}}, \frac{r}{\rho} \right)^{\frac{\alpha}{m}} \frac{16 \rho^2}{\eta^{m/2}} \omega_o^{1-m} \left( \frac{r}{\rho} \right)^{\frac{\alpha}{m}}.
\]

In both instances, by a proper, possible further redefinition of $r$, we are led to consider
\[
sup_{z \in Q_r} I^{\mu_2,\omega_o} \left( \frac{4 \rho}{\eta^{m/2}}, \frac{r}{\rho} \right)^{\frac{\alpha}{m}} \frac{16 \rho^2}{\eta^{m/2}} \omega_o^{1-m} \left( \frac{r}{\rho} \right)^{\frac{\alpha}{m}},
\]
that is, the classical parabolic, truncated Riesz potential, with no intrinsic scaling with respect to time. This proves the continuity of $u$ on $Q_{r,\omega_o^{1-m}}$. Statements concerning the continuity over a compact set, now follow by a standard covering argument.

We now deal with the proof of Proposition 4.1. Having fixed $z_o \in E_T$, we let $\varepsilon, R \in (0, 1)$ such that $Q_{2R,4R^{2-\varepsilon}}(z_o) \subset E_T$. Without loss of generality, we may assume that $z_o \equiv (0, 0)$, so that $Q_{R,R^{2-\varepsilon}}(z_o) = Q_{R,R^{2-\varepsilon}}$. Now, if
\[
osc_{Q_{R,R^{2-\varepsilon}}} u \leq \rho^{-m} \quad \text{for any } \rho \in (0, R],
\]
there is nothing to prove, since the essential oscillation of $u$ has a power-like decay. Otherwise, there exists $\rho \in (0, R]$ such that
\[
osc_{Q_{R,\rho^{2-\varepsilon}}} u > \rho^{-m}.
\]

Then, we define
\[
(4.10) \quad \mu_o^+ := \sup_{Q_{R,\rho^{2-\varepsilon}}} u, \quad \mu_o^- := \inf_{Q_{R,\rho^{2-\varepsilon}}} u, \quad \omega_o := \mu_o^+ - \mu_o^- > 0.
\]

By the preceding inequality, we have that
\[
\omega_o^{m-1} = \left( osc_{Q_{R,\rho^{2-\varepsilon}}} u \right)^{m-1} > \rho^\varepsilon,
\]
which, letting $\theta_o := \omega_o^{1-m}$, guarantees that
\[
(4.11) \quad Q_o := Q_{R,\rho^2} \subset Q_{R,\rho^{2-\varepsilon}}, \quad osc_{Q_{R,\rho^2}} u \leq \omega_o.
\]

Thus (4.11) ensures that (4.2)–(4.3) hold for $n = 0$.

Here, we remark that the role of introducing the cylinder $Q_{R,\rho^{2-\varepsilon}}$, is to guarantee that the upper bound for the oscillation in (4.11) holds true for the constructed cylinder $Q_{R,\theta_o^2}$. It will be part of the proof of Proposition 4.1 to show that at each step of the induction argument, the cylinders $Q_n$ and the essential oscillation of $u$ within them, satisfy the right geometry. Apart from this, $\varepsilon$ plays no other role in this context.

The remaining part of the section will be devoted to the proof of Proposition 4.1: we will determine constants $\delta, \nu_k \in (0, 1)$ and $C > 1$, depending only on the set of data $m, N, C_o, C_1$, and independent of $u$ and $z_o$ for which (4.2)–(4.3) hold inductively for all $n$. 
4.1. The Induction Argument. Assuming (4.2)–(4.3) hold for some \( n \in \mathbb{N}_0 \), we remove the index \( n \) by setting
\[
\rho = \rho_n, \quad \omega = \omega_n, \quad \theta = \theta_n = \omega^{1-m} Q_{\rho, \theta \rho^2} = Q_n,
\]
and
\[
\mu^+ = \sup_{Q_{\rho, \theta \rho^2}} u = \mu_n^+, \quad \mu^- = \inf_{Q_{\rho, \theta \rho^2}} u = \mu_n^-.
\]
Note that, by (4.3) we have
\[
\omega \geq \mu^+ - \mu^-.
\]
Denote by \( a, \xi \) and \( A \) fixed numbers in \((0, 1)\), and let
\[
\tilde{\theta} := A^{m-1} \theta = \left( \frac{A}{\omega} \right)^{m-1}.
\]
Then, for \( r \in (0, \frac{\rho}{2}] \) we have
\[
Q_{2r, 4\tilde{\theta}r^2} \subset Q_{\rho, \theta \rho^2}, \quad Q_{4r, 16\tilde{\theta}r^2} \subset E_T.
\]

We have the following two DeGiorgi-type results.

**Lemma 4.1.** Let \( u \) be a weak energy solution to (1.2), in \( E_T \). There exists a positive number \( \nu_- \), depending on \( a, \xi, A \) and the data \( m, N, C_o, C_1 \), such that if
\[
\left| \{ u \leq \xi \omega \} \cap Q_{2r, 4\tilde{\theta}r^2} \right| \leq \nu_- \left| Q_{2r, 4\tilde{\theta}r^2} \right|,
\]
then
\[
u_- \leq \frac{a \xi \omega}{\tilde{\theta}} \quad \text{a.e. in } Q_{r, \tilde{\theta}r^2}.
\]

**Proof.** The proof follows from the energy estimate (3.1) in Proposition 3.1; see [7, Lemma 7.1, Chapter 3]. Note that in the case considered here the constant \( C \) from the structural conditions (5.2) in Chapter 3 of [7] is 0, and therefore, the first alternative \( C \rho > 1 \) from Lemma 7.1 will never occur. \( \square \)

**Remark 4.1.** The functional dependence of \( \nu_- \) on the indicated parameters can be retrieved form [7, Chapter 3, (7.9)]. We have
\[
u_- = \gamma^{-1} a^{(m-1)} (1 - a^{N+2}) [1 + (\xi A)^{m-1}] \frac{\tilde{\theta}^m}{\left[ (\xi A)^{m-1} + \tilde{\theta}^{m-1} \right]^2},
\]
for a quantitative constant \( \gamma = \gamma(m, N, C_o, C_1) > 1 \), independent of \( a, \xi \) and \( A \). \( \square \)

**Lemma 4.2.** Let \( u \) be a weak energy solution to (1.2), in \( E_T \). Assume that \( \xi \in (0, \frac{1}{2}] \), and
\[
\frac{1}{4} \omega \leq \mu^+ - \frac{1}{4} \omega \leq \tilde{\theta} \omega.
\]
There exist constants \( \nu_+ \in (0, 1) \), depending on \( a, A \) and the data \( m, N, C_o, C_1 \), and \( B > 1 \) that depends on \( a \) and \( m, N, C_o, C_1 \), such that if
\[
\left| \{ u \geq \mu^+ - \xi \omega \} \cap Q_{2r, 4\tilde{\theta}r^2} \right| \leq \nu_+ \left| Q_{2r, 4\tilde{\theta}r^2} \right|,
\]
then either
\[
\xi \omega < B \omega_{2r, 4\tilde{\theta}r^2} \left( 4r, 16\tilde{\theta}r^2 \right)
\]
or
\[
u_+ = \frac{a \xi \omega}{\tilde{\theta}} \quad \text{a.e. in } Q_{r, \tilde{\theta}r^2}.
\]
Remark 4.2. The functional dependence of \( \nu_+ \) and \( B \) on the indicated parameters is given by

\[
\nu_+ = \left( \frac{1 - a}{\gamma} \right)^{1+\lambda}, \quad B = \frac{\gamma}{1 - a}
\]

for an arbitrary parameter \( \lambda \in (0, \frac{1}{N}] \), and a quantitative constant \( \gamma = \gamma(m, N, C_\alpha, C_1, \lambda, \lambda, \lambda) > 1 \), independent of \( a \) and \( \xi \).

\textbf{Proof.} Due to the presence of the measure \( \mu \), the classical DeGiorgi iteration scheme, as adapted to degenerate parabolic equations by DiBenedetto (see [5]), cannot be applied here, and we have to use the Kilpeläinen-Malý approach, as in [15]. We let \( B > 1 \) to be determined in a universal way in the course of the proof. In the following, we assume

\[
\xi \omega \geq B \mathbb{I}^\mu_{2r,40^2} (4r, 16\theta r^2),
\]

since otherwise the assertion of the Lemma is trivally satisfied. Let \( z_1 = (x_1, t_1) \in Q_{r, \theta r^2} \). In the following we will prove that

\[
u(\zeta_1) \leq \mu^+ - a \xi \omega,
\]

and since \( z_1 \) is an arbitrary point in \( Q_{r, \theta r^2} \), the claim of the Lemma follows. For the proof of (4.20), we shall proceed in several steps.

\textit{Step 1: Setting up an iteration scheme.} For \( j = -1, 0, 1, \ldots \) we define

\[
r_j := \frac{r}{2^j}, \quad B_j := B_{r_j}(x_1), \quad Q_j := B_j \times (t_1 - \theta r_j^2, t_1)
\]

and

\[
\alpha_j := \int_0^{r_j} \mu(Q_{q, 2r_j^2}(z_1)) \frac{d\theta}{\theta^N}.
\]

Moreover, we let

\[
a_\omega := \mu^+ - \xi \omega.
\]

For \( j \geq 0 \) we now suppose that \( a_0, \ldots, a_j \) have already been selected. Then, we choose \( a_{j+1} \) as follows: We let \( \lambda \in (0, \frac{1}{N}] \) and define

\[
K_j(a) := \frac{1}{r_j^{N+2}} \int_{B_j \cap \{ u > a_j \}} u^{m-1} \left( \frac{u - a_j}{a - a_j} \right)^{1+\lambda} \, dx dt
\]

for \( a > a_j \). Now, if

\[
K_j(a_j + \frac{1}{4}(\alpha_j - \alpha_j)) < \kappa,
\]

we define

\[
a_{j+1} := a_j + \frac{1}{4}(\alpha_j - \alpha_j).
\]

Otherwise, if

\[
\kappa \geq K_j(a_j + \frac{1}{4}(\alpha_j - \alpha_j)) \geq \kappa,
\]

we choose \( a_{j+1} > a_j + \frac{1}{4}(\alpha_j - \alpha_j) \) according to

\[
K_j(a_{j+1}) = \kappa.
\]

