

Multiple limit cycles in a Leslie–Gower-type predator–prey model considering weak Allee effect on prey*

Eduardo González-Olivares, Alejandro Rojas-Palma, Betsabé González-Yañez

Pontificia Universidad Católica de Valparaíso,
Avenida Brasil 2950, Valparaíso, Chile
eduardo.gonzalez@pucv.cl; alejandro.rojas.p@mail.ucv.cl; betsabe.gonzalez@pucv.cl

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Abstract. In this work, a modified Leslie–Gower-type predator–prey model is analyzed, considering now that the prey population is affected by a *weak Allee effect*, complementing results obtained in previous papers in which the consequences of strong Allee effect for the same model were established.

In order to simplify the calculations, a diffeomorphism is constructed to obtain a topological equivalent system for which we establish the boundedness of solutions, the nature of equilibrium points, the existence of a separatrix curve dividing the behavior of trajectories. Also, the existence of two concentric limit cycles surrounding a unique positive equilibrium point (generalized Hopf or Bautin bifurcation) is shown.

Although the equilibrium point associated to the weak Allee effect lies in the second quadrant, the model has a rich dynamics due to this phenomenon, such as it happens when a strong Allee effect is considered in prey population.

The model here analyzed has some similar behaviors with the model considering strong Allee effect, having both two limit cycles; nevertheless, they differ in the amount of positive equilibrium points and the existence in our model of a non-infinitesimal limit cycle, which exists when the positive equilibrium is a repeller node. The main results obtained are reinforced by means of some numerical simulations.

Keywords: bifurcations, limit cycle, separatrix curve, Allee effect, Leslie–Gower predator–prey models.

1 Introduction

One of the main problems in Population Dynamics is the study of the dynamical behaviour of predator–prey interactions. Varied models have been proposed for this interaction, being one of them established in the seminal work by P.H. Leslie in 1948 [28] and discussed by Leslie and Gower in 1960 [29], which is not conforming to the Lotka–Volterra framework [38].

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The principal characteristic of Leslie [32] or Leslie–Gower-type model [6] is that the equation describing the predator growth is based on a direct generalization of the logistic equation [38], assuming that the environmental carrying capacity of predators K_y is directly proportional to the available prey quantity [38], that is, $K_y(x) = nx$, where $x = x(t)$ indicates the prey population size at time $t \geq 0$ [2, 3].

This model retains the prey equation from the Volterra model [38] and the *functional response*, the change in the density of prey attacked per unit time per predator, as the prey density changes [32], is expressed by the function $h(x) = qx$ with $q > 0$. Implicitly, this assumption considers that the predator subsistence exclusively depends on prey population, i.e., the predator is specialist [38].

Meanwhile, the predator population growth rate depends on the predator population size ratio to the prey population size, a positive growth rate is predicted when the absolute prey and predator densities are significantly low [38]. In spite of anomaly, these models have been employed to describe a real predator–prey interaction without considering this objection [25].

It is well known, the Leslie–Gower model has a unique equilibrium point positive (in the interior of the first quadrant), which is globally asymptotically stable [38]. Nonetheless, it can assume that predators are generalist having an alternative food when its preferred prey is scarce. So, the environmental carrying capacity is given by $K_y(x) = nx + c$ with $c > 0$; it is said that the model is represented by a Leslie–Gower scheme, also known as modified Leslie–Gower model [4, 6].

In this work, the mathematical results obtained in a previous paper [21] are complemented in which the consequences of strong Allee effect for the Leslie–Gower-type predator–prey model were established. Any mechanism leading to a positive relationship between a component of individual fitness and the number or density of conspecifics can be named as a *Allee effect* [37].

It describes a biological scenario characterized by a positive correlation between the population sizes and its per capita growth rate at low population sizes, increasing their likelihood of extinction [13]; it has been denominated in different ways in Population Dynamics [30] and *depensation* in Fisheries Sciences [11, 17, 30].

The main characterization of Allee effect is that the per capita growth rate is positive for low population sizes. This is a common phenomenon in some animal populations and various mechanisms have been proposed as potential sources of this biological phenomena (Table 1 in [8] or Table 2.1 in [12]).

Recent ecological research suggest that two or more Allee effects can lead to these mechanisms acting simultaneously on a single population (Table 2 in [8]); the combined influence of some of these phenomena is known as *multiple (double) Allee effect* [8, 20].

This phenomenon has become crucial for population dynamics since in fact it has a surprising number of ramifications towards different branches of ecology [12], and the knowledge of this effect on simple models is essential to understand more complicated ones, specially when the weak Allee effect is considered [31].

To model the growth rate of a population affected by the Allee effect, the most simplest mathematical form will be employed, which is described by the cubic polynomial

differential equation [7, 24, 26]

$$\frac{dx}{dt} = r \left(1 - \frac{x}{K} \right) (x - m)x, \quad (1)$$

where $-K < m \ll K$ and $r, K > 0$. When $m > 0$, the population growth rate decreases if the population size is below the *threshold level* m and the population goes to extinction. In this case, equation (1) describes the *strong Allee effect* [39–41]. If $m \leq 0$, it is said that the population is affected by a *weak Allee effect* [11, 18, 41]. In Fisheries Sciences, the same phenomena are called *critical* and *pure depensation*, respectively [11, 30, 41].

The function $f(x) = dx/(x dt)$, expresses the per capita growth rate; clearly, for $-K < m \ll K$. This function grows as the population size increases from $x \geq 0$ until it attains a maximum value; then, it decreases for larger population size, as the logistic growth function. Notice that if $m < -K$, the per capita growth rate $f(x)$ is negative for all $x \geq 0$, and the growth function is of compensatory type [11] as the logistic growth function and equation (1) ceases of to represent the weak Allee effect [23]. But, mathematically the added factor $(x - m)$ have influence in the dynamics of this equation [26].

