FORMALISM FOR RELATIVE GROMOV-WITTEN INVARIANTS

ERIC KATZ

Abstract. We develop a formalism for relative Gromov-Witten invariants following Li [14, 15] that is analogous to the Symplectic Field Theory of Eliashberg, Givental, and Hofer [2]. This formalism allows us to express natural degeneration formulae in terms of generating functions and re-derive the formulae of Caporaso-Harris [1], Ran [19], and Vakil [21]. In addition, our framework gives a homology theory analogous to SFT Homology.

1. Introduction

Relative Gromov-Witten invariants following Li [14, 15] and the Symplectic Field Theory of Eliashberg, Givental, and Hofer [2] are both theories of holomorphic curves with asymptotic boundary conditions. They have different sources: the theory of relative Gromov-Witten invariants counts stable maps to a projective manifold relative a divisor and is a systematization of degeneration methods in enumerative geometry [1, 19, 21]; Symplectic Field Theory, a generalization of Floer Homology. SFT has an interesting formal structure involving a differential graded algebra whose homology is an invariant of contact structures.

In relative Gromov-Witten theory, one considers a pair (Z, D) where Z is a projective manifold and D is a smooth, possibly disconnected divisor in Z. One looks at stable maps to Z where all points of intersections of the map with D are marked and multiplicities at these points are specified. To obtain a proper moduli stack of such maps, one must allow the target to degenerate to ${}_kZ = Z \sqcup_D P_1 \sqcup_D \cdots \sqcup_D P_k$, that is, Z union a number of copies of $P = \mathbb{P}_D(N_{D/Z} \oplus 1_D)$ the projective completion of the normal bundle to D in Z. Maps with a non-smooth target are said to be *split maps*. Li constructed a moduli stack of relative maps called $\mathcal{M}(\mathcal{Z}, \Gamma)$ for Γ, a certain kind of graph, and constructed its virtual fundamental cycle. This stack has an evaluation map

$$\mathrm{Ev}_{\mathcal{M}\mathcal{Z}} : \mathcal{M}(\mathcal{Z}, \Gamma) \to Z^m \times D^r$$

where m and r are the number of interior and boundary marked points, respectively. *Relative Gromov-Witten invariants* are given by evaluating pullbacks of cohomology classes by Ev against the virtual cycle.

It is natural to break the target ${}_kZ$ as the union of ${}_lZ = Z \sqcup_D P_1 \sqcup_D \cdots \sqcup_D P_l$ and ${}_{k-l-1}P = P_{l+1} \sqcup_D \cdots \sqcup_D P_k$. In fact, such splitting is necessary to parameterize fixed loci in $\mathbb{C}^*$-localization in the sense of [11] and [7] in the relative framework [8]. If we set $X = D$, and $L = N_{D/Z}$, the normal bundle to D in Z, one is led to study stable maps into the projectivization of a line bundle $P = \mathbb{P}_X(L \oplus 1_X)$ relative to the zero and infinity sections, D_0 and D_∞ where two stable maps are declared equivalent if they can be related by a $\mathbb{C}^*$-factor dilating the fibers of $P \to X$.

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One can construct a moduli stack of such maps, $\mathcal{M}(\mathcal{A},\Gamma)$ and its virtual cycle. This moduli stack has certain natural line bundles, called the target cotangent line bundles, L^0 and L^∞ and has an evaluation map

$$\mathrm{Ev}_{\mathcal{MA}} : \mathcal{M}(\mathcal{A},\Gamma) \to X^m \times X^{r_0} \times X^{r_\infty}$$

The *rubber invariants* are obtained by evaluating pullbacks of cohomology by Ev map and powers of $c_1(\mathrm{L}^\infty)$ against the virtual cycle.

The purpose of this paper is a systematic development of the formal structure of the relative Gromov-Witten Invariants, organized in generating functions.

We note here that the rubber invariants have been introduced previously in the literature by Okounkov and Pandharipande [18] and by Graber and Vakil [8] as *maps to a non-rigid target*.

In section 2, we recall the necessary background information to describe the stacks $\mathcal{M}(\mathcal{Z},\Gamma_Z)$ and $\mathcal{M}(\mathcal{A},\Gamma_A)$. We show how to glue together such stacks to parameterize split maps in a stack $\mathcal{M}(\mathcal{Z},\Gamma_Z * \Gamma_A)$. We describe line-bundles on $\mathcal{M}(\mathcal{Z},\Gamma)$: Dil and $\mathrm{L}_{i,\mathrm{ext}}$; and line-bundles on $\mathcal{M}(\mathcal{A},\Gamma)$: Split, L^0, L^∞, $\mathrm{L}_{i,\text{not top}}$, $\mathrm{L}_{i,\text{not bot}}$. These line-bundles have geometric meaning: $\mathrm{L}_{i,\mathrm{ext}}$ is a line-bundle which has a section whose zero-stack consists of maps $f : C \to {}_kZ$ so that the ith marked point is not mapped to $Z \subset {}_kZ$ (counted with multiplicity); Split is a line-bundle whose zero-stack is all split maps; $\mathrm{L}_{i,\text{not top}}$, where i is the label of interior marked point, is a line-bundle whose zero-stack consists of all split maps $f : C \to {}_kP$ where ith marked point is not mapped to P_k; $\mathrm{L}_{i,\text{not bot}}$ is its upside-down analog.

