On measures based on the interval-valued pseudo-integrals of real functions and absolute continuity

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Abstract—One of the important mathematical tools that play a central role in many practical areas, such as mathematical economics, are real-valued functions and integration of real-valued functions with respect to interval-valued measure. The focus of this paper is on measures based on the pseudo-integration of real-valued function with respect to interval-valued measure. The absolute continuity of the first type and of the sixth type for the observed measure are investigated.

Keywords. Pseudo-operations, Interval-valued measures, Absolute continuity.

I. INTRODUCTION

As the natural generalization of the classical (single-valued) measure are considered set-valued measures and the interval-valued measure as its special case. Those measures are highly applicable in many different practical areas. The investigation of this type of measures contains, the set-valued fuzzy measure ([5]), as well as the set-valued pseudo-additive measure that is a generalization of the pseudo-additive measure in the sense of Sugeno and Murofushi (see [17]), etc. Another possible approach to this problem, with roots in the pseudo-analysis, has as the core the pseudo-integral of interval-valued functions (see [1], [2], [3]).

Due to the vast applicability of the pseudo-analysis ([9], [12], [13], [14]), the topic of this paper remains in this field. The focus of the presented construction is interval-valued measure that is now based on the pseudo-integral of a real function with respect to the interval-valued measure, rather than on interval-valued functions (see [4]). This direction of the investigation is worth following due to the fact that often, while working with uncertainty, instead of the actual values one uses intervals that incorporate some errors ([6], [8], [10], [15], [16], [20]).

The organization of the paper is the following. The second section contains preliminary notions. Interval-valued measure \( \mu_{f,M} \) via pseudo-integrals of real valued functions is given in the third section. Also, the absolute continuity of \( \sigma\oplus\)-measure is investigated. Some concluding remarks are given in the final section.

II. PRELIMINARY NOTIONS

Let us consider a closed (or semiclosed) subinterval of \([-\infty,\infty]\) denoted with \([a,b]\). Let \(\preceq\) be a total order on \([a,b]\). Pseudo-operations essential for the presented research are given by the following definitions.

Definition 1: Mapping \(\oplus: [a,b] \times [a,b] \rightarrow [a,b] \) which is commutative, non-decreasing (with respect to \(\preceq\)), associative and with a neutral element 0 is called pseudo-addition.

Definition 2: Mapping \(\odot: [a,b] \times [a,b] \rightarrow [a,b] \) that is positively non-decreasing (\(x \preceq y\) implies \(x \odot z \preceq y \odot z, z \in [a,b]_+ = \{x : x \in [a,b], 0 \preceq x\}\)), commutative, associative and with a neutral element 1 is called pseudo-multiplication.

Now, the semiring is given by the next definition.

Definition 3: Structure \(([a,b],\oplus,\odot)\) for which \(\oplus\) is a pseudo-addition and \(\odot\) is a pseudo-multiplication such that the following hold:

i) \(0 \odot x = 0\);

ii) \(x \odot (y \odot z) = (x \odot y) \odot (x \odot z)\).

is called a semiring.

For more on this subject see [13], [14].

Total order \(\preceq\) is directly connected to the choice of the pseudo-addition. If \(\oplus\) is an idempotent operation, \(\preceq\) is given by the following

\[x \preceq y\text{ if and only if }x \oplus y = y.\]

If \(([a,b],\oplus,\odot)\) is a semiring with strict pseudo-operations generated by some bijection \(g: [a,b] \rightarrow [0,\infty]\), i.e. a \(g\)-semiring ([11], [13]), then

\[x \preceq y\text{ if and only if }g(x) \leq g(y).\]

For more on this subject see [13], [14].

A. Absolute continuity for \(\sigma\oplus\)-measure

Notions essential for this paper are notions of the \(\sigma\oplus\)-measure (see [13]) and of the absolute continuity (see [5], [18], [19]).

Definition 4: Let \(\Sigma\) be a \(\sigma\)-algebra of subset of a \(X\). A set function \(\mu: \Sigma \rightarrow [a,b]_+\) is the \(\sigma\oplus\)-measure if \(\mu(\emptyset) = 0\) and

\[\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \mu(A_i) = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \mu(A_i),\]
where \((A_i)_{i \in \mathbb{N}}\) is a sequence of pairwise disjoint sets from \(\Sigma\). If \(\oplus\) is an idempotent operation, the first condition can be left out and sets from the second conditions need not be disjoint.