Such a choice is always possible since \( (a_j, \infty) \ni a \mapsto K_j(a) \) is a monotone decreasing, continuous function, \( \lim_{a \to \alpha_j} K_j(a) = +\infty \) and \( \lim_{a \to \infty} K_j(a) = 0 \). In any case, we have

\[
K_j(a_{j+1}) \leq \kappa \quad \text{and} \quad a_{j+1} \geq a_j + \frac{1}{4}(\alpha_j - \alpha_j).
\]
Step 2: A first bound on $a_j$. Here, we will prove a first rough bound on $a_j$ of the form
\begin{equation}
\tag{4.23}
 a_j < \frac{1}{2}(\mu^+ + a_{j-1}) - \frac{1}{4}B\alpha_{j-1}
\end{equation}
and
\begin{equation}
\tag{4.24}
 a_j < \mu^+ - \frac{1}{4}B\alpha_{j-1},
\end{equation}
for any $j \in \mathbb{N}$. We start proving (4.23) in the case $j = 1$. First, we observe that
\begin{equation}
\tag{4.25}
 a_o = \mu^+ - \xi \omega \leq \mu^+ - B\alpha_{j-1},
\end{equation}
by (4.19), so that
\begin{equation}
\tag{4.26}
 a := \frac{1}{2}(\mu^+ + a_o) - \frac{1}{4}B\alpha_o > a_o + \frac{1}{4}(\alpha_{j-1} - a_o).
\end{equation}
Moreover, by (4.19) and the definition of $a_o$, we obtain
\begin{equation}
\tag{4.27}
 a - a_o = \frac{1}{2}\xi \omega - \frac{1}{4}B\alpha_o \geq \frac{1}{2}\xi \omega - \frac{1}{4}\xi \omega = \frac{1}{4}\xi \omega
\end{equation}
and therefore
\begin{equation}
\tag{4.28}
 K_o(\bar{a}) = \frac{1}{r_0^{n+2}} \int \int_{\frac{1}{2}Q_o \cap \{u > a_o\}} u^{m-1} \left( \frac{u - a_o}{\bar{a} - a_o} \right)^{1+\lambda} dx \, dt \leq \frac{(\mu^+)^{m-1}}{r^{n+2}} \left( \frac{\mu^+ - (\mu^+ - \xi \omega)}{\xi \omega / 4} \right)^{1+\lambda} \left\{ \{u > a_o\} \cap Q_o \right\}
\end{equation}
\begin{equation}
\leq \frac{4^{1+\lambda}(\mu^+)^{m-1}}{r^{n+2}} |\{u > a_o\} \cap Q_{2r,4\bar{a}r^2}| \leq \frac{4^{1+\lambda}(\mu^+)^{m-1}r^{|Q_{2r,4\bar{a}r^2}|}}{r^{n+2}} \leq \gamma \omega^{m-1} \bar{\theta} \nu_+ = \gamma(N, m, A) \nu_+.
\end{equation}
Here we have used in turn (4.15), (4.14) and (4.12). Now, for fixed $\kappa \in (0, 1)$ we choose $\nu_+ \leq 2\kappa$. Note, that $\kappa$ will be chosen later in the course of the proof in a universal way. With such a choice of $\nu_+$, we have
\begin{equation}
\tag{4.29}
 K_o(\bar{a}) \leq \frac{1}{2}\kappa.
\end{equation}
Since $\bar{a} > a_o + \frac{1}{4}(\alpha_{j-1} - a_o)$, we conclude from the construction of $a_1$ that $a_o + \frac{1}{4}(\alpha_{j-1} - a_o) \leq a_1 < \bar{a}$. This proves the first bound in (4.23). The second bound follows from the first one, the definition of $a_o$, and (4.19). This proves (4.23) for $j = 1$.

Now, we let $j \in \mathbb{N}$ and assume that (4.23) holds for $1, \ldots, j$. First, we observe that by the definition of $\alpha_j$ and simple computations we have
\begin{equation}
\tag{4.30}
 \left\{ \begin{array}{l}
 \alpha_{j-1} - \alpha_j \geq \frac{1}{2N} \frac{\mu(Q_{r_j, \bar{a}r^2_j}(z_{j}))}{r_j^{N}} = \frac{1}{2N} \frac{\mu(Q_j)}{r_j^{N}}, \\
 \alpha_{j-1} - \alpha_j \leq 2N \frac{\mu(Q_{r_{j-1}, \bar{a}r^2_{j-1}}(z_{j}))}{r_{j-1}^{N}} = 2N \frac{\mu(Q_{j-1})}{r_{j-1}^{N}}.
\end{array} \right.
\end{equation}
We now let
\begin{equation}
\tag{4.31}
 \bar{a} := \frac{1}{2}(\mu^+ + a_j) - \frac{1}{4}B\alpha_j.
\end{equation}
Then, by the second inequality in (4.23), we find that
\begin{equation}
\tag{4.32}
 \bar{a} - a_j = \frac{1}{2}(\mu^+ - a_j) - \frac{1}{4}B\alpha_j > \frac{1}{4}B(\alpha_{j-1} - a_j).
\end{equation}
Moreover, using the first inequality in (4.23), we obtain
\begin{equation}
\tag{4.33}
 \bar{a} - a_j = \frac{1}{2}(\mu^+ - a_j) - \frac{1}{4}B\alpha_j = \frac{1}{2}(\mu^+ + a_j - 2a_j) - \frac{1}{4}B\alpha_j
\end{equation}
\[ \geq \frac{1}{2}(a_j - a_{j-1}) + \frac{1}{4} B(a_{j-1} - a_j) \geq \frac{1}{2}(a_j - a_{j-1}). \]

From now on we proceed much as we did in [1]. We have

\begin{equation}
K_j(\bar{a}) = \frac{1}{r_j^{N+2}} \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a}} \right) \frac{1}{m-1} \frac{u - a_j}{\bar{a} - a_j} \, dx \, dt
\end{equation}

where the constant \( \gamma \) depends only on \( m \). First, we consider the term I. We have

\[ I = \frac{\gamma(\bar{a} - a_j)^{m-1}}{r_j^{N+2}} \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a}} \right) \frac{1}{m+\lambda} \, dx \, dt. \]

By Lemma 2.3, for some fixed \( \varepsilon \in (0, 1) \) to be chosen later, we conclude that

\[ I \leq \frac{\gamma(\bar{a} - a_j)^{m-1}}{r_j^{N+2}} \left[ \varepsilon^{1+\lambda} \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a}} \right) \frac{1}{m-1} \, dx \, dt \right. \]

\[ + \gamma \varepsilon \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} \left( V_\lambda \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(\frac{m+\lambda}{m-\lambda})} \, dx \, dt \]

\[ \leq \frac{\gamma\varepsilon^{1+\lambda}}{r_j^{N+2}} \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} u^{m-1} \, dx \, dt \]

\[ + \frac{\gamma \varepsilon (\bar{a} - a_j)^{m-1}}{r_j^{N+2}} \int_{\frac{1}{2} \Omega_j \cap \{u > a_j\}} \left( V_\lambda \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(\frac{m+\lambda}{m-\lambda})} \, dx \, dt \]

\[ = \gamma \varepsilon^{1+\lambda} + \frac{\gamma \varepsilon (\bar{a} - a_j)^{m-1}}{r_j^{N+2}} \int_{\frac{1}{2} \Omega_j} \left( V_\lambda \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(\frac{m+\lambda}{m-\lambda})} \, dx \, dt, \]

where \( \gamma \) depends on \( N, m \) and \( \gamma \varepsilon \) depends on \( m, \lambda \) and \( \varepsilon \). Here, in the last line we have taken into account the fact that

\[ 1 \leq \frac{u - a_j - 1}{a_j - a_{j-1}} \quad \text{on} \quad \{u > a_j\}, \]

the inclusion \( \frac{1}{2} \Omega_j \subset \frac{1}{2} \Omega_{j-1} \) and that \( K_{j-1}(a_j) \leq \kappa \), by (4.22). To estimate the integral on the right-hand side, we apply Gagliardo-Nirenberg’s inequality from Lemma 2.1 with \( p = 2 \), \( q = \frac{2(m+\lambda)}{m-\lambda} \) and \( r = \frac{2N\lambda}{m-\lambda} \) to conclude that

\[ I \leq \gamma \varepsilon^{1+\lambda} \kappa + \gamma \varepsilon \left[ \sup_{t \in \Lambda_j} \frac{1}{r_j^{N}} \int_{\frac{1}{2} B_{r_j \times \{t\}}} \left( V_\lambda \left( \frac{u - a_j)(u - a_j)}{\bar{a} - a_j} \right) \right]^{\frac{2(m+\lambda)}{m-\lambda}} \, dx \right]^{\frac{1}{2}} \]

\[ \cdot \left( \frac{a_j - a_j}{\bar{a} - a_j} \right)^{m-1} \int_{\frac{1}{2} \Omega_j} \frac{1}{r_j^{N}} \left( V_\lambda \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(\frac{m+\lambda}{m-\lambda})} \, dx \, dt \]

\[ =: \gamma \varepsilon^{1+\lambda} \kappa + \gamma \varepsilon I_1 (I_2 + I_3), \]

with the obvious labeling for \( I_1, I_2 \) and \( I_3 \), and \( \Lambda_j := (-\bar{a} r_j^2, 0) \). In turn, we will separately estimate the appearing terms. We start with the estimate for \( I_1 \).

