Uncertain Block Replacement Policy with No Replacement at Failure

Chunxiao Zhang, Congrong Guo
College of Science, Civil Aviation University of China, Tianjin 300300, China

Abstract: The block replacement policy with no replacement at failure means that a unit is always replaced at fixed time, but not replaced at failure. In this paper, the lifetime of the unit is assumed to be an uncertain variable, the model of expected cost per unit time following no replacement at failure policy is proposed, and the existence condition of the optimal replacement time is given. Furthermore, the optimal block replacement policy of a system consisting of multiple units which operate independently of each other is discussed. At last, a numerical example on the maintenance management of spare parts in civil aviation is provided to illustrate the effectiveness of the proposed model.

Keywords: maintenance, block replacement policy, uncertainty theory

1 Introduction

One of the widely used maintenance policies is the block replacement policy, where the component is replaced at failure and also preventively at fixed times $kT (T > 0, k = 1, 2, \cdots)$. It is easy to operate, especially for multi-component systems. Barlow and Proschan firstly deduced the expected cost per unit time following a block replacement at interval $T$ over an infinite time span in their book [1]. Later, the $n$-stage block replacement was proposed in [16]. The adjustment costs, which are increasing with the age of unit, were introduced by Tilquin and Cleroux in [22]. Scarf et al [21] considered the block replacement of a two-unit system with failure dependence. Brezavsek et al [3] discussed the optimum problem of provisioning spare parts for block replacement.

However, such a block policy is rather wasteful, as sometimes it can result in the replacement of practically new units. On the other hand, in some situations, the components of a system are not monitored continuously, and the failures can be detected only at fixed period. This kind of phenomenon has received a great attention from the researchers, and various modifications have been advocated to overcome those undesirable features. The age-dependent block replacement is one of the modified block replacement policy, firstly proposed by Berg and Epstein [2], and showed that this policy is superior to
the block replacement policy in the long-run maintenance cost rate. Later Nakagawa [18] introduced an idle period without replacing failed units before the planned replacement. Murthy et al [17] proposed the policy of reusing unfailed units which were taken out in a previous block replacement. Then Kadi and Cleroux [4] appended an idle period to the used item replacement policy. Park and Yoo [20] proposed \((\tau, k)\) block replacement policy with idle count where each unit is individually replaced at failure during a specified time interval, and beyond the failure-replacement interval, failed units are left idle until a specified number of failures occur, then a block replacement is performed. For more development of block replacement policies, readers may refer to Nakagawa [19].

As discussed in the before-mentioned literature, in the original study, the lifetime of a unit is considered to be a random variable, and probability theory plays a greatly important character in the optimization of the replacement policy. The probability theory is applicable only when the obtained probability is close enough to the real frequency. But few or even no observed data is available in some situation. Therefore, we have to invite some domain experts to evaluate their belief degree. Since human tends to overweight unlikely events (Kahneman and Tversky [5]), the belief degree may have a much large range than the real frequency. Due to the particularity of the problem, it is unreasonable to adopt stochastic method. In order to deal with the involved human uncertainty, uncertainty theory was founded by Liu [6] in 2007, and was refined by Liu [10] in 2010 based on normality, duality, subadditivity and product axioms. Subsequently, a lot of researchers have contributed to the study of uncertainty theory.

Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling human uncertainty, and some applications can be found in various fields such as uncertain programming [8], uncertain statistics [10], uncertain risk analysis [11], uncertain reliability analysis [11], uncertain logic [13], uncertain inference [12], uncertain process [7], uncertain calculus [7], and uncertain finance [15]. Based on the theory of uncertain renewal process, Yao [23] proposed the uncertain block replacement policy, and Yao and Ralescu [24] investigated the uncertain age replacement policy and obtained the long-run average replacement cost. Zhang and Guo [25] applied the uncertain renewal process to the ordering problem of spare parts for aircrafts assuming the interarrival times to be uncertain variables. Zhang and Guo [26] provided some results on the optimal time of uncertain age replacement policy when the parts’ lifetimes follow different uncertainty distributions.

