More intrinsically knotted graphs

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Abstract

We demonstrate four intrinsically knotted graphs that do not contain each other, nor any previously known intrinsically knotted graph, as a minor.

1 Introduction

Throughout the paper we will take an embedded graph to mean a graph embedded in 3-space, where all of our embeddings are tame. Recall that a cycle of a spatial graph is said to be knotted if it bounds no 2-cell in $\mathbb{R}^3$. We call a graph $G$ intrinsically knotted if every spatial embedding of the graph contains a knotted cycle. In [2] it was shown that $K_{3,3,1,1}$ is intrinsically knotted by showing that the sum of the arf invariants of a certain set of cycles in a certain subgraph of $K_{3,3,1,1}$ is always odd. Using the same technique, another intrinsically knotted graph, $H$, was exhibited in [3], see Figure 1.

The graph $H$ was the only known minor-minimal intrinsically knotted graph that had the property of becoming a graph with a linkless embedding (an embedding for which all pairs of disjoint cycles form splittable links), after the removal of a certain vertex. In this paper, again using techniques from [2], we exhibit four intrinsically knotted graphs, each having a vertex whose removal results in a linklessly embeddable graph. We demonstrate that none of these graphs contain each other, nor any previously known minor-minimal intrinsically knotted graph, as a minor. We do not prove in this paper whether or not our four graphs are minor-minimal intrinsically knotted graphs. Unlike the graph $H$, the four graphs we exhibit here do contain 3-cycles. They thus lead, by $\Delta - Y$ exchanges, to even more intrinsically knotted graphs [6]. We will not further discuss these other graphs in this paper.

It remains an open question to determine all minor-minimal intrinsically knotted graphs. Because having a knotless embedding is preserved by edge
contraction [7], such a set is finite, due to the result of Robertson and Seymour [8]. The currently known minor-minimal intrinsically knotted graphs now include $K_7$ and graphs obtained by $\Delta - Y$ exchanges on $K_7$ [1, 6, 5], $K_{3,3,1,1}$ and graphs obtained by $\Delta - Y$ exchanges on $K_{3,3,1,1}$ [2, 6, 5], and the triangle-free graph $H$ [3]. This paper establishes that this list is incomplete.

2 Important lemmas

Let $D_4$ be the graph with multiple edges pictured in Figure 2. Let $v_1, v_2, v_3, v_4$ denote the four vertices of $D_4$. Let $S$ denote the collection of cycles in $D_4$ that contain all of the vertices $\{v_i : i = 1, 2, 3, 4\}$. Let $C_1, C_2, C_3,$ and $C_4$ denote the four cycles of length 2 in the graph $G$, with $C_i$ composed of the edges $e_{2i-1}$ and $e_{2i}$.  

Figure 2: The graph $D_4$.  

Figure 1: A projection of an embedding of the intrinsically knotted graph $H$, with a knotted cycle highlighted.
We define \( \alpha(K) \in \mathbb{Z}_2 \) to be the arf invariant of the knot \( K \). For background on the arf invariant, see [1] or [4]. Given two disjoint cycles, \( A \) and \( B \), in an embedded graph, we denote by \( lk(A, B) \) the mod 2 linking number of \( A \) and \( B \). Given an embedding of \( D_4 \), let \( \sigma = \sum_{s \in S} \alpha(s) \pmod{2} \). We use the following result from [10] and [2]:

**Lemma 2.1.** Given an embedding of the graph \( D_4 \), \( \sigma \neq 0 \) if and only if \( lk(C_1, C_3) \neq 0 \) and \( lk(C_2, C_4) \neq 0 \).

Thus, given an embedding of \( D_4 \), in which \( lk(C_1, C_3) \neq 0 \) and \( lk(C_2, C_4) \neq 0 \), there is a knotted cycle in this particular embedding of \( D_4 \). We will show our graphs are intrinsically knotted by showing that every embedding of them contains a subgraph that contracts onto an embedding of the graph \( D_4 \) with \( lk(C_1, C_3) \neq 0 \) and \( lk(C_2, C_4) \neq 0 \). This technique was used in [2] and [3].

