ON \( k \)-TRESTLES IN POLYHEDRAL GRAPHS

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Abstract

A \( k \)-trestle of a graph \( G \) is a 2-connected spanning subgraph of \( G \) of maximum degree at most \( k \). We show that a polyhedral graph \( G \) has a 3-trestle, if the separator-hypergraph of \( G \) contains no two different cycles joined by a path of 3-separators of length \( \geq 0 \). There are graphs not satisfying this condition that have no 3-trestles. Further, for each integer \( k \) every graph with toughness smaller than \( \frac{2}{k} \) has no \( k \)-trestle.

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1. Introduction

By Steinitz’s theorem a polyhedral graph is a planar and 3-connected graph. Let \( G \) be a connected graph. A subset \( S \) of the vertex set of \( G \) separates \( G \) if the graph \( G - S \) obtained from \( G \) by deleting the vertices of \( S \) is disconnected. If \( |S| = k \), \( S \) is said to be a \( k \)-separator of \( G \). If no
$S_p \subset S$ (a proper subset of the set $S$) separates $G$ then the $S$ is said to be a proper $k$-separator of $G$. A subgraph of $G$ is a spanning subgraph of $G$ if it contains all vertices of $G$. 2-connected spanning subgraphs in which all vertices have degree at most $k$ are called $k$-trestles. We will say that a graph $G$ is $k$-trestled if $G$ has a $k$-trestle [6]. Note that a graph $G$ has a 2-trestle if and only if $G$ is Hamiltonian.

A graph $G$ is said to be $t$-tough if for every separating set $S \subseteq V(G)$ the number $\omega(G - S)$ of components of $G - S$ is at most $\frac{|S|}{t}$. The toughness $\tau(G)$ of a non-complete graph $G$ is defined to be the largest integer $t > 0$ such that $G$ is $t$-tough. For a complete graph $G$ let $\tau(G) = \infty$. The concept of toughness was introduced by Chvátal [4]. It is easy to see that every graph with toughness less than one has no 2-trestles. The following Lemma shows that every graph has a similar property with respect to $k$-trestles, $k \geq 3$.

**Lemma 1.** Every graph $G$ with toughness $\tau(G) < \frac{2}{k}$ (where the integer $k$ is greater than one) has no $k$-trestle.

In [4] Chvátal conjectured:

**Conjecture 1 (Chvátal).** There is a real number $t_0 > 0$ such that every $t_0$-tough graph has a Hamiltonian cycle, i.e., a 2-trestle.

It seems to be interesting to consider relations between $t$-tough and $k$-trestled graphs in general. We pose the following conjecture.

**Conjecture 2.** For every integer $k$ greater than one there is a real number $t_k > 0$ such that every $t_k$-tough graph has a $k$-trestle.

There are several papers which deal with $k$-trestled polyhedral graphs. In [1] Barnette showed that there is a polyhedral graph with no 5-trestles. In [5] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle.

The well known theorem of Tutte [8] states that every 4-connected planar graph contains a Hamiltonian cycle, which means that every polyhedral graph with no 3-separators has a 2-trestle. Moreover, Tutte [8] proved

**Theorem 1.** Let $G$ be a 4-connected planar graph and let $e$ and $f$ be two edges of a facial cycle of $G$. Then $G$ has a Hamiltonian cycle through $e$ and $f$. 
Let $H_1$ and $H_2$ be two disjoint subsets of the vertex set $V(G)$ of a graph $G$. The length of a minimal path in $G$ with one end in $H_1$ and the second in $H_2$ is said to be the distance of $H_1$ and $H_2$ in $G$.

Böhmke, Harant and Tkáč in [3] showed that every maximal planar graph $G$ in which no 3-separator has any common vertex with a proper 4-separator and every two distinct 3-separators have distance at least three, has a 2-trestle. In [2] Böhmke and Harant presented examples of maximal planar graphs with no 2-trestles in which the minimal distances between two 3-separators are arbitrarily large.

Our next theorems partially supplement these results but in a more general case.

