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Research Article

Holder's Inequality ρ–Mean Continuity for Existence and Uniqueness Solution of Fractional Multi-Integrodifferential Delay System

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We herein present the detailed results for the existence and uniqueness of mild solution for multifractional order impulsive integrodifferential control equations with a nonlocal condition involving several types of semigroups of bounded linear operators, which were established on probability density functions related with the fractional differential equation. Additionally, we present the necessary and sufficient conditions to investigate Schauder's fixed point theorem with Holder's inequality ρ–mean continuity and infinite delay parameter to guarantee the uniqueness of a fixed point.

1. Introduction

The importance of investigating the solution for fractional order derivatives in integrodifferential equations with a nonlocal initial or boundary condition lies in the fact that they include several classes of fractional order integrodifferential equations, as presented in studies on the existence and uniqueness of nonlocal initial fractional order integrodifferential equations in [1–5] and in some other studies with a nonlocal boundary condition [6], as well as fractional order differential equations involving integral conditions as a boundary condition, as found in multiple papers, including those by [7, 8].

Additionally, existence and uniqueness investigated in several studies for impulsive fractional order integrodifferential problems are presented in [3, 9–14]. Integral nonlocal conditions for impulsive fractional order integrodifferential equations are also presented in [15, 16].

We consider the impulsive multifractional order integrodifferential equations with nonlocal conditions and finite delay as follows:

$$ {}^{c}D^{\alpha}x(t) = Ax(t) + Bu(t) + f\left(t, x_t, I^{\beta_1}(x(t)), I^{\beta_2}(x(t))\right), \quad t \in J = [0,b],\ t \neq t_i, \tag{1} $$

$$ \Delta x|_{t=t_i} = I_i(x(t_i)), \quad i = 1, 2, \ldots, s, \tag{2} $$

$$ x(t) = \varnothing(t) + g(x)(t), \quad t \in [-r, 0], \tag{3} $$

where ${}^{c}D^{\alpha}$ and I^{β_1} and I^{β_2} are the Caputo fractional derivative and fractional integration, respectively, of order $0 < \alpha, \beta_1, \beta_2 < 1$, the state $x(\cdot)$ is defined on the Banach space X with the norm $\|\cdot\|$, $u(\cdot)$ is the control function in Banach space $L^2(J, V)$ of admissible control functions, and V is Banach space, where $B: V \longrightarrow X$ is a linear bounded operator. $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup of operators on X generated by A. $PC(J, X) = \{x: [0,b] \longrightarrow X, x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k$ and $x(t_k^+)\}$, the impulsive functions $I_i: \mathscr{D} \longrightarrow X$, $i = 1, 2, \ldots, s$, $0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = b$. The functions

f: $J \times \mathscr{D} \times X \times X \longrightarrow X$ and g: $PC(J, X) \longrightarrow X$ are continuous and satisfies some assumptions, where $\mathscr{D} = \{\omega$: $[-r, 0] \longrightarrow X$, $\omega(t)$ is continuous for all $t \in J$ except for a finite number of points t_i at which $\omega(t_i^+)$ and $\omega(t_i^-)$ exist and $\omega(t_i) = \omega(t_i^-)\}$, the impulsive functions I_i: $\mathscr{D} \longrightarrow X$, $i = 1, 2, \ldots, s$, $0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = b$, the jump of a function x at t_i is denoted by $\Delta x(t_i)$, which is defined by $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$.

For $x \in PC(J, X)$, we denote x_t as the function in $\mathscr{D}$ given by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, and $\varnothing$ is a given function in $\mathscr{D}$.

Furthermore, $PC([-r, b], X)$ is a Banach space of all piecewise continuous functions with the norm $\|x\| = \sup\{\|x(t)\|$: $t \in [-r, b]\}$.

The scientific problem of these types of fractional integrodifferential equations that have not easy solvability even some time there are difficult to study also their behaviours for their solutions on the certain space and the description of the equation terms, however it is need more effort and practise.when the solvability of these problems have been investigated will be cover all a particular results from this problem such as [3, 10, 11].

In this paper, we first present the basic theory of existence and uniqueness of mild solution for multi-fractional order impulsive integrodifferential control equations with a nonlocal condition and infinite delay parameter (1) by defining several types of semigroups of linear-bounded operators established on probability density functions defined on $(0, \infty)$ and consider the necessary and sufficient estimators conditions, which play an important role in investigating Schauder's fixed point theorem with Holder's inequality ρ–mean continuity to guarantee the existence and uniqueness of a fixed point.

2. Preliminaries

Definition 1 (see [17]). Let $AC[0, \infty)$ be a space of absolutely continuous function. Then, the fractional integral for a function $g \in AC[0, \infty)$ of order α is defined as follows:

$$I^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t > 0,\ 0 < \alpha < 1, \tag{4}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2 (see [17]). The Caputo derivative for a function $g \in AC[0, \infty)$ of order α is defined as follows:

$${}^{c}D^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g'(s)}{(t-s)^{\alpha}} \mathrm{d}s, \quad t > 0,\ 0 < \alpha < 1, \tag{5}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 3 (see [18]). The family of bounded linear operators $T(t)$, $0 \le t < \infty$ defined on the Banach space X is a semigroup if $T(0) = I$. Here, I is the identity operator on X and $T(t + s) = T(t)T(s)$, for every t, $s \ge 0$.

Definition 4 (see [18]). Let $T(t)$ be a semigroup, then $T(t)$ is called strongly continuous and is denoted by C_0 on a Banach space X if $\|\lim_{t \downarrow 0} T(t)x - Ix\|_X = 0$.

Definition 5 (see [18]). The domain of the linear operator A: $D(A) \subseteq X \longrightarrow X$ is defined as follows. The domain $D(A) = \{x \in X$: $\lim_{t\downarrow 0}((T(t)x - x)/t)$ exists and $Ax = (\mathrm{d}T(t)/\mathrm{d}t)|_{t=0} = \lim_{t\downarrow 0}((T(t)x - x)/t)$ for $x \in D(A)$. Here, A is the generator of the semigroup $T(t)$.

Remark 1 (see [19]). We define the norm for measurable functions n: $J \longrightarrow \mathbb{R}$ as follows:

$$\|n\|_{L^p(J)} = \begin{cases} \left(\int_J |n(t)|^p \mathrm{d}t\right)^{1/p}, & 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0}\left\{\sup_{t \in J - \bar{J}} |n(t)|\right\}, & p = \infty, \end{cases} \tag{6}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. The Banach space of all Lebesgue measurable functions is $L^p(J, \mathbb{R})$.

Lemma 1 (see [20]). *If $f_1 \in L^p(J, \mathbb{R})$ and $f_2 \in L^q(J, \mathbb{R})$, then $\|f_1 f_2\| \le \|f_1\| \|f_2\|$ for $p, q \ge 1$ and $(1/q) + (1/p) = 1$, and the inequality is called Holder's inequality.*

Lemma 2 (see [19]). *Let h: $[0, b] \longrightarrow \mathbb{R}$ be a measurable function; if $|h|$ is Lebesgue integrable, then h is called the Bochner integrable.*

Lemma 3 (see [20]). *If D is a bounded, closed, and convex subset of a Banach space X and F: $D \longrightarrow D$ is completely continuous, then F has a fixed point in D.*

Lemma 4 (see [21]). *Let $\omega \in L^p(J, X)$, $1 \le p < +\infty$, then $\lim_{t \longrightarrow 0} \int_0^b \|\omega(t + r) - \omega(t)\|^p dt = 0$, and $\omega(s) = 0$ for $s \notin J$.*

In order to guarantee the existence and uniqueness of mild solution of problem (1), we introduce the following assumptions.

Assumptions

(A_1) $\{T(t)\}_{t \ge 0}$ is a strongly continuous semigroup generated by A, that is, a compact operator for every $t \ge 0$, and $\|T(t)\| \le M_1$.

(A_2) Let M_2 and M_3 be the positive constants such that $\|B\| \le M_2$ and $\|u\| \le M_3$.