The concept of the absolute continuity from classical measure theory had been generalized and it is now possible to distinguish 21 different types of absolute continuity for fuzzy measures (see [19]). The focus of this paper is on the following two types of absolute continuity for \(\sigma\oplus\) measures.

**Definition 5:** Let \(\mu\) and \(\nu\) be two arbitrary \(\sigma\oplus\) measures on \(\sigma\)-algebra \(\Sigma\) that are finite in the sense of a given semiring.

i) \(\nu\) is absolutely continuous of the first type with respect to \(\mu\), if and only if \(\nu(A) = 0\) whenever \(A \in \Sigma\) and \(\mu(A) = 0\).

ii) \(\nu\) is absolutely continuous of the sixth type with respect to \(\mu\), if and only if \(\nu(A_n) \rightarrow 0\) whenever \((A_n)_{n \in \mathbb{N}} \in \Sigma\) and \(\mu(A_n) \rightarrow 0\).

**B. Absolute continuity for measure based on the pseudo-integral**

The construction of the pseud-integral starts with pseudo-integration of an elementary function and it is analogous to the construction of the classical Lebesgue integral (see [13]). The absolute continuity for measure based on the pseudo-integral had been investigated in [18].

Let \(f : X \rightarrow [a, b]_+\) be a measurable function and let \(\mu\) be a finite \(\sigma\oplus\) measure. Then, the pseudo-integral of function \(f\) over some set \(A \in \Sigma\) with respect to \(\sigma\oplus\)-measure \(\mu\) is denoted with \(\int_A f \circ d\mu\) and

\[
\mu_f(A) = \int_A f \circ d\mu,
\]

where \(A \in \Sigma\), determines a new \(\sigma\oplus\) measure on \(\Sigma\).

It was shown in [18]) that \(\sigma\oplus\)-measure \(\mu_f\) given by (1) is absolutely continuous of the first type and of the sixth type with respect to \(\mu\).

**III. INTERVAL-VALUED MEASURE VIA PSEUDO-INTEGRALS OF REAL VALUED FUNCTIONS**

This section follows the approach to interval-valued measures from [7] that is now considered in the pseudo-analysis’ surrounding (4)).

Let \([a, b], \oplus, \odot\) be a semiring, \(X \neq \emptyset\), \(\Sigma\) a \(\sigma\)-algebra of its subsets and \((X, \Sigma, \mu)\) a measure space where \(\mu\) is the \(\sigma\oplus\)-decomposable measure.

By the pseudo-sum of sets from \(\Sigma\) the following is considered

\[
A \oplus B = \{x \oplus y \mid x \in A \text{ and } y \in B\},
\]

while the pseudo-product of a constant \(\alpha\) from \([a, b]_+\) and some set is

\[
\alpha \odot A = \{\alpha \odot x \mid x \in A\}.
\]

The pseudo-sum of sets can be extended to the countable case by

\[
\bigoplus_{i=1}^{\infty} A_i = \lim_{n \to \infty} \bigoplus_{i=1}^{n} A_i.
\]

If working with sets from \(\mathcal{I}\), where \(\mathcal{I}\) is the class of all closed subintervals of \([a, b]_+\), the interval-valued \(\sigma\oplus\) measure if \(\Pi(\emptyset) = 0\), and

\[
\Pi\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} \Pi(A_i)
\]

for \((A_i)_{i \in \mathbb{N}}\) being a sequence of pairwise disjoint sets from \(\Sigma\). Again, if \(\oplus\) is an idempotent operation, the first condition can be left out and sets from the second conditions need not be disjoint.

For sets from the class \(\mathcal{I}\) (of all closed subintervals of \([a, b]_+\)) corresponding relation "less or equal", denoted by \(\preceq_S\), is given by the next definition.

**Definition 6:** Let \(C, D \in \mathcal{I}\). Set \(C\) is less or equal than set \(D\), i.e., \(C \preceq_S D\), if for all \(x \in C\) there exists \(y \in D\) such that \(x \preceq_S y\) and if for all \(y \in D\) there exists \(x \in C\) such that \(x \preceq_S y\).