These line-bundles satisfy certain relations. On $\mathcal{M}(\mathcal{Z},\Gamma_Z)$:

$$\mathrm{ev}_i^*\mathcal{O}(D) = \mathrm{L}_{i,\mathrm{ext}};$$

and on $\mathcal{M}(\mathcal{A},\Gamma_A)$:

$$\begin{aligned} \mathrm{L}^0 \otimes \mathrm{L}^\infty &= \mathrm{Split} \\ \mathrm{L}^0 \otimes \mathrm{ev}_i^* L^\vee &= \mathrm{L}_{i,\text{not top}} \\ \mathrm{L}^\infty \otimes \mathrm{ev}_i^* L &= \mathrm{L}_{i,\text{not bot}}\,. \end{aligned}$$

In section 3, we organize intersection numbers on $\mathcal{M}(\mathcal{Z},\Gamma)$ and $\mathcal{M}(\mathcal{A},\Gamma)$ into generating functions. The intersection numbers on $\mathcal{M}(\mathcal{Z},\Gamma)$ of the form

$$\deg(\mathrm{Ev}_{\mathcal{MZ}}{}^* c \cap [\mathcal{M}(\mathcal{Z},\Gamma)]^{\mathrm{vir}})$$

are organized into the *relative potential* F which takes values in a particular graded algebra $\mathcal{F}$. The intersection numbers on $\mathcal{M}(\mathcal{A},\Gamma)$,

$$\deg(c_1(\mathrm{L}^\infty)^l \cup \mathrm{Ev}_{\mathcal{MA}}{}^* c \cap [\mathcal{M}(\mathcal{A},\Gamma)]^{\mathrm{vir}})$$

are organized into the *rubber potential* A which lies in an algebra $\mathcal{R}$. The algebra $\mathcal{R}$ acts on $\mathcal{F}$ which corresponds to joining curves in $\mathcal{M}(\mathcal{Z},\Gamma_Z)$ and $\mathcal{M}(\mathcal{A},\Gamma_A)$ to form split curves in $\mathcal{M}(\mathcal{Z},\Gamma_Z * \Gamma_A)$. Likewise, the multiplication operation in $\mathcal{R}$ corresponds to joining curves in $\mathcal{M}(\mathcal{A},\Gamma_{Ab})$ to those in $\mathcal{M}(\mathcal{A},\Gamma_{At})$.

In section 4, we prove *degeneration formulae* for the relative and rubber potentials. These degeneration formulae are differential equations satisfied by the potentials and are numerical consequences of the relations between line-bundles. Let F be the relative potential of a pair (Z,D) and let $A_{\lambda=0}$ be the rubber potential of the pair $(D, L = N_{D/Z})$ without any powers of $c_1(\mathrm{L}^\infty)$. Then, F satisfies the differential equation

$$\sum_l N_{jl} \frac{\partial F}{\partial \theta_l} = \sum_l M_{jl} \frac{\partial A_{\lambda=0}}{\partial \beta_l} \cdot F$$

where θ and β are variables dual to cohomology classes on Z and X respectively, M_{jl} and N_{jl} are matrices that keep track of cohomology information, and $\cdot$ is the action of $\mathcal{R}$ on $\mathcal{F}$.

Given a pair (X, L), the rubber potential satisfies the analogous differential equation

$$\frac{\partial}{\partial \lambda} \frac{\partial A}{\partial \beta_i} + \sum_j N_{ij} \frac{\partial A}{\partial \beta_j} = \frac{\partial A_{\lambda=0}}{\partial \beta_i} * A$$

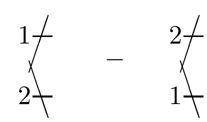
where $*$ is multiplication in $\mathcal{R}$.

In section 5, we work out several examples. We express the rational rubber potential without powers of $c_1(\mathrm{L}^\infty)$ of $(\mathbb{P}^n, \mathcal{O}(m))$ in terms of the Gromov-Witten invariants of $\mathbb{P}^n$ by a Kleiman-Bertini argument. We use this rubber potential to write down a degeneration formula for the relative Gromov-Witten potential of $(\mathbb{F}_n, D_\infty)$ and $(\mathbb{P}^n, H)$ where $D_\infty \subset \mathbb{F}_n$ is the infinity section of the rational ruled surface of degree n, and H is a hyperplane in $\mathbb{P}^n$. This immediately yields the degeneration formulae of of Caporaso-Harris [1], Ran [19], and Vakil [21], phrased in the language of differential operators as first stated by Getzler in [5].