(A_3) The function f: $J \times \mathscr{D} \times X \times X \longrightarrow X$ satisfies that for any $t \in J$, $f(t, \cdot, \cdot, \cdot)$: $\mathscr{D} \times X \longrightarrow X$ is a continuous function and for all $(\varnothing, x) \in \mathscr{D} \times X$, $f(\cdot, \varnothing, x, y)$: $J \longrightarrow X$ is strongly measurable, where

$$\mathscr{D} = \left\{ \begin{array}{c} \tau\colon [-r,\infty] \longrightarrow X\colon \tau(t) \text{ is continous every where except for a finite number of} \\ \text{points } t_i \text{ at which } \tau(t_i^+) \text{ and } \tau(t_i^-) \text{ exist and } \tau(t_i) = \tau(t_i^-) \end{array} \right\}, \tag{7}$$

and there is a positive function $\mu(\cdot) \in L^p(J, \mathbb{R}^+)$ for some p, $1 < p < \infty$ such that

$$\begin{aligned} \|f(t,x,y,z)\| &\le \mu(t)(\|x\| + \|y\| + \|z\|), \\ & x, y, z \in PC(J, X),\ t \in J. \end{aligned} \tag{8}$$

(A_4) The function $g\colon PC(J, X) \longrightarrow X$ is a continuous compact function, satisfying

$$\|g(x)\| \le L\|x\| + L',\ x \in PC(J, X), \text{ for constants,} \quad L, L' > 0. \tag{9}$$

(A_5) The operator $I_i\colon \mathscr{D} \longrightarrow X$, $i = 1, 2, \ldots, s$, is continuous and there is nondecreasing function $N_i\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\left\|I_i(x)\right\| \le N_i(\|x\|_{\mathscr{D}}), \quad i = 1, 2, \ldots, s,\ x \in \mathscr{D}. \tag{10}$$

Let $\mathscr{D}_q = \{x \in PC(J, X),\ \|x\| \le q\}$, where q a positive constant. Therefore, $\mathscr{D}_q$ is a closed, bounded, and convex subset in $\mathscr{D}$.

Definition 6. The function $x(\cdot) \in PC(J, X)$ is the mild solution of problem (1), if $x(t) = \varnothing(t) + g(x)(t)$ on $t \in [-r, 0]$ and $\Delta x|_{t=t_i} = I_i(x(t_i))$, $i = 1, 2, \ldots, s$, the restriction of $x(\cdot)$ to the interval J_i, $i = 1, 2, \ldots, s$, is continuous and the following equation is satisfied:

$$\begin{aligned} x(t) &= S_\alpha(t)[\varnothing + g(x)] \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[f\left(s, x_s, I^{\beta_1}(x(\tau)), I^{\beta_2}(x(\tau))\right)\right. \\ &\quad + Bu(s)]\mathrm{d}s \\ &\quad + \sum_{i=1}^{s} S_\alpha(t-t_i) I_i(x(t_i)), \end{aligned} \tag{11}$$

where

$$S_\alpha(t) = \int_0^\infty \Phi_\alpha(\theta) T(t^\alpha \theta)\mathrm{d}\theta, \tag{12}$$

$$\begin{aligned} T_\alpha(t) &= \alpha \int_0^\infty \theta \Phi_\alpha(\theta) T(t^\alpha\theta)\mathrm{d}\theta \\ \Phi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-(1/\alpha)} w_\alpha\left(\theta^{-(1/\alpha)}\right) \\ w_\alpha(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty} (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(\alpha n+1)}{n!}\sin(\alpha n\pi), \quad \theta \in (0,\infty), \end{aligned} \tag{13}$$

and Φ_α is the probability density function on $(0, \infty)$, with properties $\Phi_\alpha(\theta) \ge 0$ and

$$\begin{aligned} \int_0^\infty \Phi_\alpha(\theta)\mathrm{d}\theta &= 1, \\ \int_0^\infty \theta\Phi_\alpha(\theta)\mathrm{d}\theta &= \frac{1}{\Gamma(\alpha+1)}. \end{aligned} \tag{14}$$

3. Main Results

Theorem 1. *Assume that the hypotheses (A_1)–(A_5) with the condition* $M_1[(\alpha b^{\alpha-(1/p)}/(\Gamma(1+\alpha)))\ ((p-1)/(\alpha-1)p+p-1)^{(p-1)/p}(\|\sigma\|_{L^p(J,R^+)} + ((b^{\beta_1+((p-1)/p^2)})/(\Gamma(\beta_1+1)))\|\sigma\|_{L^{p^2}(J,R^+)} + ((b^{\beta_2+((p-1)/p^2)})/(\Gamma(\beta_1+1)))\|\sigma\|_{L^{p^2}(J,R^+)}) + \sum_{i=1}^{s} L_i] < 1$ *holds. Thus, for every nonlocal initial condition* $x(t) = \varnothing(t) + g(x)(t)$, $t \in [-r, 0]$, *the impulsive multifractional order integrodifferential equations with nonlocal conditions (1) has a mild solution* $x \in \mathscr{D}_q$, *where q is a positive constant, for every control* $u \in L^2(J, V)$.

Proof. For any positive constant q and $x \in \mathscr{D}_q$, from (A_1) and (A_4), we have

$$\begin{aligned} \left\|S_\alpha(t)[\varnothing + g(x)]\right\| &= \left\|\int_0^\infty \Phi_\alpha(\theta) T(t^\alpha\theta)[\varnothing + g(x)]\mathrm{d}\theta\right\| \\ &\le \int_0^\infty \Phi_\alpha(\theta)\left\|T(t^\alpha\theta)\right\|[\|\varnothing\| + \|g(x)\|]\mathrm{d}\theta \\ &\le M_1(\|\varnothing\| + Lq + \overline{L}). \end{aligned} \tag{15}$$

Therefore, $\int_0^\infty \Phi_\alpha(\theta) T(t^\alpha\theta)[\varnothing + g(x)]\mathrm{d}\theta$ exists. From (A_1), (12)–(14), and (A_3), we obtain

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]\mathrm{d}s\right\| \\
&\quad\leq \int_0^t\int_0^\infty \alpha\theta\Phi_\alpha(\theta)(t-s)^{\alpha-1}\left\|T\left((t-s)^\alpha\theta\right)\right\| \\
&\quad\times\left[\left\|f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)s+\widehat{\varnothing}(\tau))\right)\right\|+\|Bu(s)\|\right]\mathrm{d}s \\
&\quad\leq\frac{\alpha M_1}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\left[\sigma(s)\left(q+\|\widehat{\varnothing}\|_{PC}+\frac{1}{\Gamma(\beta_1)}\int_0^s(s-\tau)^{\beta_1-1}\left(q+\|\widehat{\varnothing}\|_{PC}\right)\mathrm{d}\tau\right.\right. \\
&\quad\left.\left.+\frac{1}{\Gamma(\beta_2)}\int_0^s(s-\tau)^{\beta_2-1}\left(q+\|\widehat{\varnothing}\|_{PC}\right)\mathrm{d}\tau\right)+M_2M_3\right]\mathrm{d}s \\
&\quad\leq\frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\int_0^t(t-s)^{\alpha-1}\sigma(s)\mathrm{d}s+\frac{1}{\Gamma(\beta_1+1)}\int_0^t(t-s)^{\alpha-1}\sigma(s)s^{\beta_1}\mathrm{d}s\right. \\
&\quad\left.+\frac{1}{\beta_2\Gamma(\beta_2+1)}\int_0^t(t-s)^{\alpha-1}\sigma(s)\,s^{\beta_2}\mathrm{d}s\right)+\frac{\alpha M_1}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}M_2M_3\mathrm{d}s \\
&\quad\leq\frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\left(\int_0^t(t-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_0^t(\sigma(s))^p\mathrm{d}s\right)^{1/p}\right. \\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\int_0^t(t-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_0^t\left(s^{\beta_1}\sigma(s)\right)^p\mathrm{d}s\right)^{1/p} \\
&\quad\left.+\frac{1}{\beta_2\Gamma(\beta_2+1)}\left(\int_0^t(t-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_0^t\left(s^{\beta_2}\sigma(s)\right)^p\mathrm{d}s\right)^{1/p}\right) \\
&\quad+\frac{\alpha M_1M_2M_3}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{d}s \\
&\quad\leq\frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p}t^{((\alpha-1)p+p-1)/p}\|\sigma\|_{L^p(J,R^+)}\right. \\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p}t^{((\alpha-1)p+p-1)/p}\left(\int_0^t s^{\beta_1p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_0^t(\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2} \\
&\quad\left.+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p}t^{((\alpha-1)p+p-1)/p}\left(\int_0^t s^{\beta_2p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_0^t(\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\right. \\
&\quad+\frac{\alpha M_1M_2M_3}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\mathrm{d}s \\
&\quad\leq\frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p}t^{((\alpha-1)p+p-1)/p}\left(\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{1}{1+(\beta_1p^2/(p-1))}\right)^{(p-1)/p^2}\right. \\
&\quad\cdot t^{(\beta_1p^2+p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)} \\
&\quad\left.+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{1}{1+(\beta_2p^2/(p-1))}\right)^{(p-1)/p^2}t^{(\beta_2p^2+p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\right)+\frac{M_1M_2M_3b^\alpha}{\Gamma(1+\alpha)}
\end{aligned}$$