Populations with weak Allee dynamic experimenting lower per capita growth rates at low densities, but never lead to negative per capita growth rates and therefore have no critical threshold to be exceed for population to survive [34]. To our knowledge no other work studies the weak Allee effect considering $m < 0$ in equation (1). Other articles mentioning weak Allee effect in continuous time models, only consider the special case $m = 0$ [22] or take other expression to describe this effect [27, 31, 43].

Nevertheless, the study of weak Allee effect is usually neglected since the per capita population growth rate remains positive for any population sizes such as it happens with the logistic growth [23].

Moreover, there is relatively limited evidences of strong Allee effects in natural populations, and thus for a key part of the theoretical framework for Allee effects [34] and a weak Allee effect may appear over a given range of densities [41]; so, the influence on dynamics of model is especially important to study.

Specifically, we would like to know how the weak Allee effect affect the population dynamics of both prey and predator; particularly if the weak form of this phenomenon can help to both populations avoid the extinction.

Different mathematical forms have been proposed to model the Allee effect [8] and most of them are topologically equivalent [19]. The consequences of this different mathematical formalizations has been studied for Gause-type predator–prey models in [20, 22, 35, 43, 44] and for Leslie–Gower in [21, 33]. Particularly, in a previous work, we have proved that by considering two different ways for describe the Allee effect, in a same predator–prey model, changes in the quantity of limit cycles surrounding a positive equilibrium point can occur [20]. We consider, the determination of the number of limit cycles bifurcating from a weak focus is a good criterion for the classification of predator–prey models [23].

The rest of this paper is organized as follows: The proposed model is presented in the next section; the main properties are established in Section 3, and its consequences are shown in Section 4. In addition, simulations are given to verify the main results.

2 The model

When both the Allee effect phenomenon on prey and the functional response are modelled in the most simple way, the Leslie–Gower predator–prey model is given by the following bidimensional differential equation system:

$$X_\mu: \begin{cases} \frac{dx}{dt} = (r(1 - \frac{x}{K})(x - m) - qy)x, \\ \frac{dy}{dt} = s(1 - \frac{y}{nx})y, \end{cases} \quad (2)$$

where $x = x(t)$ and $y = y(t)$ represent the prey and predators population size, respectively, for $t \geq 0$ (measured as biomass, number of individuals or density by unit of area or volume) with $\mu = (r, K, q, s, n, m) \in \mathbb{R}_+^5 \times]-K, K[$ and the parameters having different biological meanings [21].

In this work, for the analysis of system (2), we will be consider the prey population is affected by a *weak Allee effect* assuming that $-K < m < 0$, i.e., $m \in]-K, 0]$. The particular case of weak Allee effect when $m = 0$ was studied in [21]; thus, the present work complements the results obtained in our previous paper [21]. Notice that if predator is generalist (considering $c > 0$ in the predator carrying capacity K_y), the model would have a new equilibrium point $(0, c)$ over the y -axis; the implications of the added parameter will be not studied here.

For the strong Allee effect, conditions such that an equilibrium point corresponding to a positive steady state may undergo saddle-node, Hopf and Bogdanov–Takens bifurcations are given in [1] for Leslie–Gower-type predator–prey model. There the Allee effect is modelled by means of a general term that multiplies the classic logistic growth rate in the prey equation (*multiplicative Allee effect*) as equation (1). Nevertheless, we orientate our analysis to describe the behavior of system (2) by a classification in the parameter space (diagram bifurcation), mainly interested in to establish the number of limit cycles encircling a positive equilibrium point when $m < 0$.

It is clear that model (1) is a Kolmogorov-type system [15] not defined at the y -axis, particularly at the point $(0, 0)$, but both isoclines of system (1) pass through this point; however, $(0, 0)$ is a point of particular interest having a strong incidence on behavior of system (3) or vector field X_μ , which is defined on the set:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq 0\} = \mathbb{R}^+ \times \mathbb{R}_0^+.$$

The equilibrium points of system (2) in Ω are $(K, 0)$ and (x_e, nx_e) , where x_e satisfies the quadratic equation

$$rx^2 - (Kr + mr - Knq)x + Kmr = 0.$$

Then, the equation can have positive roots if and only if $Kr + mr - Knq > 0$ and $\delta = (Kr + mr - Knq)^2 - 4Kmr^2 \geq 0$, or none if $Kr + mr - Knq \leq 0$. We note that $(m, 0)$ lies in the second quadrant.

In order to simplify the calculations, we make a change of variables and a time rescaling [20, 24, 33, 36] described in the following.

Proposition 1. *System (2) is topologically equivalent to the polynomial system given by*

$$Z_\eta(u, v): \begin{cases} \frac{du}{d\tau} = ((1 - u)(u - M) - Qv)u^2, \\ \frac{dv}{d\tau} = S(u - v)v, \end{cases} \quad (3)$$

where $\eta = (Q, S, M) \in \mathbb{R}_+^2 \times]-1, 0[$ with $M = m/K$, $Q = qn/r$ and $S = s/(rK)$. Moreover, system (3) is defined on the set

$$\bar{\Omega} = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0, v \geq 0\} = \mathbb{R}_0^+ \times \mathbb{R}_0^+.$$

Proof. Let $x = Ku$ and $y = nKv$ the new dimensionless variables. Substituting into system (1) and simplifying, we obtain

$$U_\mu(x, y): \begin{cases} K \frac{du}{dt} = (r(1 - u)(Ku - m) - qnKv)Ku, \\ nK \frac{dv}{dt} = s(1 - \frac{nKv}{nKu})nKv. \end{cases}$$

Simplify and factoring

$$U_\mu(x, y): \begin{cases} \frac{du}{dt} = rK((1 - u)(u - \frac{m}{K}) - \frac{qn}{r}v)u \\ \frac{dv}{dt} = s(1 - \frac{v}{u})v. \end{cases}$$

Now, let $\tau = rK/ut$, then, $du/dt = (du/d\tau)(d\tau/dt) = (du/d\tau)(rK/u)$ and $dv/dt = (dv/d\tau)(d\tau/dt) = (dv/d\tau)(rK/u)$. Rearranging and simplifying,

$$V_\mu(x, y): \begin{cases} \frac{du}{d\tau} = ((1 - u)(u - \frac{m}{K}) - \frac{qn}{r}v)u^2, \\ \frac{dv}{d\tau} = \frac{s}{rK}(u - v)v. \end{cases}$$

