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A NOTE ON THE VERTEX-DISTINGUISHING INDEX FOR SOME CUBIC GRAPHS

Abstract. The vertex-distinguishing index of a graph G (vdi (G)) is the minimum number of colours required to colour properly the edges of a graph in such a way that any two vertices are incident with different sets of colours. We consider this parameter for some families of cubic graphs.

Keywords: edge colouring, vertex-distinguishing colouring, cubic graphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

In this paper we consider only simple graphs and we use the standard notation of graph theory. Definitions not given here may be found in [6]. Let $G = (V, E)$ be a graph of order n with the set of vertices V and the edge set E. We denote by $V_d(G)$ the set of vertices of degree d in G and $n_d(G) = |V_d(G)|$. A k-*edge-colouring* f of a graph G is an assignment of k colours to the edges of G. Let $f(e)$ be the colour of the edge e.

Denote by $F(v) = \{f(e) \colon e = uv \in E\}$ the multi-set of colours assigned to the set of edges incident to v. The colouring f is *proper* if no two adjacent edges are assigned the same colour and *vertex-distinguishing proper* (VDP for short) *colouring* if it is proper and $F(u) \neq F(v)$ for any two distinct vertices u, v.

Observe that a graph has a VDP colouring if and only it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec* graph. The minimum number of colours required to find a VDP colouring of a vdec graph G is called the *vertex-distinguishing index* and denoted by vdi (G).

The VDP colouring number was introduced and studied by Burris and Schelp in [7] and [8] (as "strong colouring") and, independently (as "observability" of a graph) by Černý, Horňák and Soták in [9, 11] and [12].

The following result has been conjectured by Burris and Schelp in [7] and [8], and proved in [4].

Theorem 1. *A vdec graph G on n vertices vertex has* $\mathrm{vdi}\,(G) \le n+1$.

Of course, this last estimation of $\mathrm{vdi}\,(G)$ cannot be improved in general as the example of complete graphs shows. However, for some families of graphs the vdi is rather closer to the maximum degree than to the order of the graph. For instance, if $\delta(G) > n/3$, then $\mathrm{vdi}\,(G) \le \Delta(G) + 5$ (see [5]).

It is easy to see that if there exists a k-colouring of G, then $\binom{k}{d} \ge n_d$ for $1 \le d \le \Delta$. So, the minimum number of colours needed for VDP colouring of a graph is given by

$$\pi(G) = \min\{k\colon \binom{k}{d} \ge n_d \text{ for } 1 \le d \le \Delta\}.$$

Burris and Shelp stated ([7] and [8]) the following conjecture which suits the Vizing's theorem on colouring index.

Conjecture 2. *Let G be a vdec graph. Then*

$$\pi(G) \le \mathrm{vdi}\,(G) \le \pi(G) + 1.$$

Conjecture 2 has been proved in a number of particular cases, including complete graphs ([7, 8] and [9]), complete bipartite graphs ([7] and [9]), unions of cycles ([1]) and others. In paper [2] it has been solved for graphs with $\Delta(G) \ge \sqrt{2\,|V(G)|} + 4$ and $\delta(G) \ge 5$. It is left to be seen if Conjecture 2 holds in remaining cases. The case of regular graphs of low degree seems to be the most interesting and the most difficult at the same time. The case of 2-regular graphs *i.e.* disjoint unions of cycles has been considered in [1]. In this paper we deal with two families of cubic graphs, ladders (Section 2) and $p \cdot K_4$ (Section 3).

2. LADDERS

Definition 3. *A ladder on $n = 2k$ vertices is a graph obtained from two copies of C_k by adding edges between the corresponding vertices of the cycles. A ladder on n vertices will be denoted by L_n. We will label the vertices of the first cycle with odd and the vertices of the second cycle with even integers, respectively (see Fig. 1).*

We shall prove the following theorem.

Theorem 4. *If k is the smallest integer such that $\binom{k}{3} \ge n$ then* $\mathrm{vdi}\,(L_n) \le k+1$.

We shall use the following results for exact value of vdi of 2-regular graphs published in [1] and [8] (see also [3]).

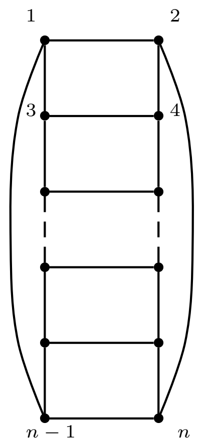

Fig. 1. A ladder L_n

Lemma 5. *A simple 2-regular graph on n vertices can be VDP coloured with k colours if and only if:*

— *k is odd and either $n = \binom{k}{2}$ or $n \leq \binom{k}{2} - 3$, or*

— *k is even and $n \leq \binom{k}{2} - \frac{k}{2}$.*

The following lemma is a simply corollary of the above lemma.

Lemma 6. *If $n = \binom{k-1}{2} + 1$ and $n \geq 3$, then there exists a VDP colouring of C_n with at most k colours.*

Proof of Theorem 4. We shall prove the theorem by induction on the order n of a graph. For the induction hypothesis we will need a slightly stronger thesis: there exists a strong colouring φ of L_n with at most $k+1$ colours such that:

$$\varphi(n-1, n-3) \neq \varphi\ (n, n-2).$$

In the first step we will consider graphs L_6, L_8 and L_{10}. The suitable colourings for L_6, L_8 are shown in Figures 2 and 3. A VDP 6-colouring of L_{10} can be easily obtained from a VDP 5-colouring of C_{10} (which exists by Lemma 5).