$$\leq \frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)b^{\alpha-(1/p)}}{\Gamma(1+\alpha)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{((p-1)/p)}\left(\|\sigma\|_{L^p(J,R^+)}+\frac{b^{\beta_1+((p-1)/p^2)}}{\Gamma(\beta_1+1)}\|\sigma\|_{L^{p^2}(J,R^+)}\right.$$
$$\left.+\frac{b^{\beta_2+((p-1)/p^2)}}{\Gamma(\beta_1+1)}\|\sigma\|_{L^{p^2}(J,R^+)}\right)+\frac{M_1M_2M_3b^{\alpha}}{\Gamma(1+\alpha)}, \tag{16}$$

for all $t\in J$. Thus, $\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s, y_s+\widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau)))+Bu(s)]ds\|$ is Lebesgue integrable with respect to $s\in[0,t]$, for all $t\in J$.

By Lemma 1, we get $\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s, y_s+\widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau)))+Bu(s)]ds\|$ is Bochner's integrable, for $s\in[0,t]$, $t\in J$.

Now, we define the operators F on $\mathscr{D}_q$ as follows:

$$(Fx)(t)=\begin{cases}\varnothing(t), & t\in[-r,0],\\ S_\alpha(t)[\varnothing+g(x)]+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[f(s, x_s, I^{\beta_1}(x(\tau)), I^{\beta_2}(x(\tau)))+Bu(s)\right]ds+\sum_{i=1}^{s} S_\alpha(t-t_i)I_i(x(t_i)), & t\in J.\end{cases} \tag{17}$$

For $\varnothing\in\mathscr{D}$, we define $\widehat{\varnothing}\in PC$ by

$$\widehat{\varnothing}(t)=\begin{cases}\varnothing(t), & t\in[-r,0],\\ S_\alpha(t)[\varnothing+g(x)], & t\in J.\end{cases} \tag{18}$$

Let $x(t)=y(t)+\widehat{\varnothing}(t)$, the operator $G: PC\longrightarrow PC$ defined as

$$(Gy)(t)=\begin{cases}0, & t\in[-r,0],\\ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[f\left(s, y_s+\widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]ds & \\ \quad+\sum_{i=1}^{s} S_\alpha(t-t_i)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right), & t\in J.\end{cases} \tag{19}$$

where $y\in PC$. Let

Clearly, since operator F has one fixed point and thus G too. So, we have

$$(Gy)(t)=\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[f\left(s, y_s+\widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]ds+\sum_{i=1}^{s} S_\alpha(t-t_i)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right), \quad t\in J. \tag{20}$$

$$q=\left(\left(\alpha M_1\|\widehat{\varnothing}\|_{PC}b^{\alpha-(1/p)}\right)/(\Gamma(1+\alpha))\right)((p-1)/((\alpha-1)p+p-1))^{(p-1)/p}\left(\|\sigma\|_{L^p(J,R^+)}+\left(b^{\beta_1+((p-1)/p^2)}/(\Gamma(\beta_1+1))\right)\|\sigma\|_{L^{p^2}(J,R^+)}\right.$$
$$\left.+\left(b^{\beta_2+((p-1)/p^2)}/(\Gamma(\beta_1+1))\right)\|\sigma\|_{L^{p^2}(J,R^+)}\right)+((M_1M_2M_3b^{\alpha})/(\Gamma(1+\alpha)))+M_1\sum_{i=1}^{s}L_i\|\widehat{\varnothing}\|_{PC}$$
$$\cdot\left(1-M_1\left[\left(\alpha b^{\alpha-(1/p)}/\Gamma(1+\alpha)\right)((p-1)/((\alpha-1)p+p-1))^{(p-1)/p}\left(\left(\|\sigma\|_{L^p(J,R^+)}+\left(b^{\beta_1+((p-1)/p^2)}/(\Gamma(\beta_1+1))\right)\|\sigma\|_{L^{p^2}(J,R^+)}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\left(b^{\beta_2+((p-1)/p^2)}/(\Gamma(\beta_1+1))\right)\|\sigma\|_{L^{p^2}(J,R^+)}\right)+\sum_{i=1}^{s}L_i\right]\right)^{-1}. \tag{21}$$

In the following, we will prove that G has a fixed point on $\mathscr{D}_q$, and then we get F has a fixed point on $\mathscr{D}_q$. □

Step 1. $\|Gy\| \le q$, then $\|Fx\| \le q$, where $y \in PC$ and $x \in \mathscr{D}_q$.

By using a method similar to the one used in (15) and (16), we have

$$\begin{aligned}
\|(Gy)(t)\| &= \Bigg\| \int_0^t (t-s)^{\alpha-1} T_\alpha (t-s)\left[f\left(s, y_s + \widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}s \\
&\quad + \sum_{i=1}^{s} S_\alpha (t-t_i) I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right)\Bigg\| \\
\|(Gy)(t)\| &= \Bigg\| \int_0^t \int_0^\infty \alpha\theta \Phi_\alpha(\theta)(t-s)^{\alpha-1} T\left((t-s)^\alpha \theta\right) \\
&\quad \times \left[f\left(s, y_s + \widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}s \\
&\quad + \sum_{i=1}^{s} \int_0^\infty \Phi_\alpha(\theta) T\left((t-t_i)^\alpha \theta\right) I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right)\Bigg\| \\
&\le \frac{\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right) b^{\alpha-(1/p)}}{\Gamma(1+\alpha)} \left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{p-1/p} \left(\|\sigma\|_{L^p(J,R^+)} + \frac{b^{\beta_1+\left(p-1/p^2\right)}}{\Gamma(\beta_1+1)} \|\sigma\|_{L^{p^2}(J,R^+)} \right. \\
&\quad \left. + \frac{b^{\beta_2+\left(p-1/p^2\right)}}{\Gamma(\beta_1+1)} \|\sigma\|_{L^{p^2}(J,R^+)} \right) + \frac{M_1 M_2 M_3 b^\alpha}{\Gamma(1+\alpha)} + M_1 \sum_{i=1}^{s} L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right) = q.
\end{aligned} \tag{22}$$

Hence, if $\|Gy\| \le q$, then $\|Fx\| \le q$, where $y \in PC$ and $x \in \mathscr{D}_q$.

Step 2. G is a completely continuous operator.

Firstly, we will show that G is continuous on $\mathscr{D}_q$.

For any $y_n, y \subseteq \mathscr{D}_q$, $n = 1, 2, \ldots$, with $\lim_{n\longrightarrow\infty} \|y_n \longrightarrow y\| = 0$, we get $\lim_{n\longrightarrow\infty} y_n(t) = y(t)$, $t \in J$.