By (4.14), we have \( \frac{1}{2} \omega \leq \mu^+ - \frac{1}{4} \omega \); since \( \xi \in (0, \frac{1}{2}], a_0 = \mu - \xi \omega \) and \( \{a_j\} \) is a monotone increasing sequence, we have \( a_j \geq a_0 = \mu^+ - \xi \omega \geq \frac{1}{2} \omega \). Moreover, by (4.23)
and (4.14), we have $a_j < \mu_+ \leq \frac{13}{12} \omega$. Therefore, we can apply Remark 3.1, and rely on the energy estimate from Proposition 3.2. Using in turn Lemma 2.3, Hölder’s inequality (note that $\lambda N \leq 1$), Lemma 2.5 for some $\varepsilon_1 \in (0, 1)$ to be chosen later in the proof, and finally the energy estimate from Proposition 3.2, we deduce

$$I_1 = \left[ \sup_{t \in A_j} \frac{1}{r_j^{N+2}} \int_{\frac{1}{2} B_j \times (t) \cap \{u > a_j\}} |V_\lambda \left( \frac{u - a_j}{\bar{a} - a_j} \right) |^{\frac{2N}{m-\lambda}} dx \right]^{\frac{m}{2}} \leq \gamma \left[ \sup_{t \in A_j} \frac{1}{r_j^{N+2}} \int_{\frac{1}{2} B_j \times (t) \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a} - a_j} \right)^{\lambda N} dx \right]^{\frac{m}{2}} \leq \gamma \left[ \sup_{t \in A_j} \frac{1}{r_j^{N+2}} \int_{\frac{1}{2} B_j \times (t) \cap \{u > a_j\}} \frac{u - a_j}{\bar{a} - a_j} dx \right]^{2\lambda} \leq \gamma^{\varepsilon_1} \left[ \sup_{t \in A_j} \frac{1}{r_j^{N+2}} \int_{\frac{1}{2} B_j \times (t) \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a} - a_j} \right) dx \right]^{2\lambda}$$

with $\gamma = \gamma(N, m, C_\omega, C_1, \lambda)$. Considering the first term in the brackets, we have

$$\frac{1}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \left( 1 + \frac{u - a_j}{\bar{a} - a_j} \right)^{1+\lambda} dx dt = \frac{1}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \left( \frac{a_j - a_{j-1}}{a_{j-1} - a_j} + \frac{u - a_j}{\bar{a} - a_j} \right)^{1+\lambda} dx dt \leq \frac{\gamma}{r_j^{N+2}} \int_{Q_{j-1} \cap \{u > a_{j-1}\}} u^{m-1} \left( \frac{u - a_{j-1}}{a_{j-1} - a_j} + \frac{u - a_j}{\bar{a} - a_j} \right)^{1+\lambda} dx dt \leq \gamma \kappa,$$

where we have taken into account (4.27) and (4.22) and $Q_j = \frac{1}{2} Q_{j-1}$. As for the other term in the brackets, by (4.26) and (4.25)

$$\frac{\mu(Q_j)}{r_j^N (\bar{a} - a_j)} \leq \frac{4\mu(Q_j)}{Br^N (\alpha_{j-1} - \alpha_j)} \leq \frac{8N}{B}.$$

Therefore, we conclude that

$$I_1 \leq \gamma^{\varepsilon_1} \left[ \frac{\kappa}{A^{m-1} \kappa_1} + \frac{1}{B} \right]^{2\lambda}.$$

Next, we estimate the term $I_2$. Using Lemma 2.3 and (4.27), we arrive at

$$I_2 = \frac{\gamma (\bar{a} - a_j)^{m-1}}{r_j^{N+2}} \int_{\frac{1}{2} Q_j \cap \{u > a_j\}} \left( \frac{u - a_j}{\bar{a} - a_j} \right)^{m-\lambda} dx dt \leq \frac{\gamma}{r_j^{N+2}} \int_{\frac{1}{2} Q_j \cap \{u > a_j\}} u^{m-1} \left( \frac{u - a_j}{\bar{a} - a_j} \right)^{1-\lambda} dx dt.$$
\[
\leq \frac{\gamma}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \left( \frac{u - a_j}{a_j - a_{j-1}} \right)^{1-\lambda} \, dx \, dt \\
\leq \frac{\gamma}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \left( \frac{u - a_{j-1}}{a_j - a_{j-1}} \right)^{1-\lambda} \, dx \, dt,
\]

where \( \gamma = \gamma(m, \lambda) \). Since the quantity in brackets on the right-hand side integral is larger than 1, we can enlarge the exponent from \( 1-\lambda \) to \( 1+\lambda \), subsequently enlarge the domain of integration to \( \frac{1}{2} Q_{j-1} \cap \{u > a_{j-1}\} \), and replace \( r_j \) by \( r_{j-1} \). This leads us to the estimate

\[
I_2 \leq \frac{\gamma}{r_{j-1}^{N+2}} \int_{\frac{1}{2} Q_{j-1} \cap \{u > a_{j-1}\}} u^{m-1} \left( \frac{u - a_{j-1}}{a_j - a_{j-1}} \right)^{1+\lambda} \, dx \, dt \leq \gamma \kappa,
\]

where in the last inequality we used again (4.22). Note that \( \gamma \) depends on \( N, m, \lambda \). At this point it remains to estimate \( I_3 \) by the energy estimate from Proposition 3.2, we obtain

\[
I_3 = \frac{(\bar{a} - a_j)^{m-1}}{r_j^{N}} \int_{Q_j \cap \{u > a_j\}} |DV \left( \frac{u - a_j}{\bar{a} - a_j} \right)|^2 \, dx \, dt
\leq \gamma \left[ \frac{1}{A^{m-1} r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \left( 1 + \frac{u - a_j}{\bar{a} - a_j} \right)^{1+\lambda} \, dx \, dt + \frac{\mu(Q_j)}{r_j^2 (\bar{a} - a_j)} \right]
\leq \gamma \left( \frac{\kappa}{A^m - 1 + \frac{1}{B}} \right),
\]

where we used (4.29) and (4.30) to estimate the two terms from the second last line. Inserting the estimates for \( I_1, I_2 \) and \( I_3 \) in the right-hand side of the inequality for \( I \), we conclude that

\[
I \leq \gamma \left( \varepsilon^{1+\lambda} + \varepsilon \left( \varepsilon^{2\lambda} + \varepsilon^{-2\lambda} (\kappa + B^{-1}) \right)^2 \right) (\kappa + B^{-1})
\]

holds true with constants \( \gamma = \gamma(N, m) \) and \( \gamma \varepsilon = \gamma(N, m, C_0, C_1, \lambda, A, \varepsilon) \).

Next, we turn our attention to the term \( II \) from the right-hand side of (4.28). Using Lemma 2.4 and (4.22), we find

\[
\begin{align*}
II & \leq \frac{\gamma \varepsilon^{1+\lambda}}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} a_j^{m-1} \, dx \, dt \\
& \quad + \frac{\gamma \varepsilon a_j^{m-1}}{r_j^{N+2}} \int_{Q_j} \left( W_{\lambda} \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(1+\lambda)} \, dx \, dt
\leq \frac{\gamma \varepsilon^{1+\lambda}}{r_j^{N+2}} \int_{Q_j \cap \{u > a_j\}} u^{m-1} \, dx \, dt \\
& \quad + \frac{\gamma \varepsilon a_j^{m-1}}{r_j^{N+2}} \int_{Q_j} \left( W_{\lambda} \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(1+\lambda)} \, dx \, dt
\leq \varepsilon^{1+\lambda} \kappa + \frac{\gamma \varepsilon a_j^{m-1}}{r_j^{N+2}} \int_{Q_j} \left( W_{\lambda} \left( \frac{u - a_j}{\bar{a} - a_j} \right) \right)^{2(1+\lambda)} \, dx \, dt,
\end{align*}
\]

where \( \gamma = \gamma(N, m) \), and \( \gamma \varepsilon = \gamma(N, \lambda, \varepsilon) \). To the integral on the right-hand side of the preceding inequality we apply the Gagliardo-Nirenberg inequality from Lemma 2.1 for the
choices \( p = 2, q = \frac{2(1+\lambda)}{1-\lambda}, r = \frac{2\lambda N}{1-\lambda} \) and \( Q = \frac{1}{2} Q_j \). This yields

\[
\Pi \leq \gamma \varepsilon^{1+\lambda} + \gamma \varepsilon \left[ \sup_{t \in A_j} \int_{\frac{1}{2} B_j \times \{u > a_j\}} \frac{\lambda N}{1-\lambda} \left( \frac{u-a_j}{a-a_j} \right) dx \right]^{\frac{2}{\lambda}}
\]

\[
\leq \gamma \varepsilon \left[ \sup_{t \in A_j} \int_{\frac{1}{2} B_j \cap \{u > a_j\}} \frac{u-a_j}{a-a_j} dx \right]^{\frac{2}{\lambda}}
\]

As in the case of the term \( I \), we now consecutively estimate the terms \( \Pi_i \) for \( i = 1, 2, 3 \). We start with the estimate of the sup-term, that is \( \Pi_1 \). In turn, we use Lemma 2.4 and Hölder’s inequality (note that \( \lambda N \leq 1 \)) to infer that

\[
\Pi_1 \leq \gamma \varepsilon^{2\lambda} + \frac{\gamma}{\varepsilon^2} \left( \frac{\kappa}{A^{m-1}} + \frac{1}{B} \right)^{2\lambda}
\]

holds true. Next, we come to the estimate of the term \( \Pi_2 \). Using again Lemma 2.4 and following the arguments from the estimation of \( I_2 \), we find that there holds:

\[
\Pi_2 \leq \gamma \varepsilon^{1+\lambda} + \gamma \varepsilon \left[ \sup_{t \in A_j} \int_{\frac{1}{2} B_j \cap \{u > a_j\}} \frac{\lambda N}{1-\lambda} \left( \frac{u-a_j}{a-a_j} \right) dx \right]^{\frac{2}{\lambda}}
\]

\[
\leq \gamma \varepsilon \left[ \sup_{t \in A_j} \int_{\frac{1}{2} B_j \cap \{u > a_j\}} \frac{u-a_j}{a-a_j} dx \right]^{\frac{2}{\lambda}}
\]

for a constant \( \gamma = \gamma(N, m, C_m, C_1, \lambda) \). Thus, it remains to bound \( \Pi_3 \). However, such a bound immediately follows from the energy estimate from Proposition 3.2 and Remark 3.1 (recall also the definition of \( \theta \) in (4.12)):

\[
\Pi_3 = \frac{a_j^{m-1}}{r_j^{N+2}} \int_{\frac{1}{2} Q_j \cap \{u > a_j\}} \left| \frac{u-a_j}{a-a_j} \right|^{1+\lambda} dx dt
\]

\[
\leq \gamma \left[ \sup_{t \in A_j} \int_{\frac{1}{2} B_j \cap \{u > a_j\}} \frac{u-a_j}{a-a_j} dx \right]^{1+\lambda}
\]

\[
\leq \gamma \left( \frac{\kappa}{A^{m-1}} + \frac{1}{B} \right).
\]

Here we have also taken into account (4.26). Altogether, we have shown that also the term \( \Pi \) can be estimated by the right-hand side of the inequality (4.31). Inserting this into (4.28), we obtain that

\[
K_j(\bar{a}) \leq \gamma \varepsilon^{1+\lambda} + \gamma \varepsilon \left[ \frac{2\lambda}{2} \varepsilon + \frac{\varepsilon}{2} \left( \kappa + \theta^{-1} \right) \right]^{2\lambda} (\kappa + B^{-1})
\]
holds true with constants $\gamma = \gamma(N, m)$ and $\gamma_{\varepsilon} = \gamma_{\varepsilon}(N, m, C_0, C_1, \lambda, A, \varepsilon)$. Note that $\varepsilon, \varepsilon_1, \kappa \in (0, 1)$ and $B > 1$ are still at our disposal. We first choose $\varepsilon$ to satisfy $\gamma \varepsilon^{1+\lambda} = \frac{1}{6}$.