For each polyhedral graph $G$ we will construct a separator-hypergraph $\mathcal{H}(G)$ with the same set of vertices, such that the edges of $\mathcal{H}(G)$ are the 3-separators of $G$. A cycle (and a path) of a hypergraph is a sequence $P_1e_1P_2e_2\cdots P_ke_kP_{k+1}$, where $P_1, P_2, \ldots, P_k$ are pairwise distinct vertices, $e_1, e_2, \ldots, e_k$ are pairwise distinct edges, the edge $e_i$ is incident with both $P_i$ and $P_{i+1}$, $1 \leq i \leq k$, and $P_{k+1} = P_1$ (and $P_{k+1} \notin \{P_1, P_2, \ldots, P_k\}$, respectively).

**Theorem 2.** Let $G$ be a polyhedral graph. Let each component of the separator-hypergraph $\mathcal{H}(G)$ have at most one cycle. Then $G$ has a 3-trestle.

**Theorem 3.** There are polyhedral graphs with more than one cycle in their separator-hypergraph which have no 3-trestles.

The polyhedral graphs constructed for Theorem 3 have separator-hypergraphs with many cycles; even 2-cycles are present.

### 2. Proofs of Theorems

**The Proof of Lemma 1.** Let $G$ be a graph with toughness $\tau(G) < \frac{2}{k}$ (where the integer $k$ is greater than one). Suppose that $G$ has a $k$-trestle $H$. Since $\tau(G) < \frac{2}{k}$ there exists a subset $S_0$ of the vertex set of $G$ ($S_0 \subset V(G)$) with

$$\frac{|S_0|}{\omega(G - S_0)} = \tau(G) < \frac{2}{k}.$$  

So $G$ contains a vertex set $S_0$ such that

$$2\omega(G - S_0) > k|S_0|.$$
If $G$ has a $k$-trestle $H$ then $S_0 \subset V(G) = V(H)$ and every vertex from $S_0$ has in $H$ a degree at most $k$. Since $H$ is 2-connected, every component of $G - S_0$ is adjacent with at least two vertices from $S_0$. This means that the following inequality holds

$$2\omega(G - S_0) \leq k|S_0|.$$  

But this contradicts the before stated inequality.

Instead of Theorem 2 we shall prove the slightly stronger but more technical Theorem 4.

**Theorem 4.** Let $G$ be a polyhedral graph. Let each component of the separator-hypergraph $H(G)$ have at most one cycle. Label a vertex in each cycle-free component of $H(G)$. Then $G$ has a 3-trestle $H$ such that every 3-valent vertex of $H$ is an unlabelled vertex of a 3-separator in $G$.

**The Proof of Theorem 4.** The proof is by induction on the number of 3-separators of the considered graphs. If $G$ has no 3-separator then $G$ is 4-connected and by Tutte’s Theorem 1 the graph $G$ has a Hamiltonian cycle. Thus $G$ has a special 3-trestle with the required properties.

Assume that Theorem 4 is true for all polyhedral graphs with at most $m$ 3-separators, $m \geq 0$. Let $G$ be a polyhedral graph with $m + 1$ 3-separators such that each component of the ”separator”-hypergraph $H(G)$ has at most one cycle.

A 3-separator $S = \{x, y, z\}$ is called elementary if one component $I(S)$ of $G - S$ has no 3-separators. W.l.o.g. we may suppose that $G$ is mapped into the plane so that $I(S)$ is the interior of the cycle $(x, y, z)$. Now we prove the following

**Claim 1.** If $S = \{x, y, z\}$ is an elementary 3-separator of $G$ then $(I(S) \cup S)_G$, the subgraph induced by $I(S) \cup S$ in $G$, contains an $x, y$-path through all vertices of $I(S) \cup S \setminus \{z\}$ avoiding $z$.

**Proof of Claim 1.** Since $S = \{x, y, z\}$ is elementary the subgraph $H := (I(S) \cup S)_G \cup (x, y, z)$ has no 3-separators and $H$ is 4-connected or $K_4$ (a complete graph on four vertices). By Tutte’s Theorem 1 the subgraph $H$ has a Hamiltonian cycle $h$ through the edges $(x, z)$ and $(z, y)$. The path $p = h \setminus \{z\}$ has the required properties, and the proof of Claim 1 is complete.
The graph $G$ obviously contains an elementary 3-separator $S = \{x, y, z\}$. This 3-separator $S$ is a hyperedge of a component $K$ of $\mathcal{H}(G)$.