In section 6, we construct a theory directly analogous to Symplectic Field Theory. One begin with a pair (X, L) and organizes a subset of the rubber invariants into a generating function H called the *Hamiltonian* that takes values in an algebra $\mathcal{H}$.

Given two interior marked points, one has the following formula among divisors on $\mathcal{M}(\mathcal{A}, \Gamma)$:

$$\mathrm{ev}_2^*(c_1(L)) - \mathrm{ev}_1^*(c_1(L)) =$$



where the figures on the right specify certain loci of split curves. As a consequence of this formula, we have in $\mathcal{H}$,

$$H^2 = 0$$

We can then define a differential on $\mathcal{H}$ by the formula

$$D^H = Hf - (-1)^{\deg f} fH.$$

The homology of this complex, called Hamiltonain Homology is an invariant of (X, L) and is the algebraic geometric analog of the SFT Homology of $S^1(L)$, the unit circle bundle of L.

In section 7, we give a direct proof of the degeneration formula for the rubber potential using the technique of virtual localization.

This paper draws most directly on the Relative Gromov-Witten Invariants constructed by J. Li [14, 15] and the Symplectic Field Theory of Eliashberg, Givental, and Hofer [2]. Other approaches to relative invariants include those of Gathmann [4], Ionel and Parker [9], and A.-M. Li and Ruan [13].

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All varieties are over $\mathbb{C}$.

2. Background

We discuss stacks of relative stable maps, $\mathcal{M}\mathcal{Z} = \mathcal{M}(\mathcal{Z}, \Gamma)$ and stacks of maps to rubber, $\mathcal{M}\mathcal{A} = \mathcal{M}(\mathcal{A}, \Gamma)$ where Γ is a particular kind of graph. The material in this section is a rephrasing of sections of [10], some of which is straightforward adaptation of [14] and [15]. While J. Li does not construct $\mathcal{M}\mathcal{A}$, our construction directly parallels his. We do change some notation from [14] to suit our purposes.

2.1. Stacks of Relative Maps.

Consider a projective manifold Z with a smooth divisor D. Wew review the construction of the stack of stable maps to Z relative to D. Given an r-tuple of positive integers $\mu = (\mu_1, \ldots, \mu_r)$, consider a marked pre-stable curves

$$(C, x_1, \ldots, x_m, p_1, \ldots, p_r)$$

and maps

$$f : C \to Z$$

so that the divisor f^*D is

$$f^*D = \sum_i \mu_i p_i.$$

To form a proper moduli stack of such maps, we must allow the target to degenerate. Let $L = N_{D/Z}$ be the normal bundle to D in Z. Let $P = \mathbb{P}(L \oplus 1_D)$ be the projective completion of L. P has two distinguished divisors, D_0 and D_∞, the zero and infinity sections of L.

Definition 2.1.1. Let ${}_kZ$ be the union of Z with k copies of P, $Z \sqcup_D P_1 \sqcup_D \cdots \sqcup_D P_k$, the scheme given by identifying $D \subset Z$ with $D_\infty \subset P_1$ and $D_0 \subset P_i$ with $D_\infty \subset P_{i+1}$ for $i = 0, 1, \ldots, k-1$.

Definition 2.1.2. Let $c : {}_kZ \to Z$ be the *collapsing map* that it the identity on Z and projects each P_i to $D \subset Z$

Note that $\mathrm{Sing}({}_kZ)$, the singular locus of ${}_kZ$ is the disjoint union of k copies of D, which we label $D_1, \ldots, D_{k-1}$ where $D_i = D_\infty \subset P_i$.

Definition 2.1.3. Let $\mathrm{Aut}({}_kZ) = (\mathbb{C}^*)^k$ be the group acting on ${}_kZ$ where each factor of $\mathbb{C}^*$ dilates the fibers of the $\mathbb{P}^1$ bundle $P_i \to X$.

Definition 2.1.4. Let $D \subset {}_kZ$ denote the divisor $D_0 \subset P_k \subset {}_kZ$.

We need to specify the appropriate data for the moduli stack of relative stable maps to (Z, D). Here we consider an algebraic curve C that is mapped to ${}_kZ$ by $f : C \to {}_kZ$ with specified tangency to D. We must specify the topology of the curve and the data of the marked points. There are two types of marked points:

(1) *interior marked points* whose image under f is not mapped to D
(2) *boundary marked points* which are mapped to D by f.

We will impose the condition that all points in C mapped to D will be marked. The data of the curve is specified as follows.

