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CONSTRUCTIVE DECOMPOSITION OF BMO FUNCTIONS AND FACTORIZATION OF A_p WEIGHTS

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ABSTRACT. We give elementary proofs of the decomposition of BMO functions on the line and factorization of A_p weights. Our method is an explicit version of a construction due to the third author.

We would like to describe an elementary construction leading to an explicit decomposition of BMO functions on $\mathbf{R}$ as bounded plus conjugate of bounded functions (i.e. Charles Fefferman's duality theorem) as well as to the factorization theorem proved in [**2**]. The method described is an explicit version of the construction in [**3**].

For simplicity we start by describing the factorization of A_2 on $\mathbf{R}$. These are the weights (i.e. nonnegative locally integrable functions) ω for which the following inequality holds:

$$\int (Mf + |Hf|)^2 \omega\, dx \leq C^2 \int |f|^2 \omega\, dx,$$

where H is the Hilbert transform and M is the Hardy-Littlewood maximal operator. They have a structural description which can be bound in [**1**]. In particular, it is known that ω^{-1} belongs with ω to A_2.

THEOREM I. *For $\omega \in A_2$, there exists weights ω_j, $j = 1, 2$, such that $\omega = \omega_1/\omega_2$ and $|H(\omega_j)| + M(\omega_j) \leq 3C\omega_j$.*

As a corollary one obtains a weak version of the Helson-Szegö theorem: for $\omega \in A_2$, we can find two bounded functions b_1, b_2 such that

$$\ln \omega = b_1 + H(b_2) \quad \text{and} \quad \|b_2\|_\infty < \pi.$$

(In the actual Helson-Szegö theorem, the last inequality is replaced by $\|b_2\|_\infty < \pi/2$. See [**1**].)

In fact, we let $W_j = \omega_j + iH(\omega_j)$. Since $\operatorname{Im} W_j \leq 3C \operatorname{Re} W_j$, $\operatorname{Re} W_j > 0$, we find

$$|W_j| \sim \omega_j \quad \text{and} \quad |\arg W_j| \leq \operatorname{arctg} 3C < \pi/2.$$

Thus $\ln \omega - \ln|W_1/W_2| = b_1$ is bounded, and

$$\ln\left|\frac{W_1}{W_2}\right| = -H\left(\arg \frac{W_1}{W_2}\right), \qquad \left|\arg \frac{W_1}{W_2}\right| = |b_2| < 2 \operatorname{arctg} 3C.$$

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Observe that since every BMO function is of the form $\ln \omega^\alpha$ for some $\omega \in A_2$, $\alpha > 0$, we obtain an explicit formula for the decomposition of a BMO function as a sum of a bounded function and the Hilbert transform of a bounded function.

The proof of the theorem is just as simple. We define, for $f \in L^2(\mathbf{R})$,

$$S(f) = \omega^{-1/2}\{|H(f\omega^{1/2})| + M(f\omega^{1/2})\} + \omega^{1/2}\{|H(f\omega^{-1/2})| + M(f\omega^{-1/2})\}.$$

Then $\|S(f)\|_{L^2(\mathbf{R})} \leqslant 2C\|f\|_{L^2(\mathbf{R})}$ so that, for u_0 any positive function in $L^2(\mathbf{R})$, the function $U = \sum_0^\infty S^j(u_0)/(3C)^j$ belongs to $L^2(\mathbf{R})$. By the countable subadditivity of S,

$$\omega^{\pm 1/2}(M + |H|)(U\omega^{\mp 1/2}) \leqslant S(U) \leqslant 3CU,$$

and the theorem follows when one sets $\omega_1 = U\omega^{1/2}$, $\omega_2 = U\omega^{-1/2}$.

Observe that the ω_j satisfy $M(\omega_j) \leqslant 3C\omega_j$, i.e. they belong to A_1. This is a particular case of the following theorem proved in [**2**] by more intricate methods.

THEOREM II. *Let $\omega \in A_p$, i.e.,*

$$\int M(f)^p \omega\, dx \leqslant C^p \int |f|^p \omega\, dx$$

and

$$\int M(f)^{p'} \omega^{1-p'}\, dx \leqslant C^{p'} \int |f|^p \omega^{1-p'}\, dx,$$

where $1/p + 1/p' = 1$. Then $\omega = \omega_1 \omega_2^{1-p}$ where $M\omega_j \leqslant 3C_j\omega$.

PROOF. We assume $s = p/p' \geqslant 1$ (i.e. $p \geqslant 2$) and define $M_s(f) = M(f^s)^{1/s}$ and

$$S(f) = \omega^{1/p} M(f\omega^{-1/p}) + \omega^{-1/ps} M_s(f\omega^{1/ps}).$$

By assumption S is bounded on $L^p(\mathbf{R})$. We define the L^p function $U = \sum_0^\infty (S^j(u_0)/(3C)^j)$, with u_0 as before and observe that, using the countable subadditivity of S, we obtain

$$M(U\omega^{-1/p}) \leqslant 3CU\omega^{-1/p} \quad \text{and} \quad M(U^s\omega^{1/p}) \leqslant 3CU^s\omega^{1/p}.$$

Write

$$\omega = (U^s\omega^{1/p})(U\omega^{-1/p})^{(1-p)} = \omega_1\omega_2^{1-p}$$

$(s = p(1 - 1/p) = p - 1)$. We remark that the proof of Theorem II carries over immediately to more general settings, e.g. those considered in [**3**].

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