This fixes $\gamma_{\varepsilon}$ in dependence on $N, m, C_0, C_1, \lambda$ and $A$. Next, we choose $B$ so large that
\begin{equation}
B \geq \frac{1}{\kappa}.
\end{equation}

This yields
\[K_j(\bar{a}) \leq \frac{1}{6} \kappa + \gamma \left[ \varepsilon_1^{2\lambda} + \varepsilon_1^{-2\lambda} \kappa^{2\lambda} \right] \kappa\]

with $\gamma = \gamma(N, m, C_0, C_1, A, \lambda)$. Now, choosing $\varepsilon_1$ in dependence on $N, m, C_0, C_1, A, \lambda$, such that $\gamma \varepsilon_1^{2\lambda} = \frac{1}{6}$, we obtain
\[K_j(\bar{a}) \leq \frac{1}{6} \kappa + \gamma \kappa^{1+2\lambda},\]

where $\gamma = \gamma(N, m, C_0, C_1, A, \lambda)$. Finally, we choose $\kappa$ in dependence on $N, m, C_0, C_1, A, \lambda$ small enough to satisfy
\[\gamma \kappa^{2\lambda} \leq \frac{1}{6}.
\]

With this choice the preceding inequality for $K_j(\bar{a})$ yields that
\[K_j(\bar{a}) \leq \frac{1}{2} \kappa.\]

Now, we recall from (4.26) that $\bar{a} > a_j + \frac{1}{4} B (\alpha_{j-1} - \alpha_j)$. Therefore, due to the construction of $a_{j+1}$ we may conclude that $a_j + \frac{1}{4} B (\alpha_{j-1} - \alpha_j) \leq a_{j+1} < \bar{a}$. This proves the first bound in (4.23). For the second bound we use the fact that $a_j < \mu^+$, which is a consequence of the second inequality in (4.23), to conclude that
\[a_{j+1} < \bar{a} = \frac{1}{2} (\mu^+ + a_j) - \frac{1}{4} B \alpha_j < \mu^+ - \frac{1}{4} B \alpha_j.
\]

This proves (4.23) for $j + 1$. Hence, we have proved the claim that (4.23) holds true for any $j \in \mathbb{N}$.

**Step 3: Improved iterative bound for $a_j$.** Here, we define
\[d_j := a_{j+1} - a_j\]

and prove that there exists a constant $\gamma$ depending only on $N, m, C_0, C_1, \lambda$ and $A$ such that
\begin{equation}
(4.33)
d_j \leq \frac{1}{2} d_{j-1} + \gamma \frac{\mu(2Q_j)}{r_{jN}^\kappa}
\end{equation}

holds true for any $j \in \mathbb{N}$.

The proof is similar to the one in Step 2 and therefore, we only sketch it. We fix $j \geq 1$.

Without loss of generality, we can assume that
\[d_j \geq \frac{1}{2} d_{j-1}, \quad d_j > \frac{1}{4} (\alpha_{j-1} - \alpha_j),\]

because otherwise, there is nothing to prove; cf. (4.25) for the case that $d_j \leq \frac{1}{4} (\alpha_{j-1} - \alpha_j)$. By the construction of $a_{j+1}$, the second inequality ensures that we have $K_j(a_{j+1}) = \kappa$.

We now work as in the proof of Step 2, but instead of estimating the term involving the measure as in (4.30), we keep the measure and replace $\bar{a} - a_j$ in the denominator by $d_j = a_{j+1} - a_j$. In this way we obtain
\[\kappa \leq (\gamma \varepsilon^{1+\lambda} + \gamma \varepsilon_1^{2\lambda} + \gamma \varepsilon_1^{-2\lambda} \kappa^{2\lambda}) \kappa + \gamma \varepsilon_1^{-2\lambda} \left[ \frac{\mu(2Q_j)}{d_j r_{jN}^\kappa} + \left( \frac{\mu(2Q_j)}{d_j r_{jN}^\kappa} \right)^{1+2\lambda} \right].\]
With the same choices for $\kappa$, $\varepsilon$, $\varepsilon_1$ as in the proof of Step 2, and the argument from the end of [1, Chapter 4.3], we conclude that

$$d_j \leq \gamma \frac{\mu(2Q_j)}{r_j^N}$$

with a constant $\gamma = \gamma(N, m, C_0, C_1, \lambda, A)$. This proves the claim.

**Step 4: Quantitative bound for $u$.** We let $J > 1$ and sum up (4.33) for $j = 1, \ldots, J - 1$. Taking into account the definition of $d_j$, and the fact that the sequence $\{a_j\}$ is a monotone increasing sequence, we deduce that

$$a_J - a_1 = \sum_{j=1}^{J-1} (a_{j+1} - a_j)$$

$$\leq \frac{1}{2} \sum_{j=1}^{J-1} (a_j - a_{j-1}) + \gamma \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}$$

$$= \frac{1}{2} (a_J - a_1) + \frac{1}{2} (a_1 - a_o) + \gamma \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}$$

$$\leq \frac{1}{2} (a_J - a_1) + \frac{1}{2} (a_1 - a_o) + \gamma \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}.$$

From this we easily obtain that

$$a_J \leq 2a_1 - a_o + \gamma \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N}.$$  \hspace{1cm} (4.34)

From the construction of $a_1$ we have two alternatives. Either

$$a_1 = a_o + \frac{1}{4} (\alpha_1 - \alpha_o),$$

or

$$a_1 > a_o + \frac{1}{4} (\alpha_1 - \alpha_o).$$

In the former case, recalling that $a_o = \mu^+ - \xi \omega$, we have

$$a_J \leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=1}^{J-1} \frac{\mu(2Q_j)}{r_j^N},$$

from which we conclude, using also (4.25)$_2$ for $j = 0$ and the definition of $Q_j$, that there holds

$$a_J \leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=0}^{J-1} \frac{\mu(2Q_{j-1})}{r_j^N}$$

$$\leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=0}^{J-1} \frac{\mu(2Q_{j-1})}{r_j^N}$$

$$\leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=0}^{\infty} \frac{\mu(Q_{j-1})}{r_j^N r_{j-1}^{N-2}}$$

$$\leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=0}^{\infty} \frac{\mu(Q_{j-1})}{r_j^N r_{j-1}^{N-2}}$$

$$\leq \mu^+ - \xi \omega + \frac{1}{2} (\alpha_1 - \alpha_o) + \gamma \sum_{j=0}^{\infty} \frac{\mu(Q_{j-1})}{r_j^N r_{j-1}^{N-2}}.$$
\[ \kappa = \frac{1}{r_o^{N+2}} \int \int_{\frac{1}{2}Q_o \cap \{u > a_o\}} u^{m-1} \left( \frac{u - a_o}{a_1 - a_o} \right)^{1+\lambda} dx dt \]
\[ \leq \alpha(N)2^{N+2} \left( \frac{13}{12} \right)^{m-1} \omega^{m-1} \left( \frac{\mu^+ - (\mu^+ - \xi \omega)}{a_1 - a_o} \right)^{1+\lambda} \nu_+ \left( \frac{\omega}{A} \right)^{1-m}, \]

implying the inequality
\[ a_1 - a_o \leq \gamma \xi \omega \left( \frac{\nu_+}{\kappa A^{1-m}} \right)^{\frac{1}{1+\lambda}} \leq \gamma \xi \omega (\nu_+)^{\frac{1}{1+\lambda}}, \]

where \( \gamma = \gamma(N, m, \sigma, \lambda, A) \). Here, we note that \( \kappa \) has already been fixed in dependence on \( N, m, \sigma, \lambda, A \). We substitute this inequality back into (4.34) and estimate \( J \sum_{j=1}^{J-1} \frac{\mu(\alpha_{a_j})}{\gamma j} \leq \gamma B^{-1} \xi \omega \), as before. In this way, we obtain
\[ a_j \leq \mu^+ - \xi \omega + \gamma \xi \omega (\nu_+)^{\frac{1}{1+\lambda}} + B^{-1} \xi \omega, \]

for a constant \( \gamma = \gamma(N, m, \sigma, \lambda, A) \). Combining the two alternatives we obtain that the preceding inequality holds true in any case for any \( J \in \mathbb{N} \). Since \( \{a_i\} \) is a monotone increasing sequence, the previous bound implies that the limit \( \lim_{j \to \infty} a_j = a_\infty \) exists, is finite, that also \( \lim_{j \to \infty} d_j = \lim_{j \to \infty} (a_j + 1 - a_j) = 0 \), and
\[ a_\infty \leq \mu^+ - \xi \omega \left[ 1 - \gamma (\nu_+)^{\frac{1}{1+\lambda}} - \gamma B^{-1} \right]. \]

Since \( a_\infty \geq a_o > 0 \) and \( Q_j \downarrow \{z_1\} \) we can conclude that
\[ \left( \frac{u(z_1)}{a_\infty} \right)^{m-1} (u(z_1) - a_\infty)^{1+\lambda} \]
\[ = \lim_{j \to \infty} \int \int_{Q_j} \left( \frac{u}{a_j} \right)^{m-1} (u - a_j)^{1+\lambda} dx dt \]
\[ = \lim_{j \to \infty} \frac{d_j^{1+\lambda}}{\alpha(N)12a_j^{m-1}} \int \int_{Q_j} u^{m-1} \left( \frac{u - a_j}{d_j} \right)^{1+\lambda} dx dt \]
\[ \leq \frac{1}{\alpha(N)\theta a_\infty^{m-1}} \lim_{j \to \infty} d_j^{1+\lambda} \int \int_{Q_j} u^{m-1} \left( \frac{u - a_j}{d_j} \right)^{1+\lambda} dx dt \]
\[ \leq \frac{1}{\alpha(N)\theta a_\infty^{m-1}} \lim_{j \to \infty} d_j^{1+\lambda} = 0. \]

Therefore, we conclude that \( u(z_1) \leq a_\infty \), and by the previous bound on \( a_\infty \) we obtain
\[ u(z_1) \leq \mu^+ - \xi \omega \left[ 1 - \gamma (\nu_+)^{\frac{1}{1+\lambda}} - \gamma B^{-1} \right]. \]

Now, we choose \( B \) large enough such that
\[ (4.35) \quad B \geq (\nu_+)^{-\frac{1}{1+\lambda}}, \]
With this choice of $B$ we have
\[ u(z_1) \leq \mu^+ - \xi \omega \left[ 1 - \gamma (\nu_+) \frac{1}{1+\lambda} \right], \]
for a constant $\gamma = \gamma (N, m, C_0, C_1, \lambda, A)$. Finally, we choose $\nu_+$ such that
\[ 1 - \gamma (\nu_+) \frac{1}{1+\lambda} \geq a \iff \nu_+ \leq \left( \frac{1 - a}{\gamma} \right)^{1+\lambda}. \]
This fixes $\nu_+$ in dependence on $N, m, C_0, C_1, \lambda, A, a$. Inserting this choice of $\nu_+$ above, we finally arrive at
\[ u(z_1) \leq \mu^+ - a \xi \omega, \]
which proves the claim (4.20).

