Directed Triangles in Digraphs

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Abstract

Let c be the smallest possible value such that every digraph on n vertices with minimum outdegree at least cn contains a directed triangle. It was conjectured by Caccetta and Häggkvist in 1978 that $c = 1/3$. Recently Bondy showed that $c \leq (2\sqrt{6}-3)/5 = 0.3797\cdots$ by using some counting arguments. In this note, we prove that $c \leq 3-\sqrt{7} = 0.3542\cdots$.

Let $G = (V, E)$ denote a digraph on n vertices. The digraphs we consider here contain no loops or multiple arcs. We also assume that G contains no digons; that is, if $(u, v) \in E$, then $(v, u) \notin E$. In 1978, Caccetta and Häggkvist [3] proposed the following conjecture:

Conjecture 1 *Any digraph on n vertices with minimum outdegree at least r contains a directed cycle of length at most $\lceil n/r \rceil$.*

A particularly interesting special case that is still open is: any digraph on n vertices with minimum outdegree at least $n/3$ contains a directed triangle. Short of proving this, one may seek a value c as small as possible such that every digraph on n vertices with minimum outdegree at least cn contains a directed triangle. This was the strategy of Caccetta and Häggkvist [3], who showed that $c \leq (3-\sqrt{5})/2 = 0.3819\cdots$ by a simple inductive argument. Recently Bondy [2] showed that $c \leq (2\sqrt{6}-3)/5 = 0.3797\cdots$ by using some counting arguments. In Theorem 1, we use induction and combine techniques from [2] and [4] to show that $c \leq 3-\sqrt{7} = 0.3542\cdots$.

Theorem 1 *If $\alpha = 3-\sqrt{7} = 0.3542\cdots$, then any digraph on n vertices with minimum outdegree at least αn contains a directed triangle.*

Proof. We use induction on n. Clearly Theorem 1 holds for $n = 3$. Now assume that Theorem 1 holds for all digraphs with fewer than n vertices and G is a counterexample with n vertices. Without loss of generality, it may be supposed that $\deg^+(u) = r = \lceil \alpha n \rceil$ for all $u \in V$. Let $N^+(u) = \{v \in V : (u, v) \in E\}$ and $N^-(u) = \{v \in V : (v, u) \in E\}$.

For any arc $(u, v) \in E$, we set:
$p(u, v) := |N^+(v) \setminus N^+(u)|$, the number of induced directed 2-paths whose first arc is (u, v);
$q(u, v) := |N^-(u) \setminus N^-(v)|$, the number of induced directed 2-paths whose last arc is (u, v);
$t(u, v) := |N^+(u) \cap N^+(v)|$, the number of transitive triangles having the arc (u, v) as 'base'.

We claim that

$$n > r + \deg^-(v) + q(u, v) + (1 - \alpha)t(u, v). \tag{1}$$

If $t(u, v) = 0$, then (1) holds because $N^+(v)$, $N^-(v)$ and $N^-(u) \setminus N^-(v)$ are pairwise-disjoint sets of cardinalities r, $\deg^-(v)$ and $q(u, v)$, respectively. If $t(u, v) > 0$, some vertex $w \in N^+(u) \cap N^+(v)$ has outdegree less than $\alpha t(u, v)$ in the subdigraph of G induced by $N^+(u) \cap N^+(v)$ (otherwise this subdigraph would contain a directed triangle, by the minimality of G). Thus w is joined to at least

$$\deg^+(w) - p(u, v) - \alpha t(u, v) = \deg^+(v) - p(u, v) - \alpha t(u, v) = (1 - \alpha)t(u, v)$$

vertices not in $N^+(v)$. Since G has no directed triangle, these vertices are neither in $N^-(v)$ nor $N^-(u) \setminus N^-(v)$. This establishes the claim. Noting that $t(u, v) = r - p(u, v)$, inequality (1) can be rewritten as:

$$\alpha t(u, v) > 2r - n + \deg^-(v) + q(u, v) - p(u, v).$$

We sum this inequality over all $(u, v) \in E$.

$$\sum_{(u,v)\in E} \alpha t(u, v) = \alpha t,$$

where t is the number of transitive triangles in G,

$$\sum_{(u,v)\in E} (2r - n) = rn(2r - n),$$

$$\sum_{(u,v)\in E} \deg^-(v) = \sum_{v\in V} (\deg^-(v))^2 \geq \frac{1}{n} \left(\sum_{v\in V} \deg^-(v) \right)^2 = r^2 n,$$

and

$$\sum_{(u,v)\in E} (q(u, v) - p(u, v)) = 0,$$

because $\sum_{(u,v)\in E} q(u, v)$ and $\sum_{(u,v)\in E} p(u, v)$ are both equal to the number of induced directed 2-paths in G. Thus $\alpha t > rn(3r - n)$. But $t \leq n \binom{r}{2}$, the number of out-2-claws of G, so $\alpha > 6 - 2n/r \geq 6 - 2/\alpha$, that is, $\alpha < 3 - \sqrt{7}$. □

Graaf, Schrijver and Seymour [5] considered a similar problem that involved both the minimum outdegree and the minimum indegree. They proved that if $\beta = 0.3487\cdots$, then any digraph on n vertices with both minimum outdegree and minimum indegree at least βn contains a directed triangle. In fact, they showed that once a value of α is found, then a value of β can be obtained from the inequality $(\frac{4}{\alpha^2} - \frac{2}{\alpha})x^2 - (\frac{24}{\alpha^2} - \frac{16}{\alpha})x + (\frac{36}{\alpha^2} - \frac{30}{\alpha} + 1) > 0$ [5, pp. 282, formula (16)]. Therefore by using $\alpha = 3 - \sqrt{7}$, one can obtain the following slight improvement.

Corollary 1 *If $\beta = 0.3477\cdots$, then any digraph on n vertices with both minimum outdegree and minimum indegree at least βn contains a directed triangle.*

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References

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