Definition 2.1.5. A *relative graph* Γ is the following data:

(1) A finite set of vertices $V(\Gamma)$
(2) A genus assignment for each vertex

$$g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$$

(3) A degree assignment for each vertex

$$d : V(\Gamma) \to B_1(Z) \equiv A_1(Z)/\sim_{\text{alg}}$$

that assigns the class of a curve modulo algebraic equivalence to each vertex.
(4) A set $R = \{1, \dots, r\}$ labelling boundary marked points together with a function assigning boundary marked points to vertices

$$a_R : R \to V(\Gamma)$$

(5) A *multiplicity assignment* for each boundary marked point

$$\mu : R \to \mathbb{Z}_{\geq 1}$$

(6) A set $M = \{1, \dots, m\}$ labelling interior marked points together with an assignment to vertices

$$a_M : M \to V(\Gamma)$$

Definition 2.1.6. Given two relative graphs Γ, Γ' are said to be isomorphic if there is a bijection

$$q : V(\Gamma) \to V(\Gamma')$$

that commutes with the maps g, d, a_R, a_M.

Definition 2.1.7. Let Γ be a relative graph. A *morphism to ${}_kZ$ of type* Γ consists of a marked curve $(C, x_1, \dots, x_{|M|}, p_1, \dots, p_{|R|})$ and a morphism $f : C \to {}_kZ$

(1) C can be written as a disjoint union of pre-stable curves C_v
(2) C_v is a connected curve of arithmetic genus $g(v)$.
(3) The map

$$(c \circ f) : C_v \to_k Z \to Z$$

has $(c \circ f)_* C_v = d(v)$.
(4) $x_i \in C_v$ for $v = a_M(i)$. These are the interior marked points.
(5) $p_i \in C_v$ for $v = a_R(i)$. These are the boundary marked points.
(6) $f^* D = \sum_{i \in R} \mu(i) p_i(s)$

Definition 2.1.8. A morphism $f : C \to {}_kZ$ is said to be *pre-deformable* if $f^{-1}(D_i)$ is the union of nodes so that for $p \in f^{-1}(D_i)$ $(i = 1, 2, \dots, k)$, the two branches of the node map to different irreducible component of ${}_kZ$ and that the order of contact to D_i are equal.

Definition 2.1.9. An *isomorphism* of morphisms f, f' to ${}_kZ$ consists of a diagram

$$\begin{array}{ccc} (C, x, p) & \xrightarrow{f} & {}_kZ \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle t} \\ (C', x', o') & \xrightarrow[f']{} & {}_kZ \end{array}$$

where h is an isomorphism of marked curves and $t \in \mathrm{Aut}({}_kZ)$.

Definition 2.1.10. A morphism to ${}_kZ$ is said to be stable if it has finitely many automorphisms.

Theorem 2.1.11. [14] *There is a Deligne-Mumford stack, $\mathcal{M}(\mathcal{Z},\Gamma)$ parameterizing pre-deformable stable morphisms to ${}_kZ$ for varying k.*

In cases where it is understood, we will write $\mathcal{MZ}$ for $\mathcal{M}(\mathcal{Z},\Gamma)$

This moduli stack is constructed from a moduli functor by considering stable maps to families of targets modelled on a sequence of spaces and divisors $(Z[0],D[0])$,$(Z[1],D[1]),\ldots$ defined inductively as follows

$$\begin{aligned} Z[0] &= Z \\ D[0] &= D \\ Z[n] &= \mathrm{Bl}_{D[n-1]\times\{0\}}(Z[n-1]\times\mathbb{A}^1) \end{aligned}$$

where $D[n]$ is the proper transform of $D[n-1]\times\mathbb{A}^1$. $Z[n]/Z$ possesses a $(\mathbb{C}^*)^k$ group of automorphisms. Note that this construction is an iteration of deformation to the normal cone. $Z[n]$ posses a map to $\mathbb{A}^n$. Given a closed point $x\in\mathbb{A}^n$, the fiber over x is $(Z[n])_x = {}_kZ$ where k is the number of zeroes among x's coordinates.

Definition 2.1.12. A map $f: C\to {}_kZ$ is said to be *split* if $k\geq 1$. The irreducible components of C that are mapped to $P_i\subset {}_kZ$ are said to be *extended components*.

Definition 2.1.13. The *evaluation map* on $\mathcal{MZ}=\mathcal{M}(\mathcal{Z},\Gamma)$ is a map

$$\mathrm{Ev}:\mathcal{MZ}\to Z^m\times D^r$$

given on a relative stable map (C,f) by

$$(x_1,\ldots,x_m,p_1,\ldots,p_r)\hookrightarrow C\to {}_kZ\to Z.$$

We will write $\mathrm{ev}_i:\mathcal{MZ}\to Z$ or $\mathrm{ev}_i:\mathcal{MZ}\to D$ to denote the evaluation map at one of the interior or boundary marked point.