Under the indicated substitution, we obtain system (3) defined on the set $\bar{\Omega}$. □

Remark 1. 1. We have constructed the diffeomorphism [10, 14] $\varphi : \bar{\Omega} \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}$, so that

$$\varphi(u, v, \tau) = \left(Ku, nKv, \frac{u}{rK}\tau \right) = (x, y, t).$$

The Jacobian matrix of φ is

$$D\varphi(u, v, \tau) = \begin{pmatrix} K & 0 & 0 \\ 0 & nK & 0 \\ \frac{1}{rK}\tau & 0 & \frac{u}{rK} \end{pmatrix},$$

and we have that $\det D\varphi(u, v, \tau) = nKu/r > 0$. Then, the diffeomorphism φ is a smooth change of variables with a rescaling of the time preserving the time orientation; thus, the vector field $X_\mu(x, y)$, is topologically equivalent to the vector field $Z_\eta = \varphi \circ X_\mu$ with $Z_\eta(u, v) = P(u, v)\partial/\partial u + Q(u, v)\partial/\partial v$ and the associated differential equation system is given by the polynomial system (3) of fourth degree and of Kolmogorov type.

2. This parameterization and time rescaling permits us to obtain a representative system, with the least amount of parameters possible, which describes the dynamical behaviors of all those systems that are topologically equivalent to system (2). Then, more important than knowing the influence of a particular parameter in the dynamical behavior of system (2), it is more useful the knowledge of the relationships between some of them for a better view of their implications in the properties of system.

The singularities of system (3) in $\tilde{\Omega}$ are $(0, 0)$, $(1, 0)$ and (u_e, u_e) , where u_e satisfies the equation

$$u^2 - (1 + M - Q)u + M = 0. \quad (4)$$

The point $(M, 0)$ lies in the second quadrant, and it has no ecological interest.

3 Main results

For the vector field Z_η or system (3) with $M < 0$, we have

Lemma 1.

1. The set $\hat{\Gamma} = \{(u, v) \in (\mathbb{R}_0^+)^2 \mid 0 \leq u \leq 1, v \geq 0\}$ is an invariant region.
2. The solutions are bounded.

Proof. 1. System (3) is of Kolmogorov type, then the axis are invariant sets. Considering $u = 1$, we have that

$$\frac{du}{d\tau} = -Qvu^2, \quad \frac{dv}{d\tau} = S(1-v)v,$$

where $du/d\tau < 0$, anything the sign of $dv/d\tau$, all trajectories point to $\hat{\Gamma}$.

2. We define the new function $w(\tau) = u(\tau) + v(\tau)/S$; the time derivative of w along a solution of system (3) is

$$\frac{dw}{d\tau} = \frac{du}{d\tau} + \frac{1}{S} \frac{dv}{d\tau} = ((1-u)(u-M) - Qv)u^2 + (u-v)v.$$

Now, $dw/d\tau + \alpha w = ((1-u)(u-M) - Qv)u^2 + (u-v)v + \alpha(u + v/S)$ with α a parameter for determine.

Collecting the expression in the right of the last equality respect to u and after some simple algebraic manipulations, we obtain

$$\begin{aligned} \frac{dw}{d\tau} + \alpha w &= -u^2(u^2 - (M+1)u) - (M+Qv)u^2 + (v+\alpha)u - \left(v^2 - \frac{\alpha}{S}v\right) \\ &= -u^2\left(u - \frac{1+M}{2}\right)^2 + \frac{(M+1)^2}{4}u^2 - \left(v - \frac{1}{2S}\alpha\right)^2 + \left(\frac{1}{2S}\alpha\right)^2 \\ &\quad - (M+Qv)u^2 + (v+\alpha)u. \end{aligned}$$

Choosing $\alpha = M/Q$ it has

$$\begin{aligned} \frac{dw}{d\tau} + \alpha w = & -\left(u - \frac{1+M}{2}\right)^2 + \frac{(M+1)^2}{4}u^2 - \left(v^2 - \frac{1}{2S} \frac{M}{Q}\right)^2 + \frac{1}{4S^2} \frac{M^2}{Q^2} \\ & - (M+Qv)\left(u - \frac{1}{2Q}\right)^2 + (M+Qv)\frac{1}{4Q^2}. \end{aligned}$$

Then, considering $0 \leq u \leq 1$ and as v satisfy the second equation of system (3), it has $0 \leq v \leq 1$; thus,

$$\frac{dw}{d\tau} + \frac{M}{Q}w \leq \frac{(M+1)^2}{4} + \frac{1}{4S^2} \frac{M^2}{Q^2} + (M+Q)\frac{1}{4Q^2}.$$

Let $\delta = (M+1)^2/4 + M^2/(4S^2Q^2) + (M+Q)/(4Q^2)$. Thus, there exists $\delta > 0$ such as

$$0 \leq \frac{dw}{d\tau} + \frac{M}{Q}w \leq \delta,$$

being a first-order linear inequality.

Applying the theorem on differential inequality [9], we obtain

$$w(u, v) \leq \frac{Q\delta}{M} + \left(w(u(0), v(0)) - \frac{Q\delta}{M}\right) \exp\left(-\frac{M}{Q}\tau\right),$$

Clearly, when $\tau \rightarrow \infty$, $0 \leq w(u, v) \leq Q\delta/M$ and the solutions are bounded. □

The result above assures that there exists a set

$$B = \left\{ (u, v) \in \bar{\Omega} \mid w(u, v) \leq \frac{Q\delta}{M} + \epsilon \ \forall \epsilon > 0 \right\}$$

such that B is the region in which all the solutions of system (3) with initial conditions in $\bar{\Omega}$ are confined. This property assures that the model is well-posed, i.e., it is no possible the existence of many predators and a little prey population size, which would be a counterintuitive situation, not sustainable ecologically.