**A. Step I: construction of the interval-valued measure \(\Pi\)**

The interval-valued \(\sigma\oplus\) measure that is used as the base for construction of the required type of interval-valued pseudo-integral of a real function is given by the following definition (4)).

**Definition 7:** Let \(\mathcal{M}\) be an arbitrary nonempty family of \(\sigma\oplus\)-decomposable measures \(\mu\). The interval-valued set-function \(\overline{\mu}_\mathcal{M} : \Sigma \rightarrow \mathcal{I}\) for the family \(\mathcal{M}\) is

\[
\overline{\mu}_\mathcal{M} = [\mu_1, \mu_r], \quad \mu_1, \mu_r \in \mathcal{M},
\]

where

\[
\mu_1(A) \preceq_S \mu(A) \preceq_S \mu_r(A)
\]

for all \(\mu\) from \(\mathcal{M}\) and all \(A\) from \(\Sigma\), if such \(\mu_1\) and \(\mu_r\) exist.

**Proposition 8:** If exists, \(\overline{\mu}_\mathcal{M}\) is an interval-valued \(\sigma\oplus\) measure.

The previous follows easily from the definitions of \(\sigma\oplus\) measures and pseudo-addition of intervals (see [4]).

**Remark 9:** Let \(\mathcal{M}_0\) be a family of \(\sigma\oplus\) measures that includes the trivial \(\sigma\oplus\) measure \(\mu_0\) of the form \(\mu_0(A) = 0\) for all \(A \in \Sigma\). If \(\mu_r\) from the previous definition for family \(\mathcal{M}_0\) exists, the interval-valued set-function \(\overline{\mu}_\mathcal{M}_0\) is

\[
\overline{\mu}_\mathcal{M}_0 = [0, \mu_r].
\]

Further on, for two \(\sigma\oplus\) measures \(\mu_1\) and \(\mu_2\) from \(\mathcal{M}\) the shorter notation \(\mu_1 \preceq_S \mu_2\) will be used if for all sets \(A\) from \(\Sigma\) holds \(\mu_1(A) \preceq_S \mu_2(A)\).
B. Step II: construction of the interval-valued pseudo-integral based on $\pi_M$

The following construction, that has been presented in [4], is an extension of the construction from [13] to the interval-valued case. The first step is to define the interval-valued pseudo-integral of an elementary function.

Further on let us assume that $\mathcal{M}$ is an arbitrary nonempty family of $\sigma$-finite measures $\mu$ such that $\mu_1$ and $\mu_r$ from Definition 7 do exist.

The interval-valued pseudo-integral of an elementary function $e$ with values $\{a_1, a_2, \ldots\}$ on sets $A_1, A_2, \ldots$, respectively, with respect to the interval-valued $\sigma$-additive $\pi_M$ is

$$\int_X e \circ d\pi_M = \bigoplus_{i=1}^{\infty} a_i \circ \pi_M(A_i).$$

It is now possible to introduce the corresponding interval-valued pseudo-integral of an arbitrary measurable function $f : X \rightarrow [a, b]$ as limit of pseudo-integrals of a sequence of elementary functions chosen in such manner that $d(\varphi_n(x), f(x)) \rightarrow 0$ uniformly while $d$ is some metric on the given semiring and $n \rightarrow \infty$.

If needed, the interval-valued pseudo-integral based on $\pi_M$ of a function $f$ on some arbitrary subset $A$ of $X$ can be easily obtained as $\int_X (f \circ \chi_A) \circ d\pi_M$, where $\chi_A$ is the pseudo-characteristic function ([13]).

Connection between interval valued pseudo-integral and the pseudo-integral is given in the next theorem.

**Theorem 10:** If $f : X \rightarrow [a, b]$ is a measurable function, then

$$\int_X f \circ d\pi_M = \left[ \int_X f \circ d\mu_1, \int_X f \circ d\mu_r \right].$$

Some of the basic properties of this integral are:

- pseudo-integral of pseudo-sum of measurable functions is pseudo-sum of pseudo-integrals;
- pseudo-integral of pseudo-product of constant and measurable function is pseudo-product of that constant and pseudo-integral of measurable function;
- pseudo-integral of a constant over some set $A$ is pseudo-product of that constant and $\pi_M(A)$;
- pseudo-integration is monotone in the following sense

$$\int_X f_1 \circ d\pi_M \preceq_s \int_X f_2 \circ d\pi_M$$

where $f_1 \preceq f_2$ and $\preceq_s$ is given by Definition 6;
- Theorem 10 and all of the previous properties also hold for integration over arbitrary $A \in \Sigma$.