By (A_3), we obtain

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} f\left(t, y_{t_n} + \widehat{\varnothing}_t, I^{\beta_1}\left(y_n(\tau)+\widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y_n(\tau)+\widehat{\varnothing}(\tau)\right)\right) \\
&= f\left(t, y_t + \widehat{\varnothing}_t, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right), \quad \text{for } t \in J.
\end{aligned} \tag{23}$$

Then,

$$\begin{aligned}
&\sup_{s\in[0,b]} \left\| f\left(s, y_{s_n} + \widehat{\varnothing}_s, I^{\beta_1}\left(y_n(\tau)+\widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y_n(\tau)+\widehat{\varnothing}(\tau)\right)\right) \right. \\
&\left. - f\left(s, y_s + \widehat{\varnothing}_s, I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)\right\| \longrightarrow 0, \\
&n \longrightarrow \infty,
\end{aligned} \tag{24}$$

and from (A_5), we obtain

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} I_i\left(y_n(t_i)+\widehat{\varnothing}(t_i)\right) = I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right), \quad i = 1, 2, \ldots, s, \\
&\sup_{s\in[0,b]} \left\| I_i\left(y_n(s_i)+\widehat{\varnothing}(s_i)\right) - I_i\left(y(s_i)+\widehat{\varnothing}(s_i)\right) \longrightarrow 0 \right\|, \\
&n \longrightarrow \infty.
\end{aligned} \tag{25}$$

Additionally, from (A_2), we obtain

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} u_n(t) = u(t), \\
&\sup_{s\in[0,b]} \left\| u_n(s) - u(s) \right\| \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{26}$$

Subsequently, we have that

$$\begin{aligned}
\|Gy_n - Gy\| &\leq \left\| \int_0^t (t-s)^{\alpha-1} T_\alpha (t-s) \left[f\left(s,\ y_{s_n} + \widehat{\varnothing}_s,\ I^{\beta_1}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right)\right) \right.\right. \\
&\quad \left.\left. - f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right)\right] \mathrm{d}s \right\| \\
&\quad + \left\| \int_0^t (t-s)^{\alpha-1} T_\alpha (t-s) B[u_n - u] \mathrm{d}s \right\| \\
&\quad + \left\| \sum_{i=1}^{s} S_\alpha (t-t_i)\left(I_i\left(y_n(t_i) + \widehat{\varnothing}(t_i)\right) - I_i\left(y(t_i) + \widehat{\varnothing}(t_i)\right)\right) \right\| \\
&\leq \left\| \int_0^t \int_0^\infty \alpha\theta\Phi_\alpha(\theta)(t-s)^{\alpha-1} T\left((t-s)^\alpha\theta\right) \left[f\left(s,\ y_{s_n} + \widehat{\varnothing}_s,\ I^{\beta_1}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right)\right) \right.\right. \\
&\quad \left.\left. - f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
&\quad + \left\| \int_0^t \int_0^\infty \alpha\theta\Phi_\alpha(\theta)(t-s)^{\alpha-1} T\left((t-s)^\alpha\theta\right) B[u_n - u] \mathrm{d}\theta \mathrm{d}s \right\| \\
&\quad + \left\| \sum_{i=1}^{s} \int_0^\infty \Phi_\alpha(\theta) T\left((t-t_i)^\alpha\theta\right)\left(I_i\left(y_n(t_i) + \widehat{\varnothing}(t_i)\right) - I_i\left(y(t_i) + \widehat{\varnothing}(t_i)\right)\right) \mathrm{d}\theta \right\| \\
&\leq \frac{\alpha M_1 b^\alpha}{\Gamma(1+\alpha)} \sup_{s\in[0,b]} \left\| f\left(s,\ y_{s_n} + \widehat{\varnothing}_s,\ I^{\beta_1}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y_n(\tau) + \widehat{\varnothing}(\tau)\right)\right) \right. \\
&\quad \left. - f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) \right\| + \frac{\alpha M_1 M_2 b^\alpha}{\Gamma(1+\alpha)} \sup_{s\in[0,b]} \|u_n - u\| + M_1 \sum_{i=1}^{s} \left\| I_i\left(y_n(t_i) \right.\right. \\
&\quad \left.\left. + \widehat{\varnothing}(t_i)\right) - I_i\left(y(t_i) + \widehat{\varnothing}(t_i)\right) \right\|.
\end{aligned} \tag{27}$$

Hence, by (16), (22), and (24), we have

$$\|Gy_n - Gy\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{28}$$

Thus, G is continuous.

Now, we will prove that $\left\{Gy,\ y \in \mathscr{D}_q\right\}$ is equicontinuous on J_i, $i = 1, 2, \ldots, s$. For any $y \in \mathscr{D}_q$ and $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned}
&\|Gy(t_2) - Gy(t_1)\| \\
&\quad \leq \left\| \int_0^{t_1} \int_0^\infty \alpha\theta\Phi_\alpha(\theta)\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] T\left((t_2-s)^\alpha\theta\right) \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) \right.\right.\right. \\
&\qquad \left.\left.\left. + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^{t_1} \int_0^\infty \alpha\theta\Phi_\alpha(\theta)(t_1-s)^{\alpha-1}\left[T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right] \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) \right.\right.\right. \\
&\qquad \left.\left.\left. + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
&\qquad + \left\| \int_{t_1}^{t_2} \int_0^\infty \alpha\theta\Phi_\alpha(\theta)(t_2-s)^{\alpha-1} T\left((t_2-s)^\alpha\theta\right) \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
&\qquad + \left\| \sum_{i=1}^{s} \int_0^\infty \Phi_\alpha(\theta)\left[S(t_2-t_i)^\alpha - S(t_1-t_i)^\alpha\right] I_i\left(y(t_i) + \widehat{\varnothing}(t_i)\right) \mathrm{d}\theta \right\| \\
&\quad = \alpha(I_1 + I_2 + I_3) + \left\| \sum_{i=1}^{s} \int_0^\infty \Phi_\alpha(\theta)\left[S(t_2-t_i)^\alpha - S(t_1-t_i)^\alpha\right] I_i\left(y(t_i) + \widehat{\varnothing}(t_i)\right) \mathrm{d}\theta \right\|,
\end{aligned} \tag{29}$$

where

$$I_1 = \left\| \int_0^{t_1} \int_0^{\infty} \theta \Phi_\alpha(\theta) \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] T\left((t_2 - s)^\alpha \theta \right) \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s) \right] \mathrm{d}\theta \mathrm{d}s \right\|$$

$$I_2 = \left\| \int_0^{t_1} \int_0^{\infty} \theta \Phi_\alpha(\theta) (t_1 - s)^{\alpha-1} \left[T\left((t_2 - s)^\alpha \theta \right) - T\left((t_1 - s)^\alpha \theta \right) \right] \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s) \right] \mathrm{d}\theta \mathrm{d}s \right\|$$

$$I_3 = \left\| \int_{t_1}^{t_2} \int_0^{\infty} \theta \Phi_\alpha(\theta) (t_2 - s)^{\alpha-1} T\left((t_2 - s)^\alpha \theta \right) \times \left[f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s) \right] \mathrm{d}\theta \mathrm{d}s \right\|$$

$$\begin{aligned}
I_1 \leq{}& \frac{M_1}{\Gamma(1+\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \times \left[\left\| f\left(s,\ y_s + \widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) \right\| + \|Bu(s)\| \right] \mathrm{d}s \\
\leq{}& \frac{M_1\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \left(\left(\int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right]^p \mathrm{d}s \right)^{(1/p)} \|\sigma\|_{L^{p/(p-1)}(J,R^+)} \right. \\
&+ \frac{1}{\Gamma(\beta_1+1)} \left(\int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right]^p \mathrm{d}s \right)^{1/p} \left(\int_0^{t_1} s^{\beta_1 p^2/(p-1)^2} \mathrm{d}s \right)^{(p-1)/p^2} \left(\int_0^{t_1} (\sigma(s))^{p^2/p-1} \mathrm{d}s \right)^{(p-1)/p^2} \\
&+ \frac{1}{\Gamma(\beta_1+1)} \left(\int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right]^p \mathrm{d}s \right)^{1/p} \left(\int_0^{t_1} s^{\beta_2 p^2/(p-1)^2} \mathrm{d}s \right)^{(p-1)^2/p^2} \left(\int_0^{t_1} (\sigma(s))^{p^2/(p-1)} \mathrm{d}s \right)^{(p-1)/p^2} \\
&+ \frac{M_1 M_2 M_3}{\Gamma(1+\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \mathrm{d}s \\
\leq{}& \frac{M_1\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \left(\|\sigma\|_{L^{(p/p-1)}(J,R^+)} + \frac{b^{\beta_1 + \left((p-1)^2/p^2\right)}}{\Gamma(\beta_1+1)} \left(\frac{1}{1 + \left(\beta_1 p^2/(p-1)^2\right)} \right)^{(p-1)^2/p^2} \|\sigma\|_{L^{p^2/p-1}(J,R^+)} \right. \\
&\left. + \frac{b^{\beta_2 + \left((p-1)^2/p^2\right)}}{\Gamma(\beta_2+1)} \left(\frac{1}{1 + \left(\beta_2 p^2/(p-1)^2\right)} \right)^{(p-1)^2/p^2} \|\sigma\|_{L^{p^2/p-1}(J,R^+)} \right) \times \left(\int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right]^p \mathrm{d}s \right)^{1/p} \\
&+ \frac{M_1 M_2 M_3}{\Gamma(1+\alpha)} \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \mathrm{d}s.
\end{aligned} \tag{30}$$

By Lagrange's mean value theorem, we have $(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \longrightarrow 0$ as $t_2 \longrightarrow t_1$ for $s \in J$. Using Lemma 4, we obtain $\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^p \mathrm{d}s \longrightarrow 0$ as $t_2 \longrightarrow t_1$. Thus, $\lim_{t_2 \longrightarrow t_1} \mathrm{I}_1 = 0$.