Theorem 2.1.14. [15] *$\mathcal{MZ}$ carries a virtual cycle of complex dimension*

$$\begin{aligned} \mathrm{vdim}\ \mathcal{MZ} &= \sum_{v\in V(\Gamma)}(\dim Z-3)(1-g(v)) \\ &\quad + < c_1(TZ)-D, d(v) >) + |R| + |M|. \end{aligned}$$

2.2. **Stack of Maps to Rubber.** In constructing $\mathcal{M}(\mathcal{Z},\Gamma)$, we had to consider stable maps to ${}_kZ$ which was Z union a chain of P's. It is useful to consider stable maps to the chain of P's subject to automorphisms. We call these maps to rubber.

Let X be a projective manifold and L a line bundle on X. Let $P=\mathbb{P}_X(L\oplus 1_X)$, and let X_0 and X_∞ denote the zero and infinity sections. We study stable maps to P relative to X_0 and X_∞ where we mod out by a $\mathbb{C}^*$-factor that dilates the fibers. Again, the target P may degenerate.

Definition 2.2.1. Let ${}_kP$ be the union of $k+1$ copies of P,

$${}_kP = P_0\sqcup_X P_1\sqcup_X\cdots\sqcup_X P_k$$

gluing $X_0\subset P_i$ to $X_\infty\subset P_{i+1}$ for $i=0,\ldots,k-1$.

${}_kP$ has distinguished divisors $D_\infty = X_\infty\subset P_0$ and $D_0=X_0\subset P_k$.

Definition 2.2.2. Let $\mathrm{Aut}({}_kP)=(\mathbb{C}^*)^{k+1}$ act on ${}_kP$ by dilating fibers of $P\to X$.

Definition 2.2.3. A *rubber graph* Γ is the following data:

(1) a finite collection of vertices $V(\Gamma)$
(2) A genus assignment for each vertex
$$g : V(\Gamma) \to \mathbb{Z}_{\geq 0}.$$
(3) A degree assignment for each vertex
$$d : V(\Gamma) \to B_1(X) = A_1(X)/_{\sim \text{alg}}.$$
(4) Sets $R_0 = \{1, \ldots, r_0\}$, $R_\infty = \{1, \ldots, r_\infty\}$ labelling boundary marked points together with a function assigning boundary marked points to vertices
$$\begin{aligned} a_0 &: R_0 \to V(\Gamma) \\ a_\infty &: R_\infty \to V(\Gamma) \end{aligned}$$
(5) A *multiplicity assignment* for boundary marked points
$$\begin{aligned} \mu^0 &: R_0 \to \mathbb{Z}_{\geq 1} \\ \mu^\infty &: R_\infty \to \mathbb{Z}_{\geq 1} \end{aligned}$$
(6) A set $M = \{1, \ldots, m\}$ labelling interior marked points together with an assignment to vertices
$$a_M : M \to V(\Gamma)$$

Definitions of morphisms to ${}_kP$ are analogous to morphisms to ${}_kZ$ with Z's replaced with P's and the following modifications. The degree assignment is
$$d(v) \in B_1(X)$$
We have marked points
$$(x_1, \ldots, x_m, p_1^0, \ldots, p_{|R_0|}^0, p_1^\infty, \ldots, p_{|R_\infty|}^\infty) \subset C$$
so that
$$\begin{aligned} f^*D_0 &= \sum_{i \in R_0} \mu^0(i) p_i^0 \\ f^*D_\infty &= \sum_{i \in R_\infty} \mu^\infty(i) p_i^\infty \end{aligned}$$

The *multiplicity condition* relates the multiplicities to D_0 and D_∞, to the degree:

Lemma 2.2.4. *If $\mathcal{M}(\mathcal{A}, \Gamma)$ is nonempty then for each vertex $v \in \Gamma$ we have*
$$\sum_{p \in a_0^{-1}(v)} \mu^0(v) - \sum_{p \in a_\infty^{-1}(v)} \mu^\infty(v) = < c_1(L), d(v) >$$

Proof. One uses $X_0 = X_\infty + \pi^* c_1(L)$ for each copy of P in the target. □

Theorem 2.2.5. *For a rubber graph Γ, $\mathcal{M}\mathcal{A} = \mathcal{M}(\mathcal{A}, \Gamma)$ is a proper Deligne-Mumford stack.*

Theorem 2.2.6. *$\mathcal{M}\mathcal{A}$ carries a virtual cycle of complex dimension*
$$\begin{aligned} \text{vdim } \mathcal{M}\mathcal{A} &= \sum_{v \in V(\Gamma)} ((\dim X - 2)(1 - g(v)) \\ &\quad + < c_1(TX), d(v) >) + |R_0| + |R_\infty| + |M| - 1. \end{aligned}$$

We have analogous evaluation maps ev_i at the interior and boundary marked points (mapping to D_0 and D_∞).