We recall that if $M \geq 0$, system (3) can have until two positive equilibrium points [21]. The number of positives singularities of system (3) differs when $M < 0$ or $M \geq 0$ as is demonstrated in the following lemma.

Lemma 2. *There exists a unique positive equilibrium point.*

Proof. Let $A = 1 + M - Q$, then $W^2 = (1 + M - Q)^2 - 4M = A^2 - 4M > 0$.

(a) For $M < 0$, equation (4) has a unique real root for any sign of A given by

$$u_e = \frac{1}{2}(A + W) = H$$

with $0 < H \leq 1$ and $u_{e1} = (A - W)/2 < 0$.

- (b) If $A = 0$, then the unique solution is $u_e = \sqrt{-M}$. Therefore, system (3) has a unique positive equilibrium point $(H, H) = (\sqrt{-M}, \sqrt{-M})$.

From each case above, system (3) has only one positive equilibrium point. \square

The Jacobian matrix of system (3) necessary to obtain the nature of equilibrium points is

$$DZ_\eta(u, v) = \begin{pmatrix} DZ_\eta(u, v)11 & -Qu^2 \\ Sv & S(u - 2v) \end{pmatrix}$$

with $DZ_\eta(u, v)11 = (-4u^2 + 3(M + 1)u - (M + Qv))u$.

3.1 Nature of equilibrium points

Lemma 3. *For all parameter values, the point $(1, 0)$ is a hyperbolic saddle point.*

Proof. Evaluating the Jacobian matrix, we obtain

$$DZ_\eta(1, 0) = \begin{pmatrix} 2M - 1 & -Q \\ 0 & S \end{pmatrix},$$

then $\det DZ_\eta(1, 0) = (2M - 1)S < 0$. \square

Lemma 4. *The point $(0, 0)$ is a non hyperbolic equilibrium point having a hyperbolic and a parabolic sector determined for the line $v = ((S + M)/S)u$, i.e., there exists a separatrix curve Σ in the phase plane that divides the behavior of trajectories.*

Proof. The Jacobian matrix is the null matrix and the proof follows as that given for the case $M > 0$ [21]. \square

Notice that the equilibrium $(0, 0)$ of the vector field Z_η is a non hyperbolic attractor point. Moreover, the straight line $v = ((S + M)/S)u$ is tangent to the separatrix curve in the phase plane dividing the behavior of the trajectories; thus, the point $(0, 0)$ is an attractor point for certain trajectories and a saddle point for others, but the trajectories above this separatrix have different α -limit sets.

As $M < 0$, the slope of this line is positive for $S > -M > 0$ and negative for $0 < S < -M$; in the last case, lies in the second quadrant.

Lemma 5. *For an open subset of parameter values, there exists a heteroclinic cycle γ_h in the first quadrant containing the equilibria $(0, 0)$ and $(1, 0)$.*

Proof. As $(1, 0)$ is saddle point and $(0, 0)$ is a non hyperbolic saddle point, then their corresponding unstable manifold $W^u(1, 0)$ and stable manifold $W^s(0, 0) = \Sigma$ are both one-dimensional objects.

Clearly, the α -limit of $W^s(0, 0)$ and the ω -limit of $W^u(1, 0)$ are bounded in the direction of the v -axis. Neither the ω -limit of $W^u(1, 0)$ is on the u -axis. Let u^* be such that $0 < u^* < 1$. Then, there are points $(u^*, v_s) \in W^s(0, 0)$ and $(u^*, v_u) \in W^u(1, 0)$

with v_s and v_u depending on the parameter values such that $v_s = s(Q, S, M)$ and $v_u = u(Q, S, M)$.

It is easy to see that, if $0 < u^* \ll 1$, then $v_s < v_u$ and the stable manifold $W^s(M, 0)$ is below the unstable manifold $W^u(1, 0)$. If $M \ll u^* < 1$, then $v_s > v_u$ and the stable manifold $W^s(M, 0)$ is above the unstable manifold $W^u(1, 0)$. Since the vector field Y_η is continuous respect to the parameters, the stable manifold $W^s(0, 0)$ intersect the unstable manifold $W^u(1, 0)$ for some parameter values.

Hence, there exists $(u^*, v^*) \in \bar{\Gamma}$ such that $v_s^* = v_u^*$. Moreover, by uniqueness of solutions of system (3), this intersection must occur along a whole trajectory γ_{1M} . Therefore, the equation $s(Q, S, M) = u(Q, S, M)$ defines a codimension-one submanifold in the parameters space for which the heteroclinic curve γ_{10} exists in \mathbb{R}_+^2 , connecting the points $(1, 0)$ and $(0, 0)$. Then, $\gamma_{10} \subset W^s(0, 0) \cap W^u(1, 0)$ and it lies entirely on a segment of the u -axis and exists for any parameter value such that $0 < H < 1$. It follows that a heteroclinic cycle γ_h exists for certain parameter values on the same submanifold. More precisely, $\gamma_h = (1, 0) \cup \gamma_{10} \cup (M, 0) \cup \gamma_{01}$. \square

Let $(u_s, v_s) \in \Sigma$ with Σ the separatrix curve determined by the stable manifold of the non hyperbolic equilibrium point $(0, 0)$. Now, we define the set

$$A = \{(u, v) \in \Gamma \mid 0 \leq u \leq 1 \text{ and } 0 \leq v \leq v_s \text{ with } v_s \in \Sigma\},$$

i.e., A is determined by the separatrix curve Σ , the u -axis and the straight line $u = 1$. Thus, the region A is compact. This implies that Poincaré–Bendixon theorem is applicable there, which will determine the nature of the unique equilibrium point (H, H) with $H = (A + W)/2$; the Jacobian matrix is

$$DZ_\eta(H, H) = \begin{pmatrix} DZ_\eta(H, H)_{11} & -QH^2 \\ SH & -SH \end{pmatrix}$$

with $DZ_\eta(H, H)_{11} = (-4H^2 + (3M + 3 - Q)H - M)H$.