Now, for $t_1 = 0$, $0 < t_2 \leq b$, clearly, $I_2 = 0$ for $t_1 > 0$ and $\varepsilon > 0$ is sufficiently small. Therefore,

$$
\begin{aligned}
I_2 \leq & \left\| \int_0^{t_1-\varepsilon} \int_0^{\infty} \theta \Phi_\alpha(\theta)(t_1-s)^{\alpha-1}\left[T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right] \times \left[f\left(s, y_s + \widehat{\varnothing}_s, I^{\beta_1}(y(\tau)\right.\right.\right. \\
& \left.\left.\left. + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
& + \left\| \int_{t_1-\varepsilon}^{t_1} \int_0^{\infty} \theta \Phi_\alpha(\theta)(t_1-s)^{\alpha-1}\left[T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right] \right. \\
& \left. \times \left[f\left(s, y_s + \widehat{\varnothing}_s, I^{\beta_1}(y(\tau) + \widehat{\varnothing}(\tau)), I^{\beta_2}(y(\tau) + \widehat{\varnothing}(\tau))\right) + Bu(s)\right] \mathrm{d}\theta \mathrm{d}s \right\| \\
\leq & \frac{\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \sup_{s\in[0,t_1-\varepsilon]} \left\|T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right\| \left[\int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1}\sigma(s)\mathrm{d}s\right. \\
& \left. + \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1}\sigma(s) \frac{s^{\beta_1}}{\beta_1\Gamma(\beta_1)}\mathrm{d}s + \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1}\sigma(s) \frac{s^{\beta_2}}{\beta_2\Gamma(\beta_2)}\mathrm{d}s\right] \\
& + \frac{1}{\Gamma(1+\alpha)} \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1} \left\| \sup_{s\in[0,t_1-\varepsilon]} T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right\| M_2 M_3 \mathrm{d}s \\
& + \frac{2M_1\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \left[\int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)\mathrm{d}s + \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1}\sigma(s) \frac{s^{\beta_1}}{\beta_1\Gamma(\beta_1)}\mathrm{d}s\right. \\
& \left. + \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1}\sigma(s) \frac{s^{\beta_2}}{\beta_2\Gamma(\beta_2)}\mathrm{d}s\right] + \frac{2M_1}{\Gamma(1+\alpha)} \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1} M_2 M_3 \mathrm{d}s \\
\leq & \frac{\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \sup_{s\in[0,t_1-\varepsilon]} \left\|T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right\| \left(\int_0^{t_1-\varepsilon} (t_1-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p} \\
& \cdot \left[\|\sigma\|_{L^p(J,R^+)} + \frac{1}{\Gamma(\beta_1+1)}\left(\int_0^{t_1-\varepsilon} s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2} \left(\int_0^{t_1-\varepsilon} (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\right. \\
& \left. + \frac{1}{\Gamma(\beta_2+1)}\left(\int_0^{t_1-\varepsilon} s^{\beta_2 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2} \left(\int_0^{t_1-\varepsilon} (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\right] \\
& + \frac{M_2 M_3}{\Gamma(1+\alpha)} \int_0^{t_1-\varepsilon} (t_1-s)^{\alpha-1} \sup_{s\in[0,t_1-\varepsilon]} \left\|T\left((t_2-s)^\alpha\theta\right) - T\left((t_1-s)^\alpha\theta\right)\right\| \mathrm{d}s \\
& + \frac{2M_1\left(q + \|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \left(\int_{t_1-\varepsilon}^{t_1} (t_1-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p} \left[\|\sigma\|_{L^p(J,R^+)} + \frac{1}{\Gamma(\beta_1+1)}\left(\int_{t_1-\varepsilon}^{t_1} s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\right. \\
& \left. \times \left(\int_{t_1-\varepsilon}^{t_1} (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2} + \frac{1}{\Gamma(\beta_2+1)}\left(\int_{t_1-\varepsilon}^{t_1} s^{\beta_2 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2} \left(\int_{t_1-\epsilon}^{t_1} (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\right] \\
& + \frac{2M_1}{\Gamma(1+\alpha)} \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1} M_2 M_3 \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)} \sup_{s\in[0,t_1-\varepsilon]} \left\|T\left((t_2-s)^{\alpha}\theta\right)-T\left((t_1-s)^{\alpha}\theta\right)\right\| \left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \\
&\cdot \left(t_1^{(((\alpha-1)p+p-1)/(p-1))}-\varepsilon^{(((\alpha-1)p+p-1)/(p-1))}\right)^{(p-1)/p} \left[\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}(t_1-\varepsilon)^{(\beta_1 p^2+p-1)/p^2}\right. \\
&\cdot \left(\frac{1}{1+\left(\beta_1 p^2/(p-1)\right)}\right)^{(p-1)/p^2} \|\sigma\|_{L^{p^2}(J,R^+)} \\
&\left.+\frac{1}{\Gamma(\beta_2+1)}(t_1-\varepsilon)^{(\beta_2 p^2+p-1)/p^2}\left(\frac{1}{1+\left(\beta_2 p^2/(p-1)\right)}\right)^{(p-1)/p^2} \|\sigma\|_{L^{p^2}(J,R^+)}\right] \\
&+\frac{M_2M_3}{\alpha\Gamma(1+\alpha)}\left(t_1^{\alpha}-\varepsilon^{\alpha}\right) \sup_{s\in[0,t_1-\varepsilon]} \left\|T\left((t_2-s)^{\alpha}\theta\right)-T\left((t_1-s)^{\alpha}\theta\right)\right\| \\
&+\frac{2M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left[\|\sigma\|_{L^p(J,R^+)}+\frac{\varepsilon^{(\beta_1 p^2+p-1)/p^2}}{\Gamma(\beta_1+1)}\right. \\
&\cdot \left(\frac{1}{1+\left(\beta_1 p^2/(p-1)\right)}\right)^{(p-1)/p^2} \|\sigma\|_{L^{p^2}(J,R^+)} \\
&\left.+\frac{\varepsilon^{(\beta_2 p^2+p-1)/p^2}}{\Gamma(\beta_2+1)}\left(\frac{1}{1+\left(\beta_2 p^2/(p-1)\right)}\right)^{(p-1)/p^2} \|\sigma\|_{L^{p^2}(J,R^+)}\right]+\frac{2M_1}{\alpha\Gamma(1+\alpha)}\varepsilon^{\alpha}M_2M_3,
\end{aligned}
\tag{31}
$$

from (A_1) implies the continuity of $\{T(t)\}_{t\geq 0}$ in the uniform operator topology, and it is clear that $I_2 \longrightarrow 0$ independent of $y\in\mathscr{D}_q$ as $t_2-t_1\longrightarrow 0$, $\epsilon\longrightarrow 0$.