Finally, a few remarks concerning the dependencies of the constants are in order. First, for the constant $\nu_+$ we imposed the smallness conditions (4.24) and (4.36), i.e.
\[ \nu_+ \leq \frac{\kappa}{2\gamma} \quad \text{and} \quad \nu_+ \leq \left( \frac{1 - a}{\gamma} \right)^{1+\lambda}. \]
Since $\kappa$ and $\gamma$ both depend on $N, m, C_0, C_1, \lambda, A$, we can choose $\nu_+$ of the form $\left( \frac{1 - a}{\gamma} \right)^{1+\lambda}$, with a constant $\gamma \geq 1$ depending on $N, m, C_0, C_1, \lambda, A$, as stated in Remark 4.2. Second, for the constant $B$ we required in (4.32) and (4.35) that
\[ B \geq \frac{1}{\kappa} \quad \text{and} \quad B \geq \left( \frac{1}{\nu_+} \right)^{1+\lambda}. \]
This means, we get the functional dependence of $B$ in the form $\frac{1}{\nu_+}$, with a constant $\gamma \geq 1$ depending on $N, m, C_0, C_1, \lambda, A$, as stated in Remark 4.2. \(\square\)

With respect to the notation of Lemma 4.1, assume
\[ r = \frac{1}{2} \theta, \quad a = \frac{1}{2}, \quad \xi = \frac{1}{2}, \quad A = 1, \quad \tilde{\theta} = \theta = \omega^{1-m}, \]
and let $\nu_*$ be the corresponding value of $\nu_-$ given by (4.13), i.e.
\[ \nu_* := \nu_-(\xi = \frac{1}{2}, a = \frac{1}{2}, A = 1). \]
Due to these choices, $\nu_*$ is now a quantity that depends only on $N, m, C_0,$ and $C_1$. Furthermore, taking into account the definition of $\mu^+$ and $\omega$, from here on we assume that
\[ \frac{1}{2} \omega \leq \mu^+ - \frac{1}{4} \omega \leq \frac{3}{4} \omega. \]
Note that this coincides with (4.14), which then holds. The left-hand inequality can be taken as holding in all cases; the case when the right-hand inequality fails to hold will be examined later. We have two alternatives, which we now discuss separately.

4.2. The first alternative. If
\[ |\{ u \leq \frac{1}{2} \omega \} \cap Q_{\theta, \theta^2} | \leq \nu_* |Q_{\theta, \theta^2}| \]
then, by Lemma 4.1 we have
\[ u \geq \frac{1}{4} \omega \quad \text{a.e. in } Q_{\frac{1}{2} \theta, \frac{1}{2} \theta^2}, \]
which implies
\[ - \inf_{Q_{\frac{1}{2} \theta, \frac{1}{2} \theta^2}} u \leq - \frac{1}{4} \omega. \]
Adding the essential supremum of $u$ over $Q_{\frac{1}{2} \theta, \frac{1}{2} \theta^2}$ on the left-hand side and $\mu^+$ on the right-hand side, by (4.37) we conclude
\[ \text{osc}_{Q_{\frac{1}{2} \theta, \frac{1}{2} \theta^2}} u \leq \mu^+ - \frac{1}{4} \omega \leq \frac{5}{8} \omega. \]
At this stage we recall that we used the short-hand notation introduced at the beginning of § 4.1, in particular that \( \varrho = \varrho_n, \theta = \theta_n \) and \( \omega = \omega_n \). We have thus proved that

\[
\text{osc}_{Q_{\frac{1}{2}\varrho_n,\theta_n}(\frac{1}{2}\varrho_n)^2} u \leq \frac{5}{6}\omega_n.
\]

This finishes the induction step in the case of the first alternative.

4.3. The second alternative. If (4.38) does not hold true, then

\[
\left| \left\{ u \leq \frac{1}{2}\omega \right\} \cap Q_{\varrho,\theta\varrho^2} \right| > \nu_* |Q_{\varrho,\theta\varrho^2}|.
\]

which we rewrite as

\[
\left| \left\{ u > \frac{1}{4}\omega \right\} \cap Q_{\varrho,\theta\varrho^2} \right| \leq (1 - \frac{1}{2}\nu_* |B_{\varrho}|.
\]

In the following, we will examine the consequences of (4.40). Due to (4.37), (4.40) yields

(4.41)

\[
\left| \left\{ u > \mu + \frac{1}{4}\omega \right\} \cap Q_{\varrho,\theta\varrho^2} \right| \leq (1 - \frac{1}{2}\nu_*) |B_{\varrho}|.
\]

We have the following

**Lemma 4.3.** There exists a time level \( \bar{s} \in [-\theta\varrho^2, -\frac{1}{2}\nu_* \theta\varrho^2] \), such that

(4.42)

\[
\left| \left\{ u(\cdot, \bar{s}) > \mu + \frac{1}{4}\omega \right\} \cap B_{\varrho} \right| \geq \frac{1 - \nu_*}{1 - \frac{1}{2}\nu_*} |B_{\varrho}|.
\]

**Proof.** If (4.42) were not to hold for some \( s \) in the indicated range, then

\[
\left| \left\{ u(\cdot, s) > \mu + \frac{1}{4}\omega \right\} \cap B_{\varrho} \right| > \frac{1 - \nu_*}{1 - \frac{1}{2}\nu_*} |B_{\varrho}| \quad \forall s \in [-\theta\varrho^2, -\frac{1}{2}\nu_* \theta\varrho^2],
\]

and therefore,

\[
\left| \left\{ u > \mu + \frac{1}{4}\omega \right\} \cap Q_{\varrho,\theta\varrho^2} \right| \geq \int_{-\theta\varrho^2}^{-\frac{1}{2}\nu_* \theta\varrho^2} \left| \left\{ u(\cdot, s) > \mu + \frac{1}{4}\omega \right\} \cap B_{\varrho} \right| ds
\]

\[
> \theta\varrho^2 \left( 1 - \frac{1}{2}\nu_* \right) |B_{\varrho}| = (1 - \nu_*) |Q_{\varrho,\theta\varrho^2}|,
\]

which contradicts (4.41). \( \square \)

In the following, whenever there is no risk of confusion, for simplicity, we will omit the reference point of the potential and we will write \( I_{\mu^2}(r, \theta) \) instead of \( I_{\mu^2}(z_o, r, \theta) \). It is a straightforward consequence of the definition, that

(4.43)

\[
I_{\mu^2}(2r, 4\theta r^2) \geq \frac{\mu(Q_{r, \theta r^2})}{r^N}.
\]

**Lemma 4.4.** There exists a positive integer \( s_1 \geq 4 \) depending only on \( N, m, C_o, C_1 \) such that either

(4.44)

\[
\omega < 2^{s_1} I_{\mu^2}(2\varrho, 4\theta \varrho^2),
\]

or

(4.45)

\[
\left| \left\{ u(\cdot, t) > \mu + \frac{\omega}{2s_1} \right\} \cap B_{\varrho} \right| \leq (1 - \frac{1}{4}\nu_*^2) |B_{\varrho}|
\]