We will also make use of the following elementary lemma, which we state without proof.

**Lemma 2.2.** Given a 6-cycle \((a, 1, b, 2, c, 3)\), any two vertices in the set \( \{a, b, c\} \) are connected to any two vertices in the set \( \{1, 2, 3\} \) by disjoint edges.

### 3 More intrinsically knotted graphs

We remark here that in our proofs, whenever we refer to linked cycles, we mean cycles that have non-zero mod 2 linking number.

![Graph](image-url)  

**Figure 3:** A linkless embedding of \( G_{14} \).
Consider the graph $G_{14}$ shown in Figure 3. It has a linkless embedding as shown. We make a new graph, $G_{15}$ by joining a new vertex, $v$, by an edge to each of the vertices \{1, 3, 4, 5, 1', 2', 3', 4', 5'\} of the graph $G_{14}$.

**Theorem 3.1.** The graph $G_{15}$ is intrinsically knotted.

**Proof.** Consider an arbitrary embedding of $G_{15}$. Temporarily insert an edge from $v$ to 2 to get an embedding of a graph we will call $G'_{15}$. The subgraph of $G'_{15}$ induced by the vertices \{v, 1, 2, 3, 4, 5, 6, 7\} is a subdivision of $K_6$, as is the subgraph induced by the vertices \{v, 1', 2', 3', 4', 5' 6', 7'\}. By a well-known result [1] [9], there will be linked cycles, within each of the embedded subdivided $K_6$'s. One of the linked cycles within the embedded subgraph induced by \{v, 1, 2, 3, 4, 5, 6, 7\} (respectively by \{v, 1', 2', 3', 4', 5', 6', 7'\}) will contain the vertex $v$. We will denote this cycle $C_1$ (respectively $C_2$). The cycle $C_1$ can pass through at most one of the vertices in \{2, 6, 7\} (respectively in \{2', 6', 7'\}). The other linked cycle, $C_2$ will either use two vertices from the set \{2, 6, 7\} (respectively $C_4$, \{2', 6', 7'\}), or will be connected to two such vertices by disjoint paths that are disjoint from $C_1$ (respectively $C_2$). Moreover, $C_1$ (respectively $C_2$) will either use or connect to one of the vertices in \{2, 6, 7\} by a path that is disjoint from $C_3$ (respectively $C_4$) and disjoint from the paths that connect $C_3$ (respectively $C_4$) to two vertices in \{2, 6, 7\}. In either case, applying Lemma 2.2 to the 6–cycle (2, 2’, 7, 7’, 6’), we get an expansion of $D_4$ with the desired linking properties. Thus, there is a knotted cycle in the embedded $G'_{15}$. If there is a knotted cycle that does not use the edge (v, 2), then there is a knotted cycle in the embedding of $G_{15}$.

Otherwise, the embedded subgraph of $G_{15}$ induced by the vertices \{v, 1, 2, 3, 4, 5, 6, 7\} does not contain linked cycles. The subgraph of $G_{15}$ induced by the vertices \{v, 1', 2', 3', 4', 5', 6', 7'\} is still a subdivision of $K_6$. In the latter induced subgraph, the linked cycle that does not pass through $v$ either uses one vertex from \{2', 6'\}, or is connected to 6' by an edge that is disjoint from the linked cycle through $v$. We will consider the case when this linked cycle is (3', 4', 6', 5') = $C_1$, which is linked to (v, 1', 2') = $C_3$; the other cases can be treated similarly. Then $v$ connects to vertex 2 via the paths (v, 5', 6', 2) and (v, 4', 6', 2). Either of these paths, together with the induced subgraph of $G_{15}$ on the vertices \{v, 1, 2, 3, 4, 5, 6, 7\}, make up an embedded subdivision of $K_6$. There thus must contain a pair of linked cycles within each embedded subgraph, and one of the linked cycles must use the path (v, 5', 6', 2) and another the path (v, 4', 6', 2). We choose $C_2$ to be the linked cycle that contains the path (v, 4', 6', 2), for otherwise vertex 5' would be
Figure 4: An example of how, in the proof of Theorem 3.1, there could be an expansion of $D_4$ with the required linking properties.