In this paper, we consider a system consisting of multiple units which operate independently of each other. This system is not monitored continuously, and its failures can be detected only at times \(kT(k = 1, 2, \cdots)\). Suppose that each unit has an identical uncertain distribution, the model a block replacement policy with no replacement at failure is discussed. The rest of this paper is organized as follows: Section 2 recalls some basic concepts and properties about uncertainty theory which will be used throughout the paper. In Section 3, block replacement policy in uncertain environment is introduced and the block replacement policy with no replacement at failure is studied. Numerical examples are given in
Section 4. Finally, some conclusions are made in Section 5.

2 Preliminaries

Let $\Gamma$ be a nonempty set. A collection $\mathcal{L}$ of $\Gamma$ is called a $\sigma$-algebra if (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then $\Lambda_1 \cup \Lambda_2 \cup \cdots \in \mathcal{L}$. Each element $\Lambda$ in the $\sigma$-algebra $\mathcal{L}$ is called an event.

Uncertain measure is a function from $\mathcal{L}$ to $[0, 1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event $\Lambda$ a number $M(\Lambda)$ which indicates the belief degree that the event $\Lambda$ will occur. In order to ensure that the number $M(\Lambda)$ has certain mathematical properties, Liu[6] proposed the following three axioms:

Axiom 1. (Normality Axiom) $M(\Gamma) = 1$ for the universal set $\Gamma$.

Axiom 2. (Duality Axiom) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.

Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$ we have

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i\right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).$$

Definition 1. (Liu [6]) The set function $M$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

Besides, in order to provide the operational law, Liu [9] defined the product uncertain measure on the product $\sigma$-algebra $\mathcal{L}$ as follows,

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, M_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. Then the product uncertain measure $M$ is an uncertain measure satisfying

$$M\left(\prod_{i=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} M_k(\Lambda_k),$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2. (Liu [6]) An uncertain variable is a measurable function $\xi$ from the uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 3. (Liu [6]) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = M\{\xi \leq x\}$$

for any real number $x$. 

3
Definition 4. (Liu [9]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$
\mathcal{M} \left( \bigcap_{i=1}^m (\xi_i \in B_i) \right) = \bigwedge_{i=1}^m \mathcal{M} \{ \xi_i \in B_i \}
$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Theorem 1. (Liu [7]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f$ is a strictly increasing function, then

$$
\xi = f(\xi_1, \xi_2, \ldots, \xi_n)
$$

is an uncertain variable with uncertainty distribution

$$
\Psi(x) = \sup_{f(x_1, x_2, \ldots, x_n) = x} \min_{1 \leq i \leq n} \Phi_i(x_i).
$$

Corollary 1. (Liu [6]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be iid uncertain variables. Then $\xi_1 + \xi_2 + \cdots + \xi_n$ and $n\xi$ are identically distributed uncertain variables. In other words, if $\Phi$ is the common uncertainty distribution of $\xi_1, \xi_2, \ldots, \xi_n$, then $\frac{1}{n} \sum_{i=1}^n \xi_i$ has also the uncertainty distribution $\Phi$ (but not independent of $\xi_1, \xi_2, \ldots, \xi_n$).

Expected value is the average of an uncertain variable in the sense of uncertain measure. It is an important index to rank uncertain variables.

Definition 5. (Liu [6]) Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$
E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_-\infty^0 M\{\xi \leq r\}dr
$$

provided that at least one of the two integrals is finite.

Theorem 2. (Liu [6]) Let $\xi$ be an uncertain variable with uncertainty distribution $\Phi$. If the expected value exists, then

$$
E[\xi] = \int_0^{+\infty} (1 - \Phi(x))dx - \int_-\infty^0 \Phi(x)dx.
$$

Generally, the expected value operator $E$ has no linearity property for arbitrary uncertain variables. But, for independent uncertain variables $\xi$ and $\eta$, Liu [6] proved that

$$
E[a\xi + b\eta] = aE[\xi] + bE[\eta]
$$

for any real numbers $a$ and $b$.

An uncertain process (Liu [7]) is essentially a sequence of uncertain variables indexed by time. As an important uncertain process, a renewal process is an uncertain process in which events occur continuously and independently of one another in uncertain times.