**Case 1.** Let $K$ have no cycle in $\mathcal{H}(G)$.

The subhypergraph $K \setminus \{S\}$ of $\mathcal{H}(G)$ has at most three cycle-free components $K_x, K_y$ and $K_z$ containing $x, y,$ and $z$, respectively. Note that some of these components can be trivial. W.l.o.g. let $K_x$ have the vertex with the label of $K$ (it may be that $x$ has this label). In $K_y$ and $K_z$ we label the vertices $y$ and $z$, respectively.

**Case 2.** Let $K$ have a cycle $C$ in $\mathcal{H}(G)$.

Note that $K$ has no label.

**Case 2.1.** Let $S \notin C$.

The subhypergraph $K \setminus \{S\}$ of $\mathcal{H}(G)$ has at most three components $K_x, K_y$ and $K_z$ containing $x, y,$ and $z$, respectively. W.l.o.g. let $C \subseteq K_x$, and $K_y$, $K_z$ are cycle-free in $\mathcal{H}(G)$. In $K_y$ and $K_z$ we label the vertices $y$ and $z$, respectively.

**Case 2.2.** Let $S \in C$.

Two vertices of $S$ belong to $C$, say, $x$ and $y$. The subhypergraph $K \setminus S$ of $\mathcal{H}(G)$ has at most two components $K_{x,y}$ and $K_z$ containing $\{x, y\}$ or $\{z\}$, respectively. The path $C \setminus \{S\} \subseteq K_{x,y}$ and both components $K_{x,y}$ and $K_z$ are cycle-free in $\mathcal{H}(G)$. We label $y$ and $z$.

In all cases we proceed in the same way.

The graphs $G_1$ and $G_2$ are obtained from $G$ by deleting the interior or the exterior of $(x, y, z)$, respectively, and adding the cycle $(x, y, z)$. Thus $G$ has a separation: $G = G_1 \cup G_2$, $G_1 \cap G_2 = (x, y, z)$, $K \setminus \{S\} \subseteq G_1$.

By the induction hypothesis $G_1$ contains a 3-trestle $T_1$ with the required properties. The degrees $\deg_{T_1}(y) = \deg_{T_1}(z) = 2$.

By Claim 1 the subgraph $G_2$ contains a $y, z$-path $T_2$ through all vertices of $G_2 \setminus \{x\}$ avoiding $x$. Then $T_1 \cup T_2$ is a 3-trestle of $G$ with the required properties. ■

**The Proof of Theorem 3.** Theorem 3 will be proved by constructing an appropriate graph. A double-cube is obtained from two disjoint copies $C_1$ and $C_2$ of the cube by identifying a face of $C_1$ with a face of $C_2$. This polyhedral graph has $n = 12$ vertices and $f = 10$ quadrangles. In each quadrangle with bounding 4-cycle $(v_0, v_1, v_2, v_3)$ we introduce a 4-cycle
(w_0, w_1, w_2, w_3) so that for every \( i \) (mod 4) a vertex \( v_i \) is connected with \( w_i \) and \( w_{i+1} \) by an edge, introduce a new vertex \( \alpha_i \) in each triangle face with bounding cycle \((v_i, v_{i+1}, w_{i+1})\) and join \( \alpha_i \) to each vertex of the bounding 3-cycle \((v_i, v_{i+1}, w_{i+1})\) by an edge.

The resulting graph \( H \) is polyhedral and its connected separator-hypergraph has more than one cycle.

We claim that \( H \) has no 3-trestle.

Suppose \( H \) has a 3-trestle \( T \). By construction each vertex \( \alpha_i \) is joined to the vertex \( v_i \) or \( v_{i+1} \) of the double-cube by at least one edge of \( T \). Thus the subgraph \( T \) has at least \( 4f \) such edges. Consequently, the double-cube has at least one vertex \( v \) of degree

\[
\deg_T(v) \geq \frac{4f}{n} = \frac{40}{12} > 3.
\]

Hence \( v \) has a degree \( \deg_T(v) \geq 4 \) and \( T \) is no 3-trestle. This contradiction shows that \( H \) has no 3-trestle.

Starting our construction with \( l \geq 3 \) cubes results in an infinite sequence of graphs satisfying Theorem 3.

\[ \square \]

References


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