Then, $\det DZ_\eta(H, H) = H^2S(4H^2 - 3H(1 + M - Q) + 2M)$ and $\text{tr } DZ_\eta(H, H) = -H(4H^2 - (3(1 + M) - 2Q)H + 2M + S)$.

As $Q = (1 - H)(H - M)/H$, the sign of $\det DZ_\eta(H, H)$ depends on the factor

$$f(H) = 4H^2 - 3H(1 + M - Q) + 2M = -M + H^2.$$

The sign of $\text{tr } DZ_\eta(H, H)$ depends on the factor

$$T = 4H^2 - (3(1 + M) - 2Q)H + 2M + S = H(2H - M - 1) + S.$$

Moreover, to determine if the positive equilibrium point is node or focus, it is necessary to know the sign of the quantity P defined by

$$\begin{aligned} P &= (\text{tr } DZ_\eta(H, H))^2 - 4 \det DZ_\eta(H, H) \\ &= (H(2H - M - 1) + S)^2 - 4H^2S(-M + H^2) \\ &= S^2 - 2H((1 - 2H)M + (2H^3 - 2H + 1))S + H^2(2H - M - 1)^2. \end{aligned}$$

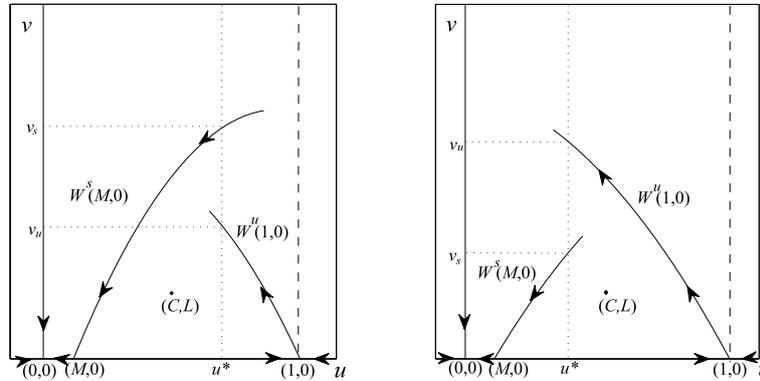


Figure 1. For $0 < H < 1$, the two possible relative positions between the stable manifold $W^s(0, 0)$ of singularity $(0, 0)$ and the unstable manifold $W^u(1, 0)$ of saddle point $(1, 0)$ are shown.

In Fig. 1 we show the relative positions of the stable manifold $W^s(0, 0)$ and the unstable manifold $W^u(1, 0)$ described in the above lemma.

Theorem 1. Let $(u^*, v_s) \in W^s(0, 0)$ and $(u^*, v_u) \in W^u(1, 0)$.

1. Supposing $v_s > v_u$, the singularity (H, H) is:
 - (i) an attractor point, if and only if $S > H(1 + M - 2H)$; moreover,
 - (a) is an attractor focus if and only if $P < 0$,
 - (b) is an attractor node if and only if $P > 0$.
 - (ii) a repeller point if and only if $S < H(1 + M - 2H)$; moreover,
 - (a) a repeller focus surrounding a stable limit cycle if and only if $P < 0$,
 - (b) is a repeller node if and only if $P > 0$.
2. Supposing that $v_s < v_u$; then (H, H) is a repeller focus or node and the equilibrium point $(0, 0)$ or else, a non-infinitesimal stable limit cycle, can be ω -limit of all trajectories of the system.

Proof. 1. Supposing that $v_s > v_u$. In the point (H, H) , we have that $\det DZ_\eta(H, H) = H^2S(-M + H^2) > 0$, then, the nature of (H, H) depends on the sign of $T = H(2H - M - 1) + S$.

(i) (H, H) is an attractor equilibrium point if and only if $S > H(1 + M - 2H)$.

(ii) (H, H) is a repeller equilibrium point if and only if $S < H(1 + M - 2H)$. As the region Λ is compact, by Hopf Bifurcation Theorem, a stable limit cycle is generated at this point. The sign of P determines if (H, H) is a focus or a node.

2. Supposing that $v_s < v_u$. In this case, the stable manifold $W^s(0, 0)$ lies under the unstable manifold $W^u(1, 0)$. Then, $T > 0$ and the point (H, H) is a repeller node. By the Existence and Uniqueness Theorem, all trajectories with initial conditions near of the point (H, H) can not intersect $W^u(1, 0)$ nor $W^s(0, 0)$. Since $(1, 0)$ is a saddle point

and $(0, 0)$ is a non hyperbolic saddle point, the trajectories have the origin $(0, 0)$ as their ω -limit, and this point is globally asymptotically stable.

Moreover, if $T > 0$, the limit cycles that appear by Hopf bifurcations grows until disappearing when the heteroclinic cycle γ_{10} joining points $(0, 0)$ and $(1, 0)$ is attained; later, this heteroclinic is broken; then, there exists a subset on the parameters space for which $(0, 0)$ is globally asymptotically stable.

So, a trajectory exists, which is originated on (H, H) and ending on $(0, 0)$, forming a new heteroclinic curve.

The above analysis considers all cases, so that the statement is proved. \square

Remark 2.

1. At least one infinitesimal limit cycle bifurcates when T changes of sign (see Fig. 2). Moreover, as v^s decreases towards v^u , this periodic orbit gets larger. At the limit when $v^s = v^u$, this periodic orbit becomes the heteroclinic cycle γ_h from Lemma 3 and then disappears as $v^s < v^u$.
2. If $M < -1$ ($m < -K$ in system (2)), the point (H, H) is an attractor point since $T > 0$ and $\text{tr } DZ_\eta(H, H) < 0$ and there are no limit cycles.

Theorem 2.

1. *There exist conditions on the parameter values for which an homoclinic curve is determined by the stable and unstable manifold non-hyperbolic equilibrium point $(0, 0)$.*
2. *There exists a non-infinitesimal limit cycle [10, 16] that bifurcates from the homoclinic surrounding the equilibrium point (H, H) .*

Proof. 1. Let $W^u(0, 0)$ and $W^s(0, 0)$ be the unstable and the stable manifold of the non-hyperbolic equilibrium point $(0, 0)$.