C. Step III: construction of the interval-valued measure $\pi^f_M$

The construction of interval-valued measure that is the core of this paper follows. Now, functions with range in $[a, b]$ are being observed.

**Definition 11:** An interval-valued set-function $\pi^f_M$ based on interval-valued pseudo-integral of a real-valued measurable function $f : X \rightarrow [a, b]$ is

$$\pi^f_M(A) = \int_A f \circ d\pi_M$$

for $A \subseteq X$.

Of course, from Theorem 10 follows the interval form of $\pi^f_M$, i.e.,

$$\pi^f_M(A) = \left[ \int_A f \circ d\mu_1, \int_A f \circ d\mu_r \right].$$

Some important properties of $\pi^f_M$ are

- maps emptyset to $[0, 0]$;
- is monotone with respect to $\preceq_s$;
- is $\sigma$-additive.

Obviously, (2) is a monotone (with respect to $\preceq_s$) interval-valued $\sigma$-additive measure (see [4]).

D. Absolute continuity for $\pi^f_M$

The absolute continuity of an interval-valued measure $\Pi_1$ from the step III with respect to finite a $\sigma$-additive measure $\Pi_2$ from the step I is given the following definition (see [18]).

**Definition 12:** Let $\Pi_1$ be an interval-valued measurable and let $\Pi_2$ be a $\sigma$-additive measure that is finite in the sense of given semiring.

i) $\Pi_1$ is absolutely continuous of the first type with respect to $\Pi_2$ if and only if $\Pi_1([0, 0])$ for all $A \in \Sigma$ such that $\Pi_2(A) = [0, 0]$.

ii) $\Pi_1$ is absolutely continuous of the sixth type with respect to $\Pi_2$ if and only if

$$D(\Pi_1(A_n), [0, 0]) \rightarrow 0$$

for all sequences $(A_n)_{n \in \mathbb{N}} \in \Sigma$ such that $\Pi_2(A_n) \rightarrow [0, 0]$, where $D$ is the pseudo-Hausdorff metric (see [18]).

The absolute continuity of two considered types for $\pi^f_M$ is given by the next proposition.

**Proposition 13:** a) $\pi^f_M(A)$ is absolutely continuous of the first type with respect to $\pi_M$;

b) $\pi^f_M$ is absolutely continuous of the sixth type with respect to $\pi_M$.

**Proof:**

a) Let us choose $A \in \Sigma$ such that $\pi_M(A) = [0, 0]$, i.e., such that both $\mu_1(A)$ and $\mu_r(A)$ are equal to 0. From

$$\pi_M(A) = \left[ \int_A f \circ d\mu_1, \int_A f \circ d\mu_r \right] = [0, 0]$$

follows the required property for the absolute continuity of the first type.
b) Let \((A_n)_{n \in \mathbb{N}} \in \Sigma\) be a sequence such that \(\Pi_2(A_n) \to [0,0]\), let \(D\) be the pseudo-Hausdorff metric (see [18]) and let \(d\) be a metric on the given semiring. Based on

\[
D(\mathcal{P}_M^f(A_n), [0,0]) = D(\left( f \circ d\mu_l, f \circ d\mu_r \right), [0,0])
\]

\[
= \max\{d(\left( f \circ d\mu_l, 0 \right), d(\left( f \circ d\mu_r, 0 \right) = 0
\]

follows the required property for the absolute continuity of the sixth type. □

IV. CONCLUSION

The focus of this paper is on the pseudo-integration of real-valued functions with respect to the interval-valued measure. The construction of the interval-valued \(\sigma-\oplus\)-measure is given and of absolute continuity for that type of measure is investigated. Due to the fact that the observed absolute continuity is connected with Vitali-Hahn-Saks type theorems ([13]), this relation will be considered in the future research.

REFERENCES