Let now $n \geq 12$. Assume, that the theorem is true for $m \leq n-1$. We have:

$$n \leq \binom{k}{3} = \binom{k-1}{3} + \binom{k-1}{2}.$$

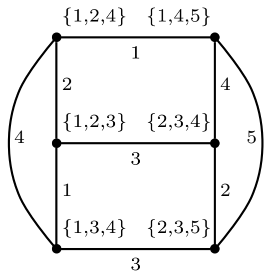

Fig. 2. 5-colouring of L_6

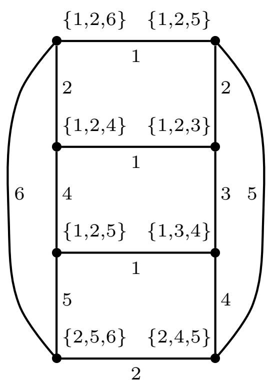

Fig. 3. 6-colouring of L_8

Let us distinguish the following two cases.

Case 1. 1°. $n < \binom{k}{3}$, or $n = \binom{k}{3}$ and $\binom{k-1}{3}$, $\binom{k-1}{2}$ are even.

We can divide $V(L_n)$ into two subsets: A containing vertices $\{1, \ldots, n_1\}$ and B containing vertices $\{n_1+1, \ldots, n\}$ in such a way that n_1 is even and $6 \le n_1 \le \binom{k-1}{3}$, $4 \le n - n_1 \le \binom{k-1}{2}$.

Let G_1 be a graph obtained from a subgraph induced by A by adding edges $\{1, n_1-1\}$ and $\{2, n_1\}$. It is an L_{n_1} of order $\le \binom{k-1}{3}$, so by induction hypothesis we know that there exists its VDP colouring φ_1 with at most k colours such that:

$$a = \varphi_1\{n_1-3, n_1-1\} \neq \varphi_1\{n_1-2, n_1\} = b \tag{1}$$

From a subgraph induced by B we choose a cycle

$$C = n_1+1, n_1+3, \ldots, n-3, n-1, n, n-2, \ldots, n_1+2, n_1+1.$$

It is a cycle of order less or equal than $\binom{k-1}{2}$ so there exists its strong colouring φ_2 with at most k colours (the conjecture has been proved for cycles). In a strong colouring of a cycle every three subsequent edges have different colours so we can choose φ_2 in such a way that:

$$\varphi_2(\{n-3,n-1\}) = a, \tag{2}$$
$$\varphi_2(\{n-1,n\}) = \varphi_1(\{n_1-1,n_1\}) = c, \tag{3}$$
$$\varphi_2(\{n-2,n\}) = b \tag{4}$$

(we may need to rename the colours).

We will now construct a strong colouring φ of L_n:

— for $e \in E(G_1) \setminus \{\{1, n_1-1\}, \{2, n_1\}\}$ we put $\varphi(e) = \varphi_1(e)$;

— for $e \in E(C)$ $\varphi(e) = \varphi_2(e)$ we put $\varphi\{1, n-1\} = \varphi_1\{1, n_1-1\}$ and $\varphi\{2, n\} = \varphi_1\{2, n_1\}$.

All other edges are given a new colour $k+1$.

By simple testing it can be seen that φ is VDP colouring of L_n with at most $k+1$ colours.

Case 2. $n = \binom{k}{3}$, and $\binom{k-1}{3}$, $\binom{k-1}{2}$ are odd.

We divide $V(L_n)$ into two subsets $A = \{1, \ldots, n_1\}$ and $B = \{n_1+1, \ldots, n\}$ in such a way, that $|A| = \binom{k-1}{3} - 1$ and $|B| = \binom{k-1}{2} + 1$.

Graph G_1 (see Case 1) is an L_{n_1} of order $\leq \binom{k-1}{3}$. So, by induction hypothesis, we know that there exists its strong colouring φ_1 with at most k colours fulfilling the additional assumption.

A cycle C chosen from the subgraph induced by B has $\binom{k-1}{2}+1$ vertices. So, by Lemma 6 we know that there exists its strong colouring with k colours. We choose it in such a way that the conditions (2), (3) i (4) are fulfilled. We continue the next steps as in Case 1 to get a VDP colouring of L_n with at most $k+1$ colours. □

3. GRAPHS pK_4

The aim of this section is to prove that Conjecture 2 holds in the case of disjoint unions of complete graphs K_4. We need the following lemma which is a simple corollary of Lemma 5.

Lemma 7. *If $1 \leq p \leq \binom{l}{2}$ then there exists a VDP colouring of pC_4 with at most $2l$ colours.*

Theorem 8. *If k is the smallest integer with $\binom{k}{3} \geq 4p$ then* $\operatorname{vdi}(pK_4) \leq k+1$.

Proof. Proof is by induction on the p. For $p=1$ we have $k=4$ and by a suitable colouring $\operatorname{vdi}(K_4) = 5 = k+1$.

For $p > 1$ we have $4p \leq \binom{k}{3} = \binom{k-1}{3} + \binom{k-1}{2}$. Denote by $r \in \{0,1,2,3\}$ such integer r that $\binom{k-1}{3} \equiv r(\pmod 4)$. Then pK_4 can be divided into two parts: G_1 containing $p_1 = \frac{\binom{k-1}{3}-r}{4}$ copies of K_4 and G_2 containing $p_2 = p - p_1$ copies of K_4. By induction hyphotesis we know that there exists a VDP colouring of $G_1 = p_1K_4$ with at most k colours. For G_2 we use VDP colouring of p_2C_4 with at most k colours (see note below) and all other edges we colour by $k+1$. Then we have VDP colouring of pK_4 with at most $k+1$ colours.

Note, that VDP k-colouring of p_2C_4 exists by Lemma 5 in case $r = 0$ or $4p_2 \leq \binom{k-1}{2}$. In case $r \neq 0$ we have k even and by Lemma 7 VDP k-colouring of p_2C_4 exists also. □

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