The trajectory determined by $W^u(0, 0)$ cannot cross the line $u = 1$ towards the right since $\bar{\Gamma}$ is an invariant region. The α -limit of stable manifold $W^s(0, 0)$ must be the point (H, H) or lies at infinity over u -axis.

Then, the ω -limit of stable manifold $W^u(0, 0)$ must be:

- (i) The point (H, H) when this point is an attractor or else a stable (attractor) limit cycle.
- (ii) The non-hyperbolic equilibrium point $(0, 0)$ when the point (H, H) is a repeller.
- (iii) The trajectory determined by the unstable manifold $W^u(1, 0)$ of the hyperbolic equilibrium point $(1, 0)$.

Therefore, there exists a set of parameter values for which the trajectories determine by $W^u(0, 0)$ and $W^s(0, 0)$ intersect forming a homoclinic curve.

2. Is straightforward by introduction of a single cross section, transverse to the homoclinic solution, that if this homoclinic is broken, then there exists a subset on the parameters space for which a stable non-infinitesimal limit cycle appears. A attracting single-round periodic orbit bifurcates inside the invariant region determined by the homoclinic loop. \square

3.2 Multiple limit cycles

The problem of determining the quantity of limit cycles, which can be born throughout the bifurcation of a weak (fine) focus [10], is not an easy task, and it is related to solving the well-known Hilbert 16th Problem (proposed by the mathematician David Hilbert in 1900) on the maximum number and relative position of limit cycles [16]. Yet this question remains unsolved in the case of a two-dimensional polynomial differential equation system, which degree must be less than or equal to $p \in \mathbb{N}$ and $p \geq 2$ [16], existing an example with four limit cycles for $p = 2$ [16].

To determine the weakness of the focus (H, H) , i.e., the number of limit cycles bifurcating of a weak focus [10, 14], in the next theorem, we will use the method for calculations of the Lyapunov numbers (or quantities) [10].

Theorem 3. *The singularity (H, H) of vector field Z_η is at least a third-order weak focus if and only if $S = H(1 + M - 2H)$ and $M^2 + (1 - 6H)M + 2H^3 = 0$.*

Proof. System (3) can be expressed by

$$Z_\nu: \begin{cases} \frac{du}{d\tau} = ((1-u)(u-M) - \frac{1}{H}(1-H)(H-M)v)u^2 \\ \frac{dv}{d\tau} = H(1+M-2H)(u-v)v, \end{cases} \quad (5)$$

where $\nu = (M, H) \in]-1, 0[\times]0, 1[$. Setting $u = U + H$ and $v = V + H$, then the new system translated to origin of coordinates system is

$$\bar{Z}_\nu: \begin{cases} \frac{dU}{d\tau} = ((1-U-H)(U+H-M) \\ \quad - \frac{1}{H}(1-H)(H-M)(V+H))(U+H)^2, \\ \frac{dV}{d\tau} = (1+M-2H)H(U-V)(V+H), \end{cases} \quad (6)$$

and the Jacobian matrix of system (5) at the point $(0, 0)$ is

$$D\bar{Z}_\nu(0, 0) = \begin{pmatrix} (1+M-2H)H^2 & -(1-H)(H-M)H \\ (1+M-2H)H^2 & -(1+M-2H)H^2 \end{pmatrix}.$$

The Jordan matrix associated to vector field Z_ν is [5]

$$J = \begin{pmatrix} 0 & -L \\ L & 0 \end{pmatrix}$$

with $L^2 = \det D\bar{Z}_\nu(0, 0) = H^3(-M + H^2)(1 + M - 2H)$ and $f_1(M, H) = 1 + M - 2H > 0$. The first Lyapunov quantity [10] is $\eta_1 = \text{tr} DZ_\eta(H, H) = \text{tr} D\bar{Z}_\nu(0, 0) = 0$, i.e., if $H(2H - M - 1) + S = 0$. Then, using the matrix change of basis [5]

$$B = \begin{pmatrix} 0 & \frac{1}{(1+M-2H)H^2} \\ -\frac{1}{L} & \frac{1}{L} \end{pmatrix}.$$

After a large algebraic calculations, we obtain

$$\bar{Z}_\nu: \begin{cases} \frac{dx}{d\tau} = -Ly - LH(1 + M - 2H)xy, \\ \frac{dy}{d\tau} = Lx + H^4(-2M + 3H^2)\frac{(1+M-2H)^2}{L}x^2 \\ \quad - H^2(-H - 2M - HM + 6H^2)(1 + M - 2H)xy \\ \quad + HL(5H - 2M - 2)y^2 + H^5(-M + 3H^2)\frac{(1+M-2H)^3}{L}x^3 \\ \quad - H^3(-H - 2M - HM + 10H^2)(1 + M - 2H)^2x^2y \\ \quad + HL(-2H - M - 2HM + 11H^2)(1 + M - 2H)xy^2 \\ \quad - L^2(-1 + 4H - M)y^3 + \frac{H^8}{L}(1 + M - 2H)^4x^4 \\ \quad - 4H^6(1 + M - 2H)^3x^3y + 6H^4L(1 + M - 2H)^2x^2y^2 \\ \quad - 4H^2L^2(1 + M - 2H)xy^3 + L^3y^4. \end{cases}$$