holds true for all \( t \in [\bar{s}, 0] \).
Proof. We let \( k = \mu^+ - \frac{1}{4} \omega \) and define
\[
H^+_k := \sup_{B_\varrho \times [\overline{s},0]} (u - k)_+ \leq \frac{1}{4} \omega.
\]
Furthermore, we let \( c = \frac{\omega}{2^\ell + 2} \) for some integer \( \ell \in \mathbb{N} \), with \( \ell \geq 2 \). In order to apply Proposition 3.3, we must have \( 0 < c < H_k \). Therefore, the integer \( \ell \) must later on be chosen large enough in a universal way; then \( s_1 \) will be \( \ell + 2 \). We will apply Proposition 3.3, on the cylinder \( B_\varrho \times [\overline{s},0] \), and with the logarithmic function \( \psi \circ u = \psi(a,b,c) \circ u \) for the choice \((a, b, c) = (H_k, k, \frac{\omega}{2^\ell + 2})\); i.e. we consider
\[
\psi(u)(x, t) := (\psi(H_k, k, \frac{\omega}{2^\ell + 2}) \circ u)(x, t) = \ln_{+}\left[ \frac{H^+_k}{H^+_k - (u(x, t) - k)_+ + \frac{\omega}{2^\ell + 2}} \right].
\]
The cutoff function \( x \mapsto \zeta(x) \in [0, 1] \) is taken to be 1 in the ball \( B_{(1 - \sigma)\varrho} \), where \( \sigma \in (0, 1) \) has to be chosen, vanishing on the boundary of \( B_\varrho \), and such that \( |D\zeta| \leq \frac{1}{\sigma \varrho} \). With these choices, we have
\[
\int_{B_{(1 - \sigma)\varrho} \times \{t\}} \psi^2(u) \, dx \leq \int_{B_\varrho \times [\overline{s},0]} \psi^2(u) \, dx + \frac{\gamma}{\sigma^2 \varrho^2} \int_{B_\varrho \times [\overline{s},0]} u^{m-1} \psi(u) \, dx \, dt
\]
\[
\quad + \frac{2^{\ell+3}}{\omega} \ln \left( \frac{2^{\ell+2} H^+_k}{\omega} \right) \int_{B_\varrho \times [\overline{s},0]} \chi_{\{u > k\}} \, d\mu,
\]
for all \( t \in [\overline{s},0] \). We now proceed to estimate all the terms separately. By their definition, and the fact that \( H^+_k \leq \frac{1}{4} \omega \), it is immediate to see that
\[
\psi(u) \leq \ln \left( \frac{H^+_k + \frac{\omega}{2^\ell + 2}}{\omega} \right) \leq \ln 2^\ell = \ell \ln 2.
\]  
For the first integrals on the right-hand side we use the preceding inequality and the fact that the logarithmic function \( \psi(u) \) vanishes whenever \( u \leq \mu^+ - \frac{1}{4} \omega + 2^{-(\ell+2)} \omega \). This leads to the following estimate of \( I_1 \):
\[
I_1 \leq \ell^2 \ln^2 2 \left\{ u(\cdot, \overline{s}) > \mu^+ - \frac{1}{4} \omega + \frac{\omega}{2^\ell + 2} \right\} \cap B_\varrho
\]
\[
\leq \ell^2 \ln^2 2 \left\{ u(\cdot, \overline{s}) > \mu^+ - \frac{1}{4} \omega \right\} \cap B_\varrho
\]
\[
\leq \ell^2 \ln^2 2 \frac{1 - \nu_*}{1 - \frac{1}{\nu_*}} |B_\varrho|,
\]
where in the last line we used Lemma 4.3. Next, we estimate the integral \( I_2 \). Here, we use again the bound for \( \psi(u) \) from above, the second inequality in (4.37) (more precisely that \( u \leq \mu^+ \leq \frac{13}{12} \omega \)), and finally \( \overline{s} \leq \theta \varrho^2 = \omega^{1-m} \varrho^2 \). This procedure yields the estimate
\[
I_2 \leq \frac{\gamma \ell \ln 2}{\sigma^2 \varrho^2} \left( \frac{13}{12} \omega \right)^{m-1} \omega^{1-m} \varrho^2 |B_\varrho| \leq \frac{\gamma \ell \ln 2}{\sigma^2} |B_\varrho|,
\]
for a constant \( \gamma = \gamma(N, m, C_\sigma, C_1) \). Finally, we come to the estimate of the integral \( I_3 \). Using the second inequality of (4.47) we obtain
\[
I_3 \leq \frac{2^{\ell+3} \ell \ln 2}{\omega} \mu(Q_\varrho, \theta \varrho^2) \leq \frac{2^{\ell+3} \ell \ln 2}{\alpha(N) \omega} \, I_2^{\prime}(2\varrho, 4\theta \varrho^2) |B_\varrho|,
\]
where we have used (4.43) for the last estimate. Assume for the moment that \( \ell \) has been chosen, and that
\[
\frac{2^\ell}{\omega} \mathbf{1}_s^\mu(2\varrho, 4\theta \varrho^2) < 1.
\]
Otherwise, (4.44) holds trivially for \( s_1 = \ell + 2 \). Then, we have \( I_3 \leq \gamma(N)\ell |B_\varrho| \). Joining the estimates for \( I_1 - I_3 \) with (4.46) we arrive at
\[
\int_{B_{(1-\sigma)\varrho} \times \{t\}} \psi^2(u) \, dx \leq \ell^2 \ln^2 2 \frac{1 - \nu_*}{1 - 2\nu_*} |B_\varrho| + \frac{\gamma \ell}{\sigma^2} |B_\varrho|,
\]
for a constant \( \gamma = \gamma(N, m, C_\alpha, C_1) \). Now we estimate the left-hand side in (4.48) from below, integrating over the smaller set \( B_{(1-\sigma)\varrho} \cap \{u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}}\} \). Since \( \psi(u)(x, t) \) is a decreasing function of \( H_{\rho}^\nu \) for any fixed \( x \) in such a set, we can estimate
\[
\psi(u)^2 \geq \ln^2 2 \frac{\sigma}{2^{r+1}} = (\ell - 1)^2 \ln^2 2.
\]
Inserting (4.49) into (4.48), and dividing through by \((\ell - 1)^2 \ln^2 2\), we obtain
\[
\left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}} \right\} \cap B_{(1-\sigma)\varrho} \right| \leq \left( \frac{\ell}{\ell - 1} \right)^2 \frac{1 - \nu_*}{1 - 2\nu_*} |B_\varrho| + \frac{\gamma \ell}{\sigma^2(\ell - 1)^2} |B_\varrho|.
\]
On the other hand, we have
\[
\left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}} \right\} \cap B_\varrho \right|
\leq \left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}} \right\} \cap B_{(1-\sigma)\varrho} \right| + \left| B_\varrho \setminus B_{(1-\sigma)\varrho} \right|
\leq \left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}} \right\} \cap B_{(1-\sigma)\varrho} \right| + N \alpha(N)\sigma |B_\varrho|.
\]
Therefore, we conclude that
\[
\left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^{r+2}} \right\} \cap B_\varrho \right| \leq \left( \frac{\ell}{\ell - 1} \right)^2 \frac{1 - \nu_*}{1 - 2\nu_*} |B_\varrho| + \frac{\gamma \ell}{\sigma^2(\ell - 1)^2} |B_\varrho| + \gamma \sigma |B_\varrho|,
\]
holds true for all \( t \in [\bar{s}, 0] \). Now choose \( \sigma \) small enough, such that \( \gamma(N)\sigma \leq \frac{3}{8} \nu_*^2 \), and then \( \ell \) so large that
\[
\left( \frac{\ell}{\ell - 1} \right)^2 < (1 - \frac{1}{2}\nu_*) (1 + \nu_*) \quad \text{and} \quad \frac{\gamma \ell}{\sigma^2(\ell - 1)^2} \leq \frac{3}{8} \nu_*^2.
\]
Now, the claim follows with \( s_1 := \ell + 2 \). Since \( \nu_* \) depends on \( N, m, C_\alpha, C_1 \), then also \( s_1 \) depends on these quantities. In particular, \( s_1 \) is independent of \( \omega, \varrho, \bar{s} \).

**Corollary 4.1.** Under the assumptions of Lemma 4.4, either
\[
\omega < 2^{s_1} \mathbf{1}_s^\mu(2\varrho, 4\theta \varrho^2),
\]
or
\[
\left| \left\{ u(\cdot, t) > \mu^+ - \frac{\omega}{2^j} \right\} \cap B_\varrho \right| \leq \left( 1 - \frac{1}{4} \nu_*^2 \right) |B_\varrho|
\]
holds true for all \( j \geq s_1 \), and for all \( t \in [-\frac{1}{2}\nu_* \varrho \varrho^2, 0] \).

We now let \( \theta_* := \frac{1}{2} \nu_* \theta \) and work in the cylinder \( Q_{\varrho, \theta_*} = B_\varrho \times (\frac{1}{2} \nu_* \theta \varrho^2, 0) \). We have the following.
Lemma 4.5. For every \( \bar{\nu} \in (0,1) \), there exists a positive integer \( q_* \), depending only on \( m, N, C_0, C_1, \) and \( \bar{\nu} \), such that either

\[
\omega < 2^{s_1+q_*} I^m_2(4\theta, 16\theta^2),
\]
or

\[
\left\{ u > \mu^+ - \frac{\omega}{2^{s_1+q_*}} \right\} \cap Q_{\theta,\bar{\nu}^*} < \bar{\nu} |Q_{\theta,\bar{\nu}}|.
\]

Proof. Without loss of generality we can assume that the first alternative of Corollary 4.1 is violated, i.e. that \( \omega \geq 2^{s_1} I^m_2(2\theta, 4\theta^2) \). Otherwise, the first alternative of the claim trivially holds for any choice of \( q_* \in \mathbb{N} \). Now, for \( q_* \) to be chosen we consider the energy estimate (3.2) for the truncated function \((u - k_j)+\) with levels

\[
k_j = \mu^+ - \frac{\omega}{2^j} \quad \text{for} \quad j = s_1, s_1 + 1, \ldots, s_1 + q_*,
\]
over the pair of cylinders

\[
Q = Q_{\theta,\bar{\nu}^*} = B_\theta \times (-\frac{1}{2} \nu_* \theta \bar{\nu}^2, 0], \quad Q' = B_{2\theta} \times (-\nu_* \theta \bar{\nu}^2, 0].
\]
The cutoff function \((x,t) \mapsto \zeta(x,t) \in [0,1]\) is taken to be 1 in \( Q \), vanishing on the parabolic boundary of \( Q' \), and such that \( |D\zeta| \leq \frac{2}{\bar{\nu}} \), and \( 0 \leq \zeta_t \leq \frac{2}{\nu_* \theta \bar{\nu}^2} \). With these stipulations, the energy estimate (3.2) takes the form

\[
C_o \left( \frac{\omega}{2} \right)^{m-1} \int_Q |D(u - k_j)+|^2 \, dx \, dt \leq \gamma \left[ \frac{\omega^{m-1}}{\nu_* \bar{\nu}^2} \int_{Q'} (u - k_j)^2 \, dx \, dt + \int_{Q'} (u - k_j)+ \, d\mu \right].
\]

Here, we used in turn the fact that \( u \leq \mu^+ \leq \frac{13}{4} \omega < 2\omega \) (see (4.37)) in order to estimate the function \( u^{m-1} \) from above, the definition \( \theta = \omega^{1-m} \), and the fact that in the left-hand side integral we only have to integrate over the set of those points in \( Q \) where \( u > k_j \).