contained in $C_2$ and in $C_1$, while, in order to get a proper expansion of $D_4$, $5'$ can only be part of $C_1$ (see Figure 4). The other cycle in the pair, $C_4$, either uses both vertices in $\{6, 7\}$, or is connected to them by disjoint paths that are disjoint from $C_2$. We thus have that $C_2$ and $C_3$ share exactly one vertex, and that $C_1$ and $C_2$ share exactly one edge. Again applying Lemma 2.2 to the 6–cycle $(2, 2', 7, 7', 6, 6')$, there exist disjoint paths connecting $C_4$ to $C_1$ and $C_3$. We thus have an expansion of $D_4$ with the required linking properties (see Figure 4).

Here is another intrinsically knotted graph, which is a variant of $G_{15}$ that was suggested to the author by Garry Bowlin. Consider the graph $H_{14}$ shown in Figure 5. It has a linkless embedding as shown. In $G_{14}$ and $H_{14}$, there are two disjoint copies of subdivided $K_5$’s, connected to each other by a 6–cycle. In $G_{14}$, a non-subdived vertex of one $K_5$ is connected to a non-subdived vertex of the other $K_5$ by an edge of the 6–cycle. In $H_{14}$, the non-subdived vertices only connect to subdivided vertices by the connecting 6–cycle.

From $H_{14}$, we make a new graph, $H_{15}$ by joining a new vertex, say $v$, by an edge to each of the vertices $\{1, 2, 3, 5, 1', 2', 3', 4', 5'\}$ in the graph $H_{14}$. By an argument similar to the one given above, we have the following:

**Theorem 3.2.** The graph $H_{15}$ is intrinsically knotted.

We obtain the intrinsically knotted graph $J_{14}$ from the graph $J_{13}$ depicted in Figure 6 by taking an additional vertex, $v$, and connecting $v$ to the vertices $\{1, 3, 4, 5, 8, 9, 10, 11, 12, 13\}$. With a little more work, we can establish the following:

**Theorem 3.3.** The graph $J_{14}$ is intrinsically knotted.

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Proof. Embed \( J_{14} \). The vertices \( \{v, 8, 9, 10, 11, 12, 13\} \) induce a \( K_{3,3,1} \) subgraph, which contains a linked \( 3 \)–cycle and \( 4 \)–cycle \([9]\). Note that the \( 4 \)–cycle of this link must use at least one of the vertices in \( \{9, 10\} \). Without loss of generality, suppose the \( 4 \)–cycle contains the edge \( (11, 9) \). The vertices \( \{v, 1, 2, 3, 4, 5, 6, 7\} \) nearly induce a subdivision of \( K_6 \), except there is an edge missing from \( v \) to \( 2 \). If there are a pair of linked cycles in this induced subgraph, then an argument analogous to that given in the proof of Theorem 3.1 shows that there is an expansion of \( D_4 \) with the desired linking properties.

Otherwise, we use the path \((v, 11, 9, 2)\) to make up for the edge missing from \( v \) to \( 2 \). We thus have an embedding of a subdivision of \( K_6 \), which must contain a non-split link on two components, call them \( A \) and \( B \). One of the
components, say \( A \), must use the path \((v, 11, 9, 2)\). The component \( B \) either uses the vertices 6 and 7, or is connected to them by disjoint paths that are disjoint from \( A \). In either case, using Lemma 2.2, we can find an expansion of \( D_4 \) with the required linking properties.

Finally, we create the graph \( J_1'4 \) from \( J_13 \) by attaching a new vertex, \( v \) to the vertices \( \{1, 2, 3, 4, 5, 11, 12, 13\} \).