By means of a time rescaling given by $T = L\tau$, we derive to the normal form [10] given by

$$\tilde{Z}_\nu: \begin{cases} \frac{dx}{dT} = -y - H(1 + M - 2H)xy, \\ \frac{dy}{dT} = x + H^4(-2M + 3H^2)\frac{(1+M-2H)^2}{L^2}x^2 \\ \quad - \frac{H^2(-H-2M-HM+6H^2)(1+M-2H)}{L}xy + H(5H - 2M - 2)y^2 \\ \quad + H^5(-M + 3H^2)\frac{(1+M-2H)^3}{L^2}x^3 \\ \quad - \frac{H^3(-H-2M-HM+10H^2)(1+M-2H)^2}{L}x^2y \\ \quad + H(-2H - M - 2HM + 11H^2)(1 + M - 2H)xy^2 \\ \quad - L(-1 + 4H - M)y^3 + \frac{H^8}{L^2}(1 + M - 2H)^4x^4 \\ \quad - \frac{4H^6(1+M-2H)^3}{L}x^3y + 6H^4(1 + M - 2H)^2x^2y^2 \\ \quad - 4H^2L(1 + M - 2H)xy^3 + L^2y^4. \end{cases}$$

Using the Mathematica package [42], we obtain that the second Lyapunov quantity [10] is

$$\eta_2 = -\frac{(1 - H)H^6(H - M)(1 + M - 2H)^2}{8L^3}f_2(M, H)$$

with $f_2(M, H) = M^2 + (1 - 6H)M + 2H^3$.

Clearly, $f_2(M, H) > 0$ for $H > 1/6$ and $\eta_2 < 0$; then, $(0, 0)$ is a one-order weak focus of vector field \tilde{Z}_ν ; thus, a unique limit cycle exists in vector field Z_η for $\eta_1 = \text{tr } DZ_\eta(H, H) > 0$ (the point (H, H) is a repeller) and $\eta_2 < 0$.

We note that $f_2(M, H)$ can change of sign for $H < 1/6$; then, the equilibrium $(0, 0)$ of vector field \tilde{Z}_ν is surrounded by at least two limit cycles. If $(0, 0)$ is a third-order weak focus of vector field \tilde{Z}_ν , then η_3 must be the first non-zero Lyapunov quantity. We obtain

$$\eta_3 = -\frac{(1 - H)H^{13}(H - M)(1 + M - 2H)^4}{192L^7}f_3(M, H)$$

with

$$f_3(M, H) = f_{30} + f_{31} + f_{32} + f_{33} + f_{34} + f_{35} + f_{36},$$

where

$$\begin{aligned}
 f_{30} &= 2H^7(103H - 542H^2 + 804H^3 - 1), \\
 f_{31} &= -H^4(1 - 115H + 1794H^2 - 7312H^3 + 8944H^4 + 1084H^5)M, \\
 f_{32} &= H^2(1 - 333H + 3997H^2 - 14015H^3 + 13016H^4 + 7312H^5 + 206H^6)M^2, \\
 f_{33} &= -H(-230 + 2474H - 7239H^2 + 904H^3 + 14015H^4 + 1794H^5 + 2H^6)M^3, \\
 f_{34} &= (-15 + 778H - 5066H^2 + 7239H^3 + 3997H^4 + 115H^5)M^4, \\
 f_{35} &= (-30 + 778H - 2474H^2 - 333H^3 - H^4)M^5, \\
 f_{36} &= (-15 + 230H + H^2)M^6.
 \end{aligned}$$

The sign of $f_3(M, H)$ could change, and therefore, η_4 must be calculated to determine the existence of a third-limit cycle. However, such task will not be tackled in this work. When $f_3(M, H) < 0$, η_3 is positive. If $\text{tr } D\tilde{Z}_\nu(0, 0) < 0$, $f_2(M, H) > 0$ and η_3 is negative, the point $(0, 0)$ of the vector field \tilde{Z}_ν is a third-order weak focus. Then, the equilibrium point (H, H) is at least a third-order weak focus of the vector field Z_η or system (3). \square

Corollary 1. *There exists at least two limit cycles surrounding the singularity (H, H) of vector field Z_η , the innermost unstable and the outermost stable if and only if $S < H(1 + M - 2H)$ and $M^2 + (1 - 6H)M + 2H^3 < 0$.*

Proof. We note that the first condition implies the negativity of $\text{tr } DZ_\eta(H, H)$, implying the point (H, H) is a local attractor; the second inequality determines the positivity of η_2 . Thus, when η_3 is negative, there exist two limit cycles. Thus, the Lyapunov quantities have different sign, and the two limit cycles are the innermost unstable and the outermost stable. \square

Remark 3. Although the calculus of weakness of the focus (H, H) requires two stringent relationships between model parameters, a tiny deviation in one of them causes the condition for the existence of one or more limit cycles. So, these relations determine a subset of measure non-zero for the existence of a unique or two limit cycles.

This is important biologically, inasmuch as in reality none of the equalities given in Theorem 11 will be possible maintain by long time, due that any little change in any of the involved parameters will mean inequality rather than equality, and hence, we are working here with a sort of structurally unstable system.

Corollary 12 implies a significant difference with the model in which the strong Allee effect is considered since this system has two limit cycles in opposite stability, i.e., the innermost stable and the outermost unstable [21].

4 Some simulations

In the following, some simulations are given to reinforces the results obtained for system (3).

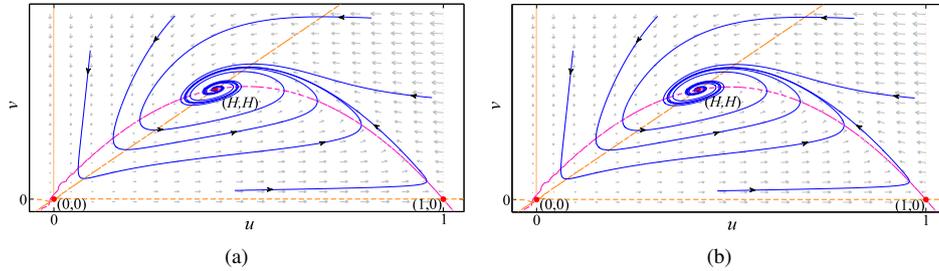


Figure 2. (a) For $H = 0.415$, $M = -0.01$ and $S = 0.1125$ ($Q = 0.5991$), the point (H, H) is an attractor focus in system (3). (b) For $H = 0.25$, $M = -0.0056$ and $S = 0.085$ ($Q = 0.765$), the point is a repeller focus surrounded by an infinitesimal stable limit cycle obtained by Hopf bifurcation.