Notably,

$$
\begin{aligned}
I_3 &\leq \frac{M_1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[\left\|f\left(s, y_s+\widehat{\varnothing}_s, I^{\beta_1}\left(y(\tau)+\widehat{\varnothing}(\tau)\right), I^{\beta_2}\left(y(\tau)+\widehat{\varnothing}(\tau)\right)\right)\right\|+\|Bu(s)\|\right]\mathrm{d}s \\
&\leq \frac{M_1}{\Gamma(1+\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left[\sigma(s)\left(\left\|y_s+\widehat{\varnothing}_s\right\|+\frac{1}{\Gamma(\beta_1)}\int_0^s(s-\tau)^{\beta_1-1}\|y(\tau)+\widehat{\varnothing}(\tau)\|\mathrm{d}\tau\right.\right. \\
&\left.\left.+\frac{1}{\Gamma(\beta_2)}\int_0^s(s-\tau)^{\beta_2-1}\|y(\tau)+\widehat{\varnothing}(\tau)\|\mathrm{d}\tau\right)+\|B\|\|u(s)\|\right]\mathrm{d}s \\
&\leq \frac{M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\left(\int_{t_1}^{t_2}(t_2-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_{t_1}^{t_2}(\sigma(s))^p\mathrm{d}s\right)^{1/p}\right. \\
&+\frac{1}{\Gamma(\beta_1+1)}\left(\int_{t_1}^{t_2}(t_2-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_{t_1}^{t_2}\left(s^{\beta_1}\sigma(s)\right)^p\mathrm{d}s\right)^{1/p} \\
&\left.+\frac{1}{\Gamma(\beta_2+1)}\left(\int_{t_1}^{t_2}(t_2-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}\left(\int_{t_1}^{t_2}\left(s^{\beta_2}\sigma(s)\right)^p\mathrm{d}s\right)^{1/p}\right)+\frac{M_1M_2M_3}{\Gamma(1+\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s
\end{aligned}
$$

$$\leq \frac{M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{1}{1+(\beta_1 p^2/(p-1))}\right)^{(p-1)/p^2}\left(t_2^{\left(\beta_1 p^2/(p-1)\right)+1}-t_1^{\left(\beta_1 p^2/(p-1)\right)+1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\right.$$

$$\left.+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{1}{1+(\beta_2 p^2/(p-1))}\right)^{p-1/p^2}\left(t_2^{\left(\beta_2 p^2/(p-1)\right)+1}-t_1^{\left(\beta_2 p^2/(p-1)\right)+1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\right)$$

$$\times\left(\int_{t_1}^{t_2}(t_2-s)^{((\alpha-1)p)/(p-1)}\mathrm{d}s\right)^{(p-1)/p}+\frac{M_1M_2M_3}{\Gamma(1+\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathrm{d}s$$

$$\leq \frac{M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)}{\Gamma(1+\alpha)}\left(\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{1}{1+(\beta_1 p^2/(p-1))}\right)^{(p-1)/p^2}\left(t_2^{\left(\beta_1 p^2/(p-1)\right)+1}-t_1^{\left(\beta_1 p^2/(p-1)\right)+1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\right.$$

$$\left.+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{1}{1+(\beta_2 p^2/(p-1))}\right)^{(p-1)/p^2}\left(t_2^{\left(\beta_2 p^2/(p-1)\right)+1}-t_1^{\left(\beta_2 p^2/(p-1)\right)+1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\right)$$

$$\cdot\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p}(t_2-t_1)^{\alpha-(1/p)}+\frac{M_1M_2M_3}{\alpha\Gamma(1+\alpha)}, \tag{32}$$

We have that $\lim_{t_2\longrightarrow t_1} I_3=0$. Thus, $Gy(t_2)-Gy(t_1)$ tends to zero independent of $y\in\mathscr{D}_q$ as $t_2-t_1\longrightarrow 0$, which means that $\left\{Gy,\ y\in\mathscr{D}_q\right\}$ is equicontinuous.

Now, we need to show that $W(t)=\left\{(Gy)(t),\ y\in\mathscr{D}_q\right\}$ is relatively compact in X and $t\in[0,b]$. So, we define an operator $G_{\varepsilon,\delta}$ on $\mathscr{D}_q$ for all $\varepsilon\in(0,t)$ and $\delta>0$ as follows:

$$\left(G_{\varepsilon,\delta}y\right)(t)=\begin{cases}0, & t\in[-r,0],\\ \int_0^{t-\varepsilon}\int_\delta^\infty \alpha\theta(t-s)^{\alpha-1}\Phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]\mathrm{d}\theta\mathrm{d}s & \\ \quad+\sum_{i=1}^s\int_\delta^\infty\Phi_\alpha(\theta)T\left((t-t_i)^\alpha\theta\right)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right), & t\in J.\end{cases}$$

$$=\begin{cases}0, & t\in[-r,0],\\ T(\epsilon^\alpha\delta)\left[\int_0^{t-\epsilon}\int_\delta^\infty\alpha\theta(t-s)^{\alpha-1}\Phi_\alpha(\theta)T\left((t-s)^\alpha\theta-\epsilon^\alpha\delta\right)\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)\right.\right. & \\ \quad\left.\left.+Bu(s)\right]\mathrm{d}\theta\mathrm{d}s+\sum_{i=1}^s\int_\delta^\infty\Phi_\alpha(\theta)T\left((t-t_i)^\alpha\theta-\epsilon^\alpha\delta\right)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right)\right], & t\in J.\end{cases} \tag{33}$$

Subsequently, from the compactness of $T(\epsilon^\alpha\delta)$, $\epsilon^\alpha\delta>0$, we obtain the set $W_{\epsilon,\delta}(t)=\left\{(G_{\epsilon,\delta}y)(t),\ y\in\ \mathscr{D}_q\right\}$ is relatively compact in X. Obviously, $W(0)$ is relatively compact in X, for all $\varepsilon\in(0,t)$ and $\delta>0$. Furthermore, for $\in\mathscr{D}_q$, we obtain