Since \( k_j \geq \mu^+ - \frac{1}{2} \omega \geq \frac{1}{2} \omega \) by the first inequality of (4.37), the estimate from below follows. The right-hand side is now easily estimated since \( (u - k_j)+ \leq \frac{\omega}{2^j} \). Therefore, we obtain

\[
C_o \left( \frac{\omega}{2} \right)^{m-1} \int_Q |D(u - k_j)+|^2 \, dx \, dt \leq \gamma \left[ \frac{\omega^{m-1}}{\nu_* \bar{\nu}^2} \left( \frac{2^j}{2} \right)^2 |Q'| + \frac{\omega}{2^j} \mu(Q') \right].
\]

Assume for the moment that \( q_* \in \mathbb{N} \) has been already determined, and that

\[
\frac{2^{s_1+q_*}}{\omega} \frac{\mu(Q_{2\theta,4\theta^2})}{\bar{\nu}^N} \leq 1.
\]

Otherwise, by (4.43) the first alternative of the claim would follow. Hence, we obtain that

\[
\int_Q |D(u - k_j)+|^2 \, dx \, dt \leq \gamma \left( \frac{2^j}{2} \right)^2 |Q|.
\]

Next, we apply DeGiorgi’s isoperimetric inequality (see [7], Chapter 2, Lemma 2.2) for \( t \in (-\frac{1}{2} \nu_* \theta \bar{\nu}^2, 0) \) over the ball \( B_\theta \) for the levels \( k_j =: k < \ell := k_j+1 \). Taking into account
Corollary 4.1, we obtain for a.e. \( t \in (-\frac{1}{2} \nu_s \theta \rho^2, 0] \) (note that the second alternative must hold, due to the assumption from the beginning of the proof)

\[
\frac{\omega}{2^{j+1}} \left| \{ u(\cdot, t) > k_{j+1} \} \cap B_\rho \right|
\]

\[
\leq \frac{\gamma \theta^{N+1}}{2^{j+1} |B_\rho|} \int_{\{ u(\cdot, t) < k_j \} \cap B_\rho} |Du(\cdot, t)| \, dx
\]

\[
\leq \frac{4 \gamma \theta^{N+1}}{\nu^2 |B_\rho|} \int_{[k_j < u(\cdot, t) < k_{j+1}] \cap B_\rho} |Du(\cdot, t)| \, dx
\]

\[
= \frac{\gamma \theta}{\nu^2} \int_{[k_j < u(\cdot, t) < k_{j+1}] \cap B_\rho} |Du(\cdot, t)| \, dx.
\]

We define

\[ A_j := \{ u > k_j \} \cap Q, \]

and integrate the preceding inequality with respect to time over \((-\frac{1}{2} \nu_s \theta \rho^2, 0]\), and apply Cauchy-Schwartz’s inequality. This gives

\[
\frac{\omega}{2^{j+1}} |A_{j+1}| \leq \frac{\gamma \theta}{\nu^2} \left( \int_0^{\frac{1}{2} \nu_s |Q|} |Du|^2 \, dx \, dt \right) \frac{1}{2} |A_j \setminus A_{j+1}| \frac{1}{2}
\]

\[
\leq \frac{\gamma \theta}{\nu^2} \left( \int Q |Du|^2 \, dx \, dt \right) \frac{1}{2} |A_j \setminus A_{j+1}| \frac{1}{2}
\]

\[
\leq \frac{\gamma}{\nu^2 \nu_a} \frac{\omega}{2^{j+1}} |Q| \frac{1}{2} |A_j \setminus A_{j+1}| \frac{1}{2}.
\]

Here, we used (4.50) in the last line. From the preceding inequality we easily get

\[ |A_{j+1}|^2 \leq \frac{\gamma}{\nu^2} |Q| |A_j \setminus A_{j+1}|. \]

We add up these recursive inequalities for \( j = s_1, s_1 + 1, \ldots, s_1 + q_\ast - 1 \) and use the fact that the right-hand side forms a telescopic series. This procedure yields the measure bound

\[ (q_\ast - 1) |A_{s_1 + q_\ast}|^2 \leq \frac{\gamma}{\nu^2} |Q|^2, \]

from which easily follows

\[ |A_{s_1 + q_\ast}| \leq \sqrt{\frac{\gamma}{\nu^2 (q_\ast - 1)}} |Q|. \]

Now, for fixed \( \tilde{\nu} \in (0, 1) \) as in the statement of the Lemma, we choose \( q_\ast \in \mathbb{N} \) according to the requirement that

\[ (4.51) \quad \sqrt{\frac{\gamma}{\nu^2 (q_\ast - 1)}} \leq \tilde{\nu}. \]

With this choice of \( q_\ast \), the claim follows. \( \square \)

Now, referring to the notation of Lemma 4.2, we let

\[ r = \frac{1}{2} \theta, \quad a = \frac{1}{2}, \quad \xi = 2^{-(s_1 + q_\ast)}, \quad A = (\frac{1}{2} \nu_a)^{\frac{1}{2s_1}}, \quad \tilde{\theta} = \frac{1}{2} \nu_s \theta. \]

Applying Lemma 4.2, and taking into account the first alternative of Lemma 4.5, we conclude that either

\[ (4.52) \quad \omega < C |\int_{2Q, 2^{s_1} \theta \rho^2} (4\theta \rho^2), \quad C := \max\{ B, 2^{s_1 + q_\ast} \} \]
where $B$ is the constant in (4.16), or
\begin{equation}
(4.53) \quad u \leq \mu^+ - \frac{\omega}{2s+q_*+1} \quad \text{a.e. in } B_{\frac{1}{2} \varrho} \times \left(-\frac{1}{2} \nu_* \theta \left(\frac{1}{2} \varrho\right)^2, 0\right],
\end{equation}
provided $\varrho$ is chosen equal to $\nu_*$ as defined in (4.18), of course with the choices $\alpha = \frac{1}{2}$ and $A = \left(\frac{1}{2} \nu_*\right)^{\frac{1}{2+q_*}}$, and then in turn the integer $q_*$ is chosen according to (4.51). Then, $C$ depends only on $m, N, C_\varrho, C_1$. The estimate (4.53) can be rewritten as
\[
\sup_{Q_{\frac{1}{2} \varrho, \theta} \subset B_{\frac{1}{2} \varrho}} \text{osc}\ u \leq \mu^+ - \frac{\omega}{2s+q_*+1}.
\]

But this implies that either (4.52) holds true, or
\[
\text{osc}\ u = \sup_{Q_{\frac{1}{2} \varrho, \theta} \subset B_{\frac{1}{2} \varrho}} u - \inf_{Q_{\frac{1}{2} \varrho, \theta} \subset B_{\frac{1}{2} \varrho}} u \leq \mu^+ - \frac{\omega}{2s+q_*+1} - \mu^-
\]
\[
\leq \left(1 - \frac{1}{2s+q_*+1}\right) \omega.
\]

Now, recalling the abbreviations we introduced at the beginning of § 4.1, in particular that $\varrho = \varrho_n$, $\omega = \omega_n$ and $\theta = \theta_n = \omega_n^{1-m}$, we have thus proved that either
\[
\omega_n \leq C \frac{m}{4} \sup_{Q_{\varrho_n} \subset B_{\varrho_n}} \left(4 \varrho_n, 16 \theta_n \varrho_n^2\right),
\]
or
\[
\text{osc}_{Q_{\varrho_n} \subset B_{\varrho_n}} u \leq \left(1 - \frac{1}{2s+q_*+1}\right) \omega_n
\]
holds true. Since $\text{osc}_{Q_{\varrho_n} \subset B_{\varrho_n}} u \leq \omega_n$ by our induction assumption, we conclude that in any case there holds
\begin{equation}
(4.54) \quad \text{osc}_{Q_{\varrho_n} \subset B_{\varrho_n}} u \leq \max \left\{\left(1 - \frac{1}{2s+q_*+1}\right) \omega_n, C \frac{m}{4} \sup_{Q_{\varrho_n} \subset B_{\varrho_n}} \left(4 \varrho_n, 16 \theta_n \varrho_n^2\right)\right\}.
\end{equation}

4.4. Pasting the two alternatives together. Recalling (4.1), we define
\[
\delta := 1 - \frac{1}{2s+q_*+1}, \quad \omega_{n+1} := \max \left\{\delta \omega_n, C \frac{m}{4} \sup_{Q_{\varrho_n} \subset B_{\varrho_n}} \left(4 \varrho_n, 16 \theta_n \varrho_n^2\right)\right\}.
\]

Moreover, in view of the estimates for the oscillation of $u$ from the first alternative in (4.39) and from the second alternative in (4.54), we need to define $\eta \in (0, 1)$, in such a way that
\[
Q_{n+1} = B_{\theta \varrho_n} \times \left(-\omega_{n+1}^{1-m}(\eta \varrho_n)^2, 0\right] \subset B_{\frac{1}{2} \varrho_n} \times \left(-\frac{1}{2} \nu_* \omega_{n+1}^{1-m}(\frac{1}{2} \varrho_n)^2, 0\right].
\]

Since $m \geq 1$, by its definition, we always have $\omega_{n+1}^{1-m} \leq (\delta \omega_n)^{1-m}$. Therefore, the requirement is satisfied, if we set
\[
\eta := \sqrt{\frac{1}{2} \nu_* \delta^{m-1}} < \frac{1}{2}.
\]

With these choices for $\delta$ and $\eta$, we can paste together the first alternative (4.39) (note that $\delta \geq \frac{3}{2}$) and the second alternative (4.54) to conclude that
\[
\text{osc}_{Q_{n+1}} u \leq \omega_{n+1}
\]
holds true. The induction argument is now completed, provided the assumption (4.37) is satisfied.
4.5. **The Proof of Proposition 4.1 concluded.** If (4.37) does not hold for some index $n$, then we have

$$\mu_n^+ > \frac{13}{12} \omega_n,$$

which implies that $\mu_n^- \geq \frac{1}{12} \omega_n$. However, this inequality implies that $u$ is uniformly bounded away from zero in $Q_n$, and therefore equation (1.2) under the structure condition (1.3) is non-degenerate in $Q_n$, and behaves like a quasilinear parabolic equation with growth of order 2, with a measure data right-hand side, as considered, for example, in [8, 9]. By these results, $u$ is continuous in $Q_n$.

We briefly outline, how to make this quantitative, following the same approach used in [7], Appendix B, § 13. Assume first that (4.37) fails to hold for $n = 0$. Then, with $\mu_0^+$ and $\omega_o$ defined by (4.10), modify the construction of $Q_o$ in (4.11) as follows: Instead of $Q_o$ we consider the smaller cylinder

$$Q_{\theta^o, \theta^o, e^o_2} = B_{e^o} \times (-\theta^o e^o_2, 0] \subset Q_o$$

where $\theta^o := \left(\frac{12}{13} \mu_o^+ \right)^{\frac{1}{m-1}}$.

Next, we introduce the change of variables by letting

$$\Phi(x, s) := (x, \theta^o s).$$

Then $\Phi$ maps $Q_{\theta^o, \theta^o, e^o_2}$ into $Q_{\theta^o, \theta^o, e^o_2}$. Then, we scale $u$ down to the new cylinder by letting

$$v(x, s) := \frac{u(x, \theta^o s)}{\mu_o^+}$$

for $(x, s) \in Q_{\theta^o, \theta^o, e^o_2}$.