Block replacement policy in uncertain environment is proposed by Yao [24]. It means that a unit is always replaced at failure or periodically with time $s$. Assume that the lifetimes of the units are iid
uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution $\Phi$. Then replacement times before the given time $s$ form an uncertain renewal process $N_t$. Let $a$ denote the “failure replacement” cost of replacing a unit when it fails earlier than $s$, and $b$ the “planned replacement” cost of replacing a unit at planned time $s$. It is clear that the cost of one period is $aN_s + b$ and the average cost is
\[
\frac{aN_s + b}{s}.
\]
Then the expected cost rate is
\[
E \left[ \frac{aN_s + b}{s} \right] = \frac{1}{s} (aE[N_s] + b).
\]

3 Block Replacement Policy with No Replacement at Failure

In some situation, a unit is not monitored continuously, and its failures can be detected only at times $kT (k = 1, 2, \cdots)$ and some maintenance is done. Assume that a unit is always replaced at times $kT$, but it is not replaced at failure, and hence, it remains in failed state for the time interval from a failure to its detection. Let $c_1$ be the downtime cost per unit of time elapsed between a failure and its replacement, and $c_2$ be the cost of planned replacement. The lifetime of a unit is denoted by $\xi$ which is a positive uncertain variable, then the average cost of one cycle is
\[
C(T, \xi) = \frac{1}{T} \left[ c_1 (T - \xi) + c_2 \right].
\]

**Theorem 3.** Let $\xi$ be a positive uncertain variable with an uncertainty distribution $\Phi$. If the cost function is
\[
C(T, x) = \frac{1}{T} \left[ c_1 (T - x) + c_2 \right],
\]
then the uncertain variable $C(T, \xi)$ has an uncertainty distribution
\[
\Psi(x) = \begin{cases} 
0, & \text{if } x < \frac{c_2}{T} \\
1 - \Phi \left( T + \frac{c_2}{c_1} - \frac{T}{c_1} x \right), & \text{if } \frac{c_2}{T} \leq x < \frac{c_2}{c_1} + \frac{c_2}{T}\ \\
1, & \text{if } x \geq \frac{c_2}{c_1} + \frac{c_2}{T}.
\end{cases}
\]

**Proof:** Since $C(T, \xi) \geq \frac{c_2}{T}$, we have
\[
\Psi(x) = \mathcal{M}\{C(T, \xi) \leq x\} = 0
\]
for any $x \in \left( -\infty, \frac{c_2}{T} \right)$. If $x \in \left[ \frac{c_2}{T}, \frac{c_2}{c_1} + \frac{c_2}{T} \right)$, then
\[
\Psi(x) = \mathcal{M}\{C(T, \xi) \leq x\} = \mathcal{M}\left\{ \xi \geq T + \frac{c_2}{c_1} - \frac{T}{c_1} x \right\} = 1 - \Phi \left( T + \frac{c_2}{c_1} - \frac{T}{c_1} x \right).
\]
If $x \geq \frac{c_2}{c_1} + \frac{c_2}{T}$, then
\[
\Psi(x) = \mathcal{M}\{C(T, \xi) \leq x\} = 1.
\]
The theorem is proved.
Theorem 4. Let $\xi$ be a positive uncertain variable with an uncertainty distribution $\Phi$. If the cost function is

$$C(T, \xi) = \frac{1}{T} \left[ c_1[T - \xi]^+ + c_2 \right],$$

then

$$E[C(T, \xi)] = \frac{c_2}{T} + \frac{c_1}{T} \int_0^T \Phi(x)dx.$$ 

Proof: From Theorem 3, we have

$$E[C(T, \xi)] = \int_0^{+\infty} [1 - \Psi(x)]dx$$

$$= \int_0^c 1dx + \int_c^{c + \frac{1}{T}} \Phi \left( T + \frac{c_2}{c_1} - \frac{T}{c_1}x \right) dx + \int_{c + \frac{1}{T}}^{+\infty} 0dx$$

$$= \frac{c_2}{T} + \int_c^{c + \frac{1}{T}} \Phi \left( T + \frac{c_2}{c_1} - \frac{T}{c_1}x \right) dx$$

$$= \frac{c_2}{T} + \frac{c_1}{T} \int_0^T \Phi(x)dx.$$ 

The theorem is verified.

Differentiating $E[C(T, \xi)]$ with respect to $T$ and setting it equal to zero, we can get

$$T \Phi(T) - \int_0^T \Phi(x)dx = \frac{c_2}{c_1}. \quad (1)$$

Thus, if there exists an optimum time $T^*$ that uniquely satisfies the equation (1), the resulting expected cost per unit of time is

$$E[C(T^*, \xi)] = c_1 \Phi(T^*).$$

Next, we discuss the existence of $T^*$.