**Theorem 3.4.** The graph \( J_1'4 \) is intrinsically knotted.

**Proof.** Embed \( J_1'4 \). The vertices \( \{v, 1, 2, 3, 4, 5, 6, 7\} \) induce a subdivision of \( K_6 \). Let \( C_1 \) stand for the linked cycle through \( v \), and \( C_3 \) the other linked cycle in this induced subgraph. As before, \( C_3 \) uses or is connected by two disjoint paths disjoint from \( C_1 \), to two of the three vertices in the set \( \{2, 6, 7\} \). Here we will consider the case where \( C_3 = (1, 7, 5, 6, 4) \); other cases are treated similarly. In this case, the subgraph induced by \( \{v, 8, 9, 10, 11, 12, 13\} \), together with the paths \((v, 4, 6, 8)\), \((v, 4, 6, 9)\) and \((v, 1, 7, 10)\) make up a graph that contains \( K_{3,3,1} \) as a minor. In the embedding, it contains two linked cycles, one of which uses one of the three paths described in the previous sentence. We call this linked cycle \( C_2 \), and the other linked cycle \( C_4 \). We thus have that \( C_1 \) and \( C_2 \) share exactly the vertex \( v \), and that \( C_2 \) and \( C_3 \) share exactly an edge. We will have an expansion of \( D_4 \) provided we can connect \( C_4 \) to \( C_1 \) and \( C_3 \) by disjoint paths. We can do this via Lemma 2.2, because \( C_4 \) uses two of the three vertices in \( \{8, 9, 10\} \), and \( C_3 \) uses vertices 6 and 7 (one of which is not part of \( C_2 \)), and \( C_1 \) uses vertex 2.

At present, we do not know if \( H_{15} \), \( G_{15} \), \( J_{14} \), or \( J_{14}' \) are minor minimal with respect to intrinsic knotting. We do, however, know the following:

**Theorem 3.5.** The graphs \( H_{15} \), \( G_{15} \), \( J_{14} \), and \( J_{14}' \) do not contain each other as a minor, nor do they contain in any previously known intrinsically knotted graph as a minor.

**Proof.** Since \( H_{15} \) and \( G_{15} \) each have the same number of vertices and edges, one cannot be a minor of the other. Since \( J_{14} \) and \( J_{14}' \) each have the same number of vertices, the only way for one to be a minor of the other is for \( J_{14}' \) to be a subgraph of \( J_{14} \), but this is not true by inspection. By a theorem in [3], all other previously known minor-minimal intrinsically knotted graphs, except \( H \), have the property that removing any vertex results in an intrinsically linked graph. Since every graph in \( \{H_{15}, G_{15}, J_{14}, J_{14}'\} \) has a vertex
that, when removed, yields a graph that has a linkless embedding, the only known minor-minimal intrinsically knotted graph that may possibly be a minor of $H_{15}$, $G_{15}$, $J_{14}$ or $J'_{14}$ is $H$.

We now show that $H$ cannot be a minor of $G_{15}$. The proofs that $H$ is not a minor of $H_{15}$, $J_{14}$ or of $J'_{14}$, and that $J_{14}$ or $J'_{14}$ are not minors of $H_{15}$ or $G_{15}$ are all similar and we thus omit them.

We first note that both $H$ and $G_{15}$ are 4-connected. Furthermore, there are exactly two sets of 4 vertices in $H$ whose removal will disconnect $H$. Those two sets are \{v, 2, 4, 6\} and \{v, 7, 9, 11\}. The graph $H - \{v, 2, 4, 6\}$ has four components; one is $K_{3,3}$, and the other three components are all points. The graph $H - \{v, 7, 9, 11\}$ has four components that are also $K_{3,3}$ and three points. There are exactly two sets of 4 vertices in $G_{15}$ whose removal will disconnect $G_{15}$. Those two sets are \{v, 2', 6', 7'\} and \{v, 2, 6, 7\}. The graph $G_{15} - \{v, 2', 6', 7'\}$ has two components, one of which is a subdivision of $K_5$, and the other is a connected graph on 4 vertices. If $G_{15}$ contained $H$ as a minor, one of the components of $G_{15} - \{v, 2', 6', 7'\}$ would have to contain $K_{3,3}$ as a minor. Thus $G_{15}$ does not contain $H$ as a minor.

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References


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