$$\begin{aligned}
\left\|(Gy)(t)-\left(G_{\varepsilon,\delta}y\right)(t)\right\| &\le \alpha\left\|\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\Phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\times\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]\mathrm{d}\theta\mathrm{d}s\right\| \\
&+\alpha\left\|\int_0^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\Phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),\right.\right.\right. \\
&\left.I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\Big]\mathrm{d}\theta\mathrm{d}s-\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\Phi_\alpha(\theta)T\left((t-s)^\alpha\theta\right) \\
&\left[f\left(s,\ y_s+\widehat{\varnothing}_s,\ I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]\mathrm{d}\theta\mathrm{d}s \\
&+\left\|\sum_{i=1}^s\int_0^\delta \Phi_\alpha(\theta)T\left((t-t_i)^\alpha\theta\right)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right)\mathrm{d}\theta\right\| \\
&\le \alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_0^t (t-s)^{\alpha-1}\left(\sigma(s)\left(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\right)+M_2M_3\right)\mathrm{d}s\right]\int_0^\delta\theta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&+\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\left(\sigma(s)\left(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\right)+M_2M_3\right)\mathrm{d}s\right]\int_0^\infty\theta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&\le \alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_0^t (t-s)^{\alpha-1}\left(\sigma(s)\left(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\right)+M_2M_3\right)\mathrm{d}s\right]\int_0^\delta\theta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&+\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_{t-\epsilon}^t (t-s)^{\alpha-1}\left(\sigma(s)\left(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\right)+M_2M_3\right)\mathrm{d}s\right] \\
&+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&\le \alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_0^t (t-s)^{\alpha-1}\sigma(s)\mathrm{d}s+\int_0^t (t-s)^{\alpha-1}\sigma(s)\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}\mathrm{d}s\right. \\
&\left.+\int_0^t (t-s)^{\alpha-1}\sigma(s)\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\mathrm{d}s+\int_0^t (t-s)^{\alpha-1}M_2M_3\mathrm{d}s\right]\int_0^\delta\theta\Phi_\alpha(\theta)\mathrm{d}\theta \\
&+\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left[\int_{t-\epsilon}^t (t-s)^{\alpha-1}\sigma(s)\mathrm{d}s+\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\sigma(s)\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}\mathrm{d}s\right. \\
&\left.+\int_{t-\epsilon}^t (t-s)^{\alpha-1}\sigma(s)\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\mathrm{d}s+\int_{t-\epsilon}^t (t-s)^{\alpha-1}M_2M_3\mathrm{d}s\right] \\
&+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta\Phi_\alpha(\theta)\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\|\sigma\|_{L^p(J,R^+)}\right.\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\left(\int_0^t s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_0^t (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\left(\int_0^t s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_0^t (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\left.+M_2M_3\int_0^t (t-s)^{\alpha-1}\mathrm{d}s\right)\\
&\quad\times\left(\int_0^\delta \theta\Phi_\alpha(\theta)\mathrm{d}\theta\right)+\alpha M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\|\sigma\|_{L^p(J,R^+)}\right.\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left(\int_{t-\varepsilon}^t s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_{t-\varepsilon}^t (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left(\int_{t-\varepsilon}^t s^{\beta_1 p^2/(p-1)}\mathrm{d}s\right)^{(p-1)/p^2}\left(\int_{t-\varepsilon}^t (\sigma(s))^{p^2}\mathrm{d}s\right)^{1/p^2}\left.+M_2M_3\int_{t-\varepsilon}^t (t-s)^{\alpha-1}\mathrm{d}s\right)\\
&\quad+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta \Phi_\alpha(\theta)\mathrm{d}\theta \leq M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\|\sigma\|_{L^p(J,R^+)}\right.\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\left(\frac{p-1}{\beta_1p^2+p-1}t^{\beta_1 p^2/(p-1)}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\\
&\quad+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\left(\frac{p-1}{\beta_2p^2+p-1}t^{(\beta_2 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\left.+M_2M_3t^\alpha\right)\\
&\quad\times\left(\int_0^\delta \theta\Phi_\alpha(\theta)\mathrm{d}\theta\right)+M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\|\sigma\|_{L^p(J,R^+)}\right.\\
&\quad+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left(\frac{p-1}{\beta_1p^2+p-1}\epsilon^{(\beta_1 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\\
&\quad+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left(\frac{p-1}{\beta_2p^2+p-1}\epsilon^{(\beta_2 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\|\sigma\|_{L^{p^2}(J,R^+)}\left.+M_2M_3t^\alpha\right)\\
&\quad+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta \Phi_\alpha(\theta)\mathrm{d}\theta\\
&\leq M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} t^{((\alpha-1)p+p-1)/p}\left[\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{\beta_1p^2+p-1}t^{(\beta_1 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\right.\right.\\
&\quad\times\|\sigma\|_{L^{p^2}(J,R^+)}+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{p-1}{\beta_2p^2+p-1}t^{(\beta_2 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\left.\|\sigma\|_{L^{p^2}(J,R^+)}\right]\left.+M_2M_3t^\alpha\right)\left(\int_0^\delta \theta\Phi_\alpha(\theta)\mathrm{d}\theta\right)\\
&\quad+M_1\left(q+\|\widehat{\varnothing}\|_{PC}\right)\left(\left(\frac{p-1}{(\alpha-1)p+p-1}\right)^{(p-1)/p} \varepsilon^{((\alpha-1)p+p-1)/p}\left[\|\sigma\|_{L^p(J,R^+)}+\frac{1}{\Gamma(\beta_1+1)}\left(\frac{p-1}{\beta_1p^2+p-1}\epsilon^{(\beta_1 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\right.\right.\\
&\quad\times\|\sigma\|_{L^{p^2}(J,R^+)}+\frac{1}{\Gamma(\beta_2+1)}\left(\frac{p-1}{\beta_2p^2+p-1}\epsilon^{(\beta_2 p^2+p-1)/p-1}\right)^{(p-1)/p^2}\left.\|\sigma\|_{L^{p^2}(J,R^+)}\right]\left.+M_2M_3t^\alpha\right)\\
&\quad+M_1\sum_{i=1}^s L_i\left(q+\|\widehat{\varnothing}\|_{PC}\right)\int_0^\delta \Phi_\alpha(\theta)\mathrm{d}\theta.
\end{aligned}
\tag{34}
$$

Thus, the sets $W_{\varepsilon,\delta}(t)=\left\{(G_{\varepsilon,\delta}y)(t),\ y\in\mathscr{D}_q\right\}$ are relatively compact and arbitrarily close to the set $W(t)=\left\{(Gy)(t),\ y\in\mathscr{D}_q\right\}$, for $t\in(0,b]$.

The set $W(t)=\left\{(Gy)(t),\ y\in\mathscr{D}_q\right\}$ is uniformly bounded from (22), and therefore, according to the Ascoli–Arzela theorem, it is relatively compact. Because G is continuous on

$\mathscr{D}_q$ considered G has been completely continuous. Subsequently, by Schauder's fixed point theorem, G has a fixed point $\in \mathscr{D}_q$. Thus, $x = y + \widehat{\varnothing}$ is a fixed point of F in $\mathscr{D}_q$, and multifractional nonlocal impulsive control system (1) has a mild solution and completes the proof.

Now, let us obtain a unique result for the impulsive nonlocal control fractional system (1) by using the Banach contraction principle.

Theorem 2. *We assume that the hypotheses* (A_1)–(A_5) *with the condition* $[(M_1Lb^{\alpha}/\alpha)(1+(s^{\beta_1}/\Gamma(\beta_1+1))+(s^{\beta_2}/\Gamma(\beta_2+1)))+M_1\sum_{i=1}^{s}N_i]<1$ *holds. Thus, for every nonlocal initial condition* $x(t)=\varnothing(t)+g(x)(t)$, $t\in[-r,0]$, *the impulsive multifractional order integrodifferential equations with nonlocal conditions (1) have a unique mild solution* $x\in\mathscr{D}_q$, *where* q *is a positive constant, for every control* $u\in L^2(J,V)$.

Proof. Consider the operator F on $\mathscr{D}_q$ defined as follows:

$$(Fx)(t)=\begin{cases}\varnothing(t), & t\in[-r,0],\\ S_{\alpha}(t)[\varnothing+g(x)]+\int_0^t(t-s)^{\alpha-1}T_{\alpha}(t-s)\left[f\left(s,x_s,I^{\beta_1}(x(\tau)),I^{\beta_2}(x(\tau))\right)+Bu(s)\right]\mathrm{d}s & \\ \quad +\sum_{i=1}^{s}S_{\alpha}(t-t_i)I_i(x(t_i)), & t\in J.\end{cases} \tag{35}$$

Let $x(t)=y(t)+\widehat{\varnothing}(t)$. We define the operator $G\colon PC\longrightarrow PC$ as follows:

$$(Gy)(t)=\begin{cases}0, & t\in[-r,0],\\ \int_0^t(t-s)^{\alpha-1}T_{\alpha}(t-s)\left[f\left(s,y_s+\widehat{\varnothing}_s,I^{\beta_1}(y(\tau)+\widehat{\varnothing}(\tau)),I^{\beta_2}(y(\tau)+\widehat{\varnothing}(\tau))\right)+Bu(s)\right]\mathrm{d}s & \\ \quad +\sum_{i=1}^{s}S_{\alpha}(t-t_i)I_i\left(y(t_i)+\widehat{\varnothing}(t_i)\right), & t\in J.\end{cases} \tag{36}$$

Clearly, the operator F has a unique fixed point and so as G. Thus, it can be seen that, for y_1, $y_2\in PC([-r,b],\mathbb{R})$ and for all $t\in[-r,0]$, we get $(Gy_1)(t)-(Gy_2)(t)=0$. For $t\in[t_k,t_{k+1}]$, we have the following:

$$\begin{aligned}\|(Gy_1)(t)-(Gy_2)(t)\| &= \Bigg\|\int_0^t(t-s)^{\alpha-1}T_{\alpha}(t-s)\Big[f\Big(s,y_{1s}+\widehat{\varnothing}_s,I^{\beta_1}\big(y_1(\tau)+\widehat{\varnothing}(\tau)\big),\\ &\quad I^{\beta_2}\big(y_1(\tau)+\widehat{\varnothing}(\tau)\big)\Big)+Bu(s)\Big]\mathrm{d}s+\sum_{i=1}^{s}S_{\alpha}(t-t_i)I_i\big(y_1(t_i)+\widehat{\varnothing}(t_i)\big)\\ &\quad -\int_0^t(t-s)^{\alpha-1}T_{\alpha}(t-s)\Big[f\Big(s,y_{2s}+\widehat{\varnothing}_s,I^{\beta_1}\big(y_2(\tau)+\widehat{\varnothing}(\tau)\big),\\ &\quad I^{\beta_2}\big(y_2(\tau)+\widehat{\varnothing}(\tau)\big)\Big)+Bu(s)\Big]\mathrm{d}s-\sum_{i=1}^{s}S_{\alpha}(t-t_i)I_i\big(y_2(t_i)+\widehat{\varnothing}(t_i)\big)\Bigg\|\\ &\le M_1\int_0^t(t-s)^{\alpha-1}L\Big(\|y_1-y_2\|+\big\|I^{\beta_1}(y_1(\tau)-y_2(\tau))\big\|\\ &\quad +\big\|I^{\beta_2}(y_1(\tau)-y_2(\tau))\big\|\Big)\mathrm{d}s+M_1\sum_{i=1}^{s}N_i\|y_1(t_i)-y_2(t_i)\|\\ &\le M_1L\int_0^t(t-s)^{\alpha-1}\Bigg(\|y_1-y_2\|+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}\|y_1(s)-y_2(s)\|\\ &\quad +\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\|y_1(s)-y_2(s)\|\Bigg)\mathrm{d}s+M_1\sum_{i=1}^{s}\|N_iy_1(t_i)-y_2(t_i)\|\\ &\le\Bigg[M_1L\int_0^t(t-s)^{\alpha-1}\Bigg(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\Bigg)\mathrm{d}s+M_1\sum_{i=1}^{s}N_i\Bigg]\sup_{t\in[-r,b]}\|y_1(t)-y_2(t)\|\\ &\le\Bigg[\frac{M_1Lb^{\alpha}}{\alpha}\Bigg(1+\frac{s^{\beta_1}}{\Gamma(\beta_1+1)}+\frac{s^{\beta_2}}{\Gamma(\beta_2+1)}\Bigg)+M_1\sum_{i=1}^{s}N_i\Bigg]\|y_1(t)-y_2(t)\|.\end{aligned} \tag{37}$$

Subsequently, G is a contraction operator on $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, s$, and by Banach contraction principle, G has a unique fixed point on $\mathscr{D}_q$, which means that F has a unique fixed point on $\mathscr{D}_q$. □

4. Illustrative Example

Consider the impulsive multifractional order integrodifferential equations with nonlocal conditions and finite delay of the form:

$$\frac{\partial^\alpha}{\partial t^\alpha} z(t,\tau) = \frac{\partial^\beta}{\partial t^\beta} z(t,\tau) + n^*(\tau)u(t,\tau) + f(t, z(t-\bar{t},\tau), I^{\beta_1}k_1(t, w(s, \tau-\bar{t}))\mathrm{d}s, I^{\beta_2}k_2 \cdot (t, w(s, \tau-\bar{t})), \tag{38}$$

For $\tau \in [0,\pi]$, $t \in [0,b]$, $0 < \alpha, \beta, \beta_1, \beta_2 \le 1$,

$$z(t^+,\tau) - z(t^-,\tau) = I_i(z(t^-,\tau)), \quad \tau \in (0,\pi], \; i = 1, \ldots, k, \tag{39}$$

$$z_0(\tau) = \varnothing(\tau) + \int_0^b h(s)\log(1 + |z(s,\tau)|)\mathrm{d}s, \quad t \in [-r, 0], \quad \tau \in [0,\pi], \tag{40}$$

where $r > 0$, $I_i > 0$, $i = 1, \ldots, k$,

$$\varnothing \in \mathscr{D} = \{\varphi\colon [-r,b] \longrightarrow R, \varphi \text{ is continous every where except for a countablenumber of points at which } \varphi(s^-), \varphi(s^+) \text{exist with } \varphi(s^-) = \varphi(s^+)\}$$
$$0 = t_0 < t_1 < t_2 < \cdots < t_{k+1} = b, \; z(t_i^+) = \lim_{(k,\tau)\longrightarrow(0^+,\tau)} z(t_i + h, \tau), \tag{41}$$

$z(t_i^-) = \lim_{(k,\tau)\longrightarrow(0^-,\tau)} z(t_i + h,\tau)$. Let $X = L^2[0,\pi]$ and $A\colon X \longrightarrow X$ be defined by $Aw = w^\beta$ with the $D(A) = \left\{ \begin{array}{c} w \in X\colon w \text{ is absolutely continuous } w^\beta \in X, \\ w(\tau) = w(0) = 0 \end{array} \right\}$, which satisfies the following:

(A_1) It is well known that A is an infinitesimal generator of a semigroup $T(t)$ defined by $T(t)w(s) = w(t+s)$ for each $w \in X$ and $t \ge 0$, $T(t)$ is a compact strongly continuous semigroup on X.

(A_2) B: $X \longrightarrow X$ is a bounded linear control operator which is defined by (Bu) $(\tau) = n^*(\tau)u(\tau)$ for $\tau \in [0,\pi]$. $u \in L^2(J,X)$ and $\|B\| \le M_2$, $\|u\| \le M_3$.

$$\begin{aligned}(A_3)\ F(t,z,h_1,h_2)(\tau) &= f(t, z(\tau,t), h_1(\tau,t), h_2(\tau,t)) \\ h_1(\tau,t) &= I^{\beta_1}k_1(t, w(s, \tau-\bar{t}))\mathrm{d}s \\ h_2(\tau,t) &= I^{\beta_2}k_2(t, w(s, \tau-\bar{t}))\mathrm{d}s.\end{aligned} \tag{42}$$

Take $f(t, z(\tau,t), h_1(\tau,t), h_2(\tau,t)) = c_0 \sin(z(t)), c_0$ is a constant, F is continuous and for all $z, h_1 \in \mathscr{D} \times X$, $F(., z, h_1, h_2)\colon J \longrightarrow X$ is strongly measurable:

$$\begin{aligned}&\|F(t,z,h_1,h_2)(\tau)\| = c_0\|\sin(z(t))\| \le c_0, \\ &\text{where } \mu(t)(\|z\| + \|h_1\| + \|h_2\|) = c_0, \; z, h_1, h_2 \in PC(J, X).\end{aligned} \tag{43}$$

(A_4) $g\colon PC([a,b]\colon X) \longrightarrow X$ is a continuous function which is defined by

$$\begin{aligned}g(\sigma)(\tau) &= \int_0^b h(s)\log(1 + |\sigma(s)(\tau)|)\mathrm{d}s, \\ &\sigma \in PC([0,b]\colon X)\sigma(s)(\tau) = z(s,\tau).\end{aligned} \tag{44}$$

Then, g is a compact operator and $\|g(\sigma)(\tau)\| = \|\int_0^b h(s)\log(1 + |\sigma(s)(\tau)|)\mathrm{d}s\| \le |h(s)|\log 1 + |\sigma(s)(\tau)| \le \|\sigma(s)(\tau)\|$. Hence, $L = 1$ and $L' = 0$.

(A_5) $I_i\colon \mathscr{D} \longrightarrow X$ is a continuous for each $i = 1, \ldots, k$ such that

$I_i(z)(\tau) = \int_{[0,\pi]} \rho_i(\tau,y)\cos^2(z(y)\mathrm{d}y$, $z \in X$ for each $i = 1, \ldots, k$. Then,

$$\left\|\int_{[0,\pi]} \rho_i(\tau,y)\cos^2(z(y)\mathrm{d}y\right\| \le \left\|\int_{[0,\pi]} \rho_i(\tau,y)\mathrm{d}y\right\| \|z(y)\|. \tag{45}$$

Let $N_i\colon R^+ \longrightarrow R^+$ $L_i(\|z(y)\|_{\mathscr{D}}) = \|\int_{[0,\pi]} \rho_i(\tau,y)\mathrm{d}y\|$, $i = 1, \ldots, k$, nondecreasing function.

(A_6) Choose the Condition

$$\left[\frac{M_1 b^\alpha}{\alpha}\left(1 + \frac{b^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{b^{\beta_2}}{\Gamma(\beta_2+1)}\right) + M_1 \sum_{i=1}^k N_i\right] < 1, \quad 0 < \alpha, \beta_1, \beta_2 \le 1. \tag{46}$$

Therefore, the nonlocal multifractional partial impulsive differential systems (38)–(40) can be written in the abstract forms (1)–(3), and all conditions of Theorems 1 and 2 are satisfied.

5. Conclusion

The existence and uniqueness of multifractional order impulsive integrodifferential equations with nonlocal conditions and infinite delay using Schauder's fixed point theorem required certain types of semigroups defined on probability density functions and Holder's inequality ρ-mean continuity as well as some necessary and sufficient estimators conditions that play an important role in guaranteeing the solution.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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