Definition 2.2.7. The *evaluation map* on $\mathcal{MA} = \mathcal{M}(\mathcal{A}, \Gamma)$ is

$$\mathrm{Ev} : \mathcal{MA} \to X^n \times X^{|R_0|} \times X^{|R_\infty|}$$

This moduli stack is constructed from a moduli functor by considering stable maps to families of targets modelled on a sequence of spaces and divisors $(A[0], D_0[0], D_\infty[0])$,$(A[1], D_0[1], D_\infty[1]), \ldots$ where

$$\begin{aligned} A[0] &= P \\ D_0[0] &= X_0 \\ D_\infty[0] &= X_\infty \\ A[n] &= \mathrm{Bl}_{D_0[n-1]\times\{0\}}(A[n-1] \times \mathbb{A}^1) \end{aligned}$$

where $D_0[n]$ is the proper transform of $D_0[n-1] \times \mathbb{A}^1$. and $D_\infty[n]$ is the inverse image of $D_\infty[n-1] \times \mathbb{A}^1$. $A[n]/X$ possesses a $(\mathbb{C}^*)^{k+1}$ group of automorphisms

Definition 2.2.8. A *split map* in $\mathcal{MA}$ is a map $f : C \to {}_kP$ where $k \geq 1$, that is, the target is not smooth.

Definition 2.2.9. For a map $f : C \to {}_kP$ in $\mathcal{MA}$, the irreducible components of C that are mapped to P_k are said to be the *top components* while the components of C that are mapped to P_0 are said to be the *bottom components*.

We should explain our top/bottom convention. In Z, moving towards D is considered moving towards the top. In P, D_0 is considered the top while D_∞ is the bottom. This slightly odd convention makes sense in that the most natural choice for (X, L) is $(D, N_{D/Z})$. In this case, the zero section of P is identified with D and the normal bundle to D_0 in P is equal to the normal bundle to D in Z. Therefore, $D_0 \subset P$ like $D \subset Z$ is on top.

2.3. **Trivial Cylinders.**

We will single out certain connected components of curves parameterized by $\mathcal{MA}$. These are the so called trivial cylinders which will be significant when we encode the data of the moduli space into generating functions.

Definition 2.3.1. Let Γ be a rubber graph. A vertex v is said to correspond to a *trivial cylinder* of degree r if

(1) $g(v) = 0$.
(2) $d(v) = 0$
(3) $a_0^{-1}(v)$ is a single point.
(4) $a_\infty^{-1}(v)$ is a single point.
(5) $\mu^0(a_0^{-1}(v)) = \mu^\infty(a_\infty^{-1}(v)) = r$.
(6) $A_M^{-1}(v)$ is empty.

A trivial cylinder corresponds to a connected component of a map to rubber. This map is from a chain of k $\mathbb{P}^1$'s to ${}_kP$ where each $\mathbb{P}^1$ is mapped to a fiber of $P_i \to X$ and is of degree r and totally ramified at X_0 and X_∞.

Note that if Γ has a single vertex corresponding to a trivial cylinder, then there are no rubber maps of type Γ that are not invariant under the $\mathbb{C}^*$-action that dilates the fibers of P. Therefore, there are no stable rubber maps and the moduli space is empty. This does not rule out morphisms of type Γ which has a component which is a trivial cylinder. In fact, one can add a trivial cylinder component to any family.

2.4. **Gluing Moduli Stacks.** Consider a projective manifold Z, together with a smooth divisor D. We will consider a relative moduli stack $\mathcal{M}(\mathcal{Z},\Gamma_Z)$ corresponding to (Z,D) and a rubber moduli stack $\mathcal{M}(\mathcal{A},\Gamma_A)$ corresponding to $(X = D, L = N_{D/Z})$ where $N_{D/Z}$ is the normal bundle to D in Z. One can join maps in $\mathcal{M}\mathcal{Z}$ to maps in $\mathcal{M}\mathcal{A}$ if certain conditions are met. Likewise under particular conditions can join maps in $\mathcal{M}\mathcal{A}_1 = \mathcal{M}(\mathcal{A},\Gamma_1)$ to maps in $\mathcal{M}\mathcal{A}_2 = \mathcal{M}(\mathcal{A},\Gamma_2)$. We make these conditions precise below.

Definition 2.4.1. Let Γ_Z be a relative graph and Γ_A be a rubber graph. Suppose that $L : RZ \to RA_\infty$ is a bijection from the labelling sets for boundary marked points in Γ_Z to the labelling sets for boundary marked points mapping to D_∞ in Γ_A so that

$$\mu_Z(q) = \mu_A^\infty(L(q)).$$

Let

$$J : M_Z \sqcup M_A \to \{1,\dots,|M_Z| + |M_A|\}$$

be a bijection between the labelling sets of the interior marked points and a set of $|M_Z| + |M_A|$ elements. We call the data (Γ_A,Γ_Z,L,J) a *graph join quadruple*.

Colloquially, we've matched boundary marked points on Γ_Z and Γ_A with the same multiplicity.