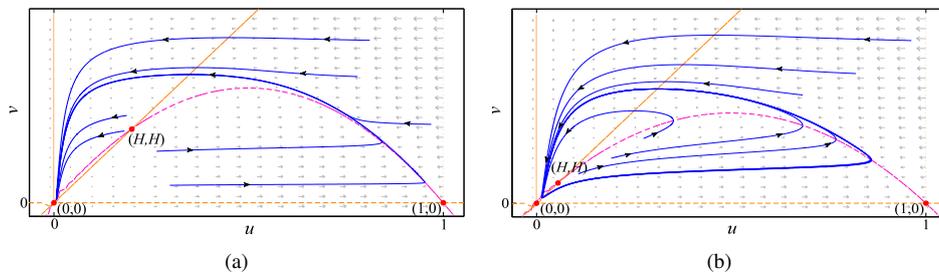


Figure 3. (a) For $H = 0.2$, $M = -0.001$ and $S = 0.015$ ($Q = 0.804$), the point (H, H) is a repeller node and $(0, 0)$ is global attractor. Then, the associated point $(0, 0)$ in which system (2) is not defined is also globally asymptotically stable by the diffeomorphism φ . (b) For $H = 0.055$, $M = -0.00506419$ and $S = 0.04$ ($Q = 1.032$), the equilibrium point (H, H) is a repeller node, but surrounded by a non-infinitesimal limit cycle obtained by homoclinic bifurcation (Theorem 8), whose existence cannot be proved by the Hopf bifurcation theorem.

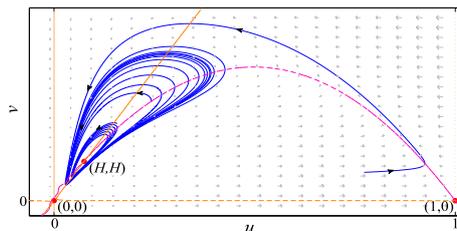


Figure 4. For $H = 0.075$, $M = -0.0050644$ and $S = 0.0645$ ($Q = 0.9875$), system (3) has two limit cycles, the outermost stable and the innermost unstable, confirming the proof of Theorem 11.

5 Discussion

In this work, we have analyzed a simple Leslie–Gower predator–prey model, considering the phenomenon known as weak Allee effect showing a significant impact in the dynamics of model. The main results have been expressed in function of the parameter $S = s/(rK)$,

indicating as the relation among r , s and K is important in the dynamical behavior of system (2).

It is well known that Leslie–Gower model is structurally stable as it has a unique equilibrium point at the interior of the first quadrant that is globally asymptotically stable. Ecologically speaking, this means the prey and predator species coexist for any parameter values.

We have modified the Leslie–Gower model by introducing a new factor, always positive, in the prey growth rate to describe the weak Allee effect and a compensatory curve of growth is obtained [11] as logistic growth function. However, the new model here studied has a rich and varied dynamics, where two of them are not topologically equivalent, unlike the original Leslie–Gower model.

We have shown the importance of the singularity $(0, 0)$ in the dynamics of the modified Leslie–Gower model, demonstrating the existence of a bistability phenomenon since $(0, 0)$ is always an attractor in addition to either a stable positive equilibrium point or a stable limit cycle for determined parameter values.

Furthermore, the inclusion of weak Allee effect has a significant consequence in the dynamics of system since until two limit cycles can appear surrounding the unique positive equilibrium point. Limit cycles do not exist in the original Leslie–Gower model since the unique positive equilibrium point is globally asymptotically stable for a wide set of parameter values.

For system (3), we proved the existence of a separatrix curve determined by the stable manifold of point $(0, 0)$ dividing the behavior of trajectories. This implies that the model is highly sensitive to the initial conditions. Moreover, for a wide set of the parameter values, the model can have two different attractors, and there exist solutions having the origin as its ω -limit. This implies in system (2) the possibility of depletion for both species is high, yet the ratio x/y is small.

We have shown that the weak Allee effect may be a destabilizing force in predator–prey models since the equilibrium point of the Leslie–Gower model changes from stable to unstable or otherwise. The analyzed model involves a difference with the special case of weak Allee effect when $m = 0$ [21], respect to the quantity of limit cycles bifurcating of a fine (weak) focus.

System (3) has a unique positive equilibrium point when $m < 0$, which is surrounded by two limit cycles, for certain parameter values, the innermost unstable and the outermost stable. Meanwhile, system (3) has a unique limit cycle, when $m = 0$ [21], which can disappear when the other parameters vary; however, the trajectories of system (3) can also have different ω -limit sets strongly depending on the initial conditions.

The Leslie–Gower-type predator–prey model, considering strong Allee effect studied in [21] and the weak Allee effect here analyzed, have an similar behavior with respect to the non-uniqueness of the limit cycle, but differs in the amount of positive equilibrium points. Assuming $M > 0$, there may be up to two positive equilibrium points with the possibility of homoclinic orbits [21].

Moreover, for the model with weak Allee, we show by simulations, that a non-infinitesimal limit cycle [10] surrounding the unique positive equilibrium point exists; the proof of the existence of this limit cycle is left as an open problem.

Furthermore, it can see the varied and rich dynamic that has the modified Leslie–Gower model here studied in comparison with the Gause-type predator–prey model using the same function to model the Allee effect and the linear functional response, which is described in the Kot’s book [26].

Taking the Leslie–Gower-type predator–prey model [32, 38], interesting works could be analyzed in future in which: i) a generalist predators be considered, or ii) other expressions to the Allee effect be considered in the prey growth function.

In short, the consideration of weak Allee effect on prey population has a high impact on the dynamics of the classical Leslie–Gower model, resulting in a new system with various different mathematical behaviors, which have adequate biological meanings. Thus, considering the obtained results, we affirm that the number of limit cycles bifurcating from a weak focus, it is a good mathematical criterion for the classification of predator–prey models.

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