Theorem 5. Let $\xi$ be a positive uncertain variable with an uncertainty distribution $\Phi$. Assume

$$C(T, x) = \frac{1}{T} \left[ c_1[T - x]^+ + c_2 \right].$$

If

$$\lim_{T \to +\infty} \left[ T \Phi(T) - \int_0^T \Phi(x)dx \right] > \frac{c_2}{c_1}, \quad (2)$$

then there exists a finite and unique $T^*(0 < T^* < +\infty)$ that satisfies

$$T^* \Phi(T^*) - \int_0^{T^*} \Phi(x)dx = \frac{c_2}{c_1}.$$ 

Proof: Let

$$Q(T) = T \Phi(T) - \int_0^T \Phi(x)dx.$$ 

It is easy to prove that $\lim_{T \to 0} Q(T) = Q(0) = 0$. And $Q(T)$ is increasing because for any $\Delta T > 0$,

$$Q(T + \Delta T) - Q(T) = [x \Phi(x)]_{T}^{T+\Delta T} - \int_{T}^{T+\Delta T} \Phi(x)dx \geq 0.$$
If \( Q(+\infty) > \frac{c_2}{c_1} \), then, from the monotonicity and the continuity of \( Q(T) \), there exists a finite and unique \( T^*(0 < T^* < +\infty) \) that satisfies equation (1) and minimizes \( E[C(T, \xi)] \).

The theorem is verified.

Next, we consider a system consisting of \( n \) units that operate independently of each other and have identical uncertainty distribution \( \Phi \). It is assumed that each unit remains in failed state if it fails before the planned replacement and are all replaced together at times \( kT (k = 1, 2, \cdots) \). Let \( \xi_1, \xi_2, \cdots, \xi_n \) be the lifetimes of the units. Then the average cost per unit of time in one cycle is

\[
C(T, n, \xi_1, \xi_2, \cdots, \xi_n) = \frac{1}{T} \left[ c_1 \left( \sum_{i=1}^{n} [T - \xi_i]^+ \right) + c_2 \right].
\]

**Theorem 6.** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be a sequence of iid positive uncertain variables with a common uncertainty distribution \( \Phi \). Assume

\[
C(T, n, x_1, x_2, \cdots, x_n) = \frac{1}{T} \left[ c_1 \left( \sum_{i=1}^{n} [T - x_i]^+ \right) + c_2 \right].
\]

Then the uncertain variable \( C(T, n, \xi_1, \xi_2, \cdots, \xi_n) \) has an uncertainty distribution

\[
\Psi_n(x) = \begin{cases} 
0, & \text{if } x < \frac{c_2}{T} \\
1 - \Phi \left( T - \frac{T x - c_2}{n c_1} \right), & \text{if } \frac{c_2}{T} \leq x < nc_1 + \frac{c_2}{T} \\
1, & \text{if } x \geq nc_1 + \frac{c_2}{T}.
\end{cases}
\]

**Proof:** It is easy to prove that the uncertain variable \( T \wedge \xi_1 \) has an uncertainty distribution

\[
\Upsilon(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
\Phi(x), & \text{if } 0 < x \leq T \\
1, & \text{if } x > T.
\end{cases}
\]
Thus,

\[ \Psi_n(x) = M \{ C(T, n, \xi_1, \xi_2, \cdots, \xi_n) \leq x \} \]

\[ = M \left\{ \frac{1}{T} \left[ c_1 \left( \sum_{i=1}^{n} (T - T \wedge \xi_i) \right) + c_2 \right] \leq x \right\} \]

\[ = M \left\{ \frac{1}{T} \left[ c_1 \left( nT - \sum_{i=1}^{n} T \wedge \xi_i \right) + c_2 \right] \leq x \right\} \]

\[ = M \left\{ \sum_{i=1}^{n} T \wedge \xi_i \geq nT - \frac{T x - c_2}{c_1} \right\} \]

\[ = 1 - M \left\{ \sum_{i=1}^{n} T \wedge \xi_i \leq nT - \frac{T x - c_2}{c_1} \right\} \]

\[ = 1 - T \left( T - \frac{T x - c_2}{n c_1} \right) \]

\[ = \begin{cases} 
0, & \text{if } x < \frac{c_2}{T} \\
1 - \Phi \left( T - \frac{T x - c_2}{n c_1} \right), & \text{if } \frac{c_2}{T} \leq x < nc_1 + \frac{c_2}{T} \\
1, & \text{if } x \geq nc_1 + \frac{c_2}{T} 
\end{cases} \]

The theorem is verified.