Definition 2.4.2. Define the *graph join* $\Gamma_A *_{L,J} \Gamma_Z$ to be the following relative graph. Let the graph Δ be obtained by taking as vertices the vertices of Γ_Z and Γ_A and for every $q \in RZ$, place an edge between the vertices corresponding to q and $L(q)$. Let $\Gamma_A *_{L,J} \Gamma_Z$ be given as follows. The vertices of $\Gamma = \Gamma_A *_{L,J} \Gamma_Z$ are the connected components of Δ. Let $b_Z : V(\Gamma_Z) \to V(\Gamma)$, and $b_A : V(\Gamma_A) \to V(\Gamma)$ be the functions taking vertices of Γ_A and Γ_A to the components in Δ containing them. For $v \in V(\Gamma)$, let Δ_v be the connected component of Δ corresponding to v. Define the data for $\Gamma_A *_{L,J} \Gamma_Z$ as follows:

(1) $g(v) = (\sum_{w\in b_Z^{-1}(v)} g(w)) + (\sum_{w\in b_A^{-1}(v)} g(w)) + \dim(h^1(\Delta_v))$
(2) $d(v) = (\sum_{w\in b_Z^{-1}(v)} d(w)) + (\sum_{w\in b_A^{-1}(v)} i_*d(w))$ where $i : X \to Z$ is the inclusion and $i_* : B_1(X) \to B_1(Z)$ is the induced map.
(3) $R = RA_0$ with $a_R : R \to V(\Gamma)$ given by
$$a_R = b_A \circ a_0$$
(4) $\mu : R \to \mathbb{Z}_{\geq 1}$ given by
$$\mu_R = \mu^0$$
(5) $M = \{1,\dots,|M_Z| + |M_A|\}$ with assignment function $a : M \to V(\Gamma)$ given for $k \in J(M_Z)$ by
$$a(k) = b_Z \circ a_{MZ} \circ J^{-1}$$
while for $k \in J(M_A)$ by
$$a(k) = b_A \circ a_{MA} \circ J^{-1}$$

Given (Γ_Z,Γ_A,L,J) as above, consider the evaluation map at the boundary marked points on $\mathcal{M}(\mathcal{Z},\Gamma_Z)$ followed by a map $L_* : D^r \to D^r$ which reorders the products of D^r according to L:

$$L_* \circ Ev_R : \mathcal{M}(\mathcal{Z},\Gamma_Z) \to D^r \to D^r$$

and the evaluation map at the boundary marked points mapping to $D_\infty \cong X$ on $\mathcal{M}(\mathcal{A}, \Gamma_A)$,

$$\mathrm{Ev}_{R_\infty} : \mathcal{M}(\mathcal{A}, \Gamma_A) \to D^r.$$

Theorem 2.4.3. [14] *There is a morphism*

$$\Phi_{\Gamma_Z,\Gamma_A,L,J} : \mathcal{M}(\mathcal{A}, \Gamma_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_Z) \to \mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z).$$

Definition 2.4.4. Let the stack $\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Gamma_A \sqcup_{L,J} \Gamma_Z)$ be the image stack of Φ in $\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)$.

Definition 2.4.5. An *automorphism of RZ* is a permutation

$$\sigma : RZ \to RZ$$

so that $\mu_Z(\sigma(i)) = \mu_Z(i)$ and $a_{RZ}(\sigma(i)) = a_{RZ}(i)$. The group of all such automorphisms is denoted by $\mathrm{Aut}_{\Gamma_Z}(RZ)$. Likewise, we define $\mathrm{Aut}_{\Gamma_A}(RA_0)$ and $\mathrm{Aut}_{\Gamma_A}(RA_\infty)$.

Given $L : RZ \to RA_\infty$, we may define $\mathrm{Aut}_{\Gamma_A,\Gamma_Z,L}(RZ, RA_\infty)$ as the subgroup of $\mathrm{Aut}_{\Gamma_Z}(RZ) \times \mathrm{Aut}_{\Gamma_A}(RA_\infty)$ such that for $(\sigma, \tau) \in \mathrm{Aut}_{\Gamma_Z}(RZ) \times \mathrm{Aut}_{\Gamma_A}(RA_\infty)$ we have $L(\sigma(i)) = \tau(L(i))$ for $1 \le i \le |RZ|$.

Lemma 2.4.6. *(*[14]*, Prop 4.13) Φ is finite and étale onto its image of degree equal to*

$$|\mathrm{Aut}_{\Gamma_A,\Gamma_Z,L}(RZ, RA_\infty)|$$

at every integral substack of $\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Gamma_A \sqcup_{L,J} \Gamma_Z)$.