The vector-field $A$ and the measure $\mu$ are also transformed by defining

$$\tilde{A}(x, s, \xi) := \frac{\theta^*}{\mu_o^+} A(x, \theta^* s, u(x, \theta^* s), \mu_o^+ \xi), \quad \text{and} \quad \tilde{\mu} := \frac{\theta^*}{\mu_o^+} \Phi^* \mu,$$

where $\Phi^* \mu$ denotes the pull-back of the measure $\mu$. Now, it is straightforward to check that $v$ satisfies

$$v_t - \text{div} \tilde{A}(x, s, Dv) = \tilde{\mu} \quad \text{weakly in } Q_{\theta^o, \theta^o, e^o_2}.$$

Moreover, the transformed vector-field $\tilde{A} : Q_{\theta^o, \theta^o, e^o_2} \times \mathbb{R}^N \to \mathbb{R}^N$ fulfills for a.e $(x, t) \in Q_{\theta^o, \theta^o, e^o_2}$ and every $\xi \in \mathbb{R}^N$ the monotonicity and boundedness condition

$$\begin{align*}
\left\{ \begin{array}{l}
\tilde{A}(x, s, \xi) \cdot \xi \geq \left(\frac{1}{12}\right)^{m-1} m C_o |\xi|^2, \\
|\tilde{A}(x, s, \xi)| \leq \left(\frac{13}{12}\right)^{m-1} m C_1 |\xi|,
\end{array} \right.
\end{align*}$$

for the same structural constants $m \geq 1, C_o, C_1$ as in (1.3). Therefore, $v$ is a weak solution of a non-singular parabolic equation in the parabolic cylinder $Q_{\theta^o, \theta^o, e^o_2}$. By the classical theory, there exist $\delta_o \in (0, 1)$ and $\gamma > 1$, that can be determined a priori only in terms of the data $N, m, C_o, C_1$, and which are independent of $\theta^o$ and $\mu_o^+$, and a sequence of radii $\rho_t = 4^{-t} \rho_o$, such that

$$\text{osc}_{Q_{\rho_{t+1}} v} \leq \delta_o \text{osc}_{Q_{\rho_t}} v + \frac{\gamma}{(\mu_o^+)^m} \|u\|_{L^1(Q_{\rho_{t+1}}, \rho_{t+1})} \left(4\rho_t, 16\rho_t^2\right).$$

Returning to the function $u$ and the cylinder $Q_{\theta^o, \theta^o, e^o_2}$, this establishes the induction argument for this sequence of cylinders.

Now, suppose that (4.37) continues to hold at each step until $n - 1$, and that it fails to hold at step $n$. In this case, we modify the construction by considering the smaller cylinder

$$Q_{\theta^o, \theta^o, e^o_2} = B_{e^o} \times (-\theta^o e^o_2, 0] \subset Q_n$$

where $\theta^o_n := \left(\frac{12}{13} \mu_n^+ \right)^{1-m}$.

As in the case $n = 0$ we use a change of variables similar to (4.55) to transform the equation into a non-singular one, to which the classical theory, which we have just discussed, can be applied. This completes the proof of Proposition 4.1. \[\square\]
Remark 4.3. Roughly speaking, if at step \(n\) (4.37) does not hold true, then on \(Q_n\) the solution \(u\) is bounded away from zero and therefore the equation behaves like a non-singular parabolic equation with measure data on the right-hand side, to which the classical regularity theory can be applied. Hence, at scale \(n\), i.e. on \(Q_n\), the behavior changes from degenerate to non-degenerate, and \(Q_n\) becomes a non-intrinsic cylinder. At such a scale, which could be called switching scale, the standard parabolic theory is applicable and yields the decay of the oscillation on a sequence of non-intrinsic cylinders (i.e. standard parabolic cylinders). More precisely, the same arguments given in the final part of the proof of Theorem 1.1 hold true, and the continuity of \(u\) remains a direct consequence of the uniform vanishing on compact sets \(E_\alpha \subset E_T\) of the potential \(I^\mu_2(z, r, r^2)\).  

\[\square\]

5. Applications

In this final chapter we list some simple consequences of Theorem 1.1. A first, actually immediate, corollary concerns measures which have integrable densities.

**Corollary 5.1.** Let \(u\) be a non-negative, locally bounded, weak energy solution of the porous medium equation (1.2) in the sense of Definition 1.1, where the vector-field \(A\) fulfills the growth and ellipticity conditions (1.3). Assume that \(\mu \in L^{2N/2} (E_T)\). If the functions

\[z \to \int_0^r \|\mu\|_{L^{2N/2}(Q_\varphi(z))} \frac{d\varphi}{\varphi}\]

converge locally uniformly to zero in \(E_T\) as \(r \to 0\), then \(u\) is locally continuous in \(E_T\).

A further, also rather straightforward, corollary of Theorem 1.1, concerns measures with special density properties. Let us consider a function \(h : [0, \infty) \to [0, \infty)\), such that

\[(5.1) \quad \int_0^r h(\varphi) \frac{d\varphi}{\varphi} < \infty \quad \text{for some} \quad r > 0.
\]

Then we have

**Corollary 5.2.** Let \(u\) be a non-negative, locally bounded, weak energy solution of the porous medium equation (1.2) in the sense of Definition 1.1, where the vector-field \(A\) fulfills the growth and ellipticity conditions (1.3). Assume that \(\mu\) satisfies

\[\mu(Q_{\varphi, \varphi^2}(z)) \leq C \varphi^N h(\varphi)
\]

for every cylinder \(Q_{\varphi, \varphi^2}(z) \in E_T\), where the function \(h(\cdot)\) satisfies (5.1); then \(u\) is locally continuous in \(E_T\).

The preceding Corollary covers for example the case of measures \(\mu\) satisfying

\[\mu(Q_{\varphi, \varphi^2}(z)) \leq \gamma \varphi^{N+\varepsilon} \quad \text{for some} \quad \varepsilon > 0,
\]

whenever \(Q_{\varphi, \varphi^2}(z) \in E_T\). As fas as we know, this result is new.

A third consequence concerns measures, which have densities in Lorentz spaces. In order to explain the result we have in mind, we need to recall a few basic definitions relevant to Lorentz spaces. Let \(\mu : E_T \to \mathbb{R}\) be a measurable map such that

\[\left\lfloor \left\lfloor z \in E_T : |\mu(z)| > \sigma \right\rfloor \right\rfloor < \infty \quad \text{for} \sigma > 0.
\]

We assume that \(\mu\) is extended to the whole \(\mathbb{R}^{N+1}\) letting \(\mu \equiv 0\) outside \(E_T\). The decreasing rearrangement \(\mu^* : [0, \infty) \to [0, \infty]\) is pointwise defined by

\[
\mu^*(s) := \sup \left\{ \sigma \geq 0 : \left\lfloor \left\lfloor z \in E_T : |\mu(z)| > \sigma \right\rfloor \right\rfloor > s \right\}.
\]
Now, the usual definition of the Lorentz space $L(p, q)$, for $p \in (0, \infty)$ and $q \in (0, \infty)$, is as follows:

$$ (5.2) \quad \mu \in L(p, q) \iff [\mu]_{p, q} \overset{\text{def}}{=} \left( \frac{q}{p} \int_0^\infty \left( \mu^*(\varrho)^\frac{1}{q} \right)^q \frac{d\varrho}{\varrho} \right)^\frac{1}{q} < \infty. $$

Later on we will need a characterization of Lorentz spaces, using an averaged version of $\mu^*$. This characterization goes back to Hunt [10]. For $s > 0$, we consider the following maximal operator

$$ \mu(s)^{**} \overset{\text{def}}{=} \frac{1}{s} \int_0^s \mu^*(\sigma) \, d\sigma. $$

With $\mu^{**}$ at hand, for $q < \infty$ one defines

$$ (5.3) \quad \|\mu\|_{\gamma, q} \overset{\text{def}}{=} \left( \frac{q}{p} \int_0^\infty \left( \mu^{**}(\varrho)^\frac{1}{q} \right)^q \frac{d\varrho}{\varrho} \right)^\frac{1}{q}. $$

Then, [17, Theorem 3.21] shows that

$$ [\mu]_{p, q} \leq \|\mu\|_{p, q} \leq \gamma(p, q)[\mu]_{p, q} $$

holds true for $p > 1$. Therefore it is natural to work with the quantity $\|\cdot\|_{p, q}$ when dealing with Lorentz spaces, at least when $p > 1$. Since in our application the index $p$ is always greater then one (actually we have $p = \frac{N+2}{2}$), we can use this second more convenient characterization. We note that the quantity $\|\cdot\|_{p, q}$ makes $L(p, q)$ to a Banach space when $p > 1$. From [9, §3.1, Lemma 2] we recall the following.

**Lemma 5.1.** Assume that $\mu \in L\left(\frac{N+2}{2}, 1\right)$. Then for every $r > 0$ there holds

$$ \sup_{z \in E_T} I^\mu_2(z, r, r^2) \leq \gamma(N) \int_0^{2\alpha(N)r^{N+2}} \mu^{**}(s) s^{\frac{2}{N+2}} \frac{d\varrho}{\varrho}. $$

Theorem 1.1 has now the following immediate corollary:

**Corollary 5.3.** Let $u$ be a non-negative, locally bounded, weak energy solution of the porous medium equation (1.2) in the sense of Definition 1.1, where the vector-field $A$ fulfills the growth and ellipticity conditions (1.3). Assume that $\mu \in L\left(\frac{N+2}{2}, 1\right)$ holds locally in $E_T$. Then $u$ is locally continuous in $E_T$.

**Proof.** Without loss of generality, we may assume that $\mu$ is extended to $\mathbb{R}^{N+1}$ by zero outside of $E_T$ and that $\mu \in L\left(\frac{N+2}{2}, 1\right)_{(\mathbb{R}^{N+1})^\circ}$. In order to conclude the claim, we only have to switch from $E_T$ to any subset $E_o \subset E_T$, and then apply the following argument. This localization of $\mu$ to $E_o$ is always possible, by setting $\mu$ to zero outside $E_o$. Now the proof goes as follows: Since $\mu \in L\left(\frac{N+2}{2}, 1\right)$ we infer from (5.3) that

$$ \int_0^\infty \mu^{**}(s) s^{\frac{2}{N+2}} \frac{d\varrho}{\varrho} < \infty. $$

Therefore, by Lemma 5.3 we can conclude that

$$ \lim_{r \downarrow 0} \sup_{z \in E_T} I^\mu_2(z, r, r^2) \leq \lim_{r \downarrow 0} \gamma(N) \int_0^{2\alpha(N)r^{N+2}} \mu^{**}(s) s^{\frac{2}{N+2}} \frac{d\varrho}{\varrho} = 0. $$

Now, the claim follows from Theorem 1.1. \qed
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