**Theorem 7.** Let \( \xi_1, \xi_2, \cdots, \xi_n \) be a sequence of iid positive uncertain variables with a common uncertainty distribution \( \Phi \). If the cost function is

\[ C(T, n, x_1, x_2, \cdots, x_n) = \frac{1}{T} \left[ c_1 \left( \sum_{i=1}^{n} [T - x_i]^+ \right) + c_2 \right], \]

then

\[ E[C(T, n, \xi_1, \xi_2, \cdots, \xi_n)] = \frac{1}{T} \left[ nc_1 \int_{0}^{T} \Phi(x)dx + c_2 \right] \]

**Proof:** From Theory 6, we have

\[ E[C(T, n, \xi_1, \xi_2, \cdots, \xi_n)] = \int_{0}^{c_2/T} 1dx + \int_{nc_1+c_2/T}^{c_2} \Phi(T - \frac{T x - c_2}{nc_1})dx + \int_{nc_1+c_2/T}^{\infty} 0dx \]

\[ = \frac{c_2}{T} + \frac{nc_1}{T} \int_{0}^{T} \Phi(x)dx \]

\[ = \frac{1}{T} \left[ nc_1 \int_{0}^{T} \Phi(x)dx + c_2 \right]. \]

The theorem is verified.

4 Application

In this section, we consider the replacement of a lighting system for an airport. Ceiling lamp in the terminal is an important component part of the lighting system of an airport. For the sake of saving
source of energy in a long time, more and more airports prefer to LED lights, although its installation is costly. Assume that one set of ceiling light consists of \( n = 1800 \) LED lamps which operate independently with each other, and the installation cost of one set is \( c_2 = $34,600 \). If the replacement cycle is \( T \), when any one lamp fails in the interval \((kT, (k + 1)T), k = 0, 1, \cdots\), the cost per unit time caused by various kinds of salvage is \( c_1 = $11 \). The data of the lifetimes of LED lamps are unavailable because this kind of LED light is new, then the lifetimes are assumed to be independent uncertain variables with a common linear uncertainty distribution \( \mathcal{L}(a, b) (a > 0) \) (Liu [14]), that is

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x - a)/(b - a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b.
\end{cases}
\]

We invite some experts in the field of LED light to evaluate the possible lifetimes in order to obtain the parameters \( a \) and \( b \).

According to the experts’ belief by using the principle of least squares given by Liu [14], we obtain the unknown parameters of the uncertainty distribution: \( \hat{a} = 20,000, \hat{b} = 50,000 \). Thus the cost function is

\[
C(T, n, \xi_1, \xi_2, \cdots, \xi_n) = \frac{1}{T} \left[ 11 \times \left( \sum_{i=1}^{n} [T - \xi_i]^+ \right) + 34600 \right].
\]

After a verification, the condition that

\[
\lim_{T \to +\infty} n[T\Phi(T) - \int_0^T \Phi(x)dx] > \frac{c_2}{c_1}
\]

is satisfied. Therefore, the optimal block replacement time is

\[
T^* = \sqrt{a^2 + \frac{2c_2(b - a)}{nc_1}} = 20002.62.
\]

That is, the set of lamp bulbs are replaced every 20002.62 hours.

### 5 Conclusions

In this paper, uncertain block replacement policy with no replacement at failure was studied. The lifetime of a unit was considered to be an uncertain variable to describe human uncertainty. Considering the expected replacement cost per unit time as an optimality criterion, we obtain the analytical expression of the expected cost rate in one cycle, and give the condition of the existence of the optimal replacement time. As an illustration, we applied the model to the replacement of an airport’s lighting system.

### Acknowledgements

This work is supported by the Fundamental Research Funds for the Central Universities under Grant no.ZXH2012K005. The research is also supported by Uncertainty Theory Laboratory of Department of
Mathematical Sciences, Tsinghua University, Beijing 100084, China.

References