Definition 2.4.7. Let $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$ be a graph join quadruple Let the *boundary multiplicity* $m(\Upsilon)$ be given by

$$m(\Upsilon) = \prod_{i \in RZ} \mu_Z(i).$$

Definition 2.4.8. Two quadruples are said to be *join-equivalent* if they give the same image under Φ

Proposition 2.4.9. *Consider a join-equivalence class of quadruples*

$$[\Upsilon] = [(\Gamma_A, \Gamma_Z, L, J)].$$

Let $N = \mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)$. Let

$$M_\Upsilon = \coprod_{(\Gamma'_Z, \Gamma'_A, L', J')} \mathcal{M}(\mathcal{A}, \Gamma'_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma'_Z)$$

where the disjoint union is over quadruples join-equivalent to Υ. Then $\Phi_{[\Upsilon]} : M \to N$ is an étale map of degree

$$|M_Z|!|M_A|!(|RZ|!)^2$$

Proof. This follows from the previous lemma and that there are

$$|M_Z|!|M_A|! \frac{(|RZ|!)^2}{|\mathrm{Aut}_{\Gamma_A,\Gamma_Z,L}(RZ, RA_\infty)|}$$

elements in $(\Gamma_Z, \Gamma_A, L, J)$'s graph join-equivalence class. □

Likewise, we may define graph-join for rubber graphs, Γ_t, Γ_b (where t and b stand for top and bottom). Let $L : R_{b0} \to R_{t\infty}$ be a bijective function satisfying

$$\mu_b^0(q) = \mu_t^\infty(L(q)).$$

Let

$$J : M_b \sqcup M_t \to \{1, \ldots, |M_b| + |M_t|\}$$

be a bijective map. Then we define the *graph join*, a rubber graph $\Gamma = \Gamma_t *_{L,J} \Gamma_b$ as above, except that instead of condition (3) above, we have

$$R_0 = R_{t0}, \quad a_0 = b_{A_t} \circ a_{0t}$$
$$\mu^0 = \mu_t^0$$
$$R_\infty = R_{b\infty}, \quad a_\infty = b_{A_b} \circ a_\infty b$$
$$\mu^\infty = \mu_b^\infty.$$

Definition 2.4.10. Let $\Upsilon = (\Gamma_{A_t}, \Gamma_{A_b}, L, J)$ be a quadruple describing a decomposition in $\mathcal{M}(\mathcal{A}, \Gamma_{A_t} *_{L,J} \Gamma_{A_b})$. Define $m(\Upsilon)$ by

$$m(\Upsilon) = \prod_{i \in RA_t} \mu_t^\infty(i).$$

Now, let $r = |R_{b0}| = |R_{t\infty}|$. Exactly as above, we have

Theorem 2.4.11. [14] *There is a morphism*

$$\Phi : \mathcal{M}(\mathcal{A}, \Gamma_{A_t}) \times_{D^r} \mathcal{M}(\mathcal{A}, \Gamma_{A_b}) \to \mathcal{M}(\mathcal{A}, \Gamma_{A_t} *_{L,J} \Gamma_{A_b}),$$

étale of degree $|\operatorname{Aut}_{\Gamma_{A_b}, \Gamma_{A_t}, L}(RA_{b0}, RA_{t\infty})|$

Corollary 2.4.12. *Consider a moduli stack* $N = \mathcal{M}(\mathcal{A}, \Gamma_{A_b} \sqcup_{L,J} \Gamma_{A_t})$. *Let*

$$M = \coprod_{(\Gamma'_{A_t}, \Gamma'_{A}, L', J')} \mathcal{M}(\mathcal{A}, \Gamma'_{A_b}) \times_{D^r} \mathcal{M}(\mathcal{A}, \Gamma'_{A_t})$$

where the disjoint union over $(\Gamma_{At}, \Gamma Ab, L, J)$*'s join-equivalence class. The* $M \to N$ *is an étale map of degree*

$$|M_{A_b}|!|M_{A_t}|!(|R_{A_b 0}|!)^2$$

2.5. **Line Bundles on Moduli Stacks.**

The moduli stacks carry line-bundles with particular geometric meaning.

Given a relative graph Γ_Z, $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$ has canonically defined line-bundles

(1) Dil, a line bundle that has a section whose zero stack is supported on all split curves.
(2) $\mathrm{L}_{i,\mathrm{ext}}$ where i is a distinguished interior marked point, a line bundle that has a section whose zero stack is supported on split curves where i lies on an extended component.

It will be shown that $c_1(\mathrm{Dil})$ on $\mathcal{M}\mathcal{Z}$ is (counted with multiplicity) the locus of split maps and $c_1(\mathrm{L}_{i,\mathrm{ext}})$ is a weighted count of split maps with i on an extended component.

For Γ_A, $\mathcal{M}(\mathcal{A}, \Gamma_A)$ has the following line-bundles

(1) L^0, the target cotangent line-bundle at X_0. $c_1(\mathrm{L}^0) = \Psi_0$, the target Ψ class of [3].
(2) L^∞, the target cotangent line-bundle at X_∞ which is L^0's upside-down analog. $c_1(\mathrm{L}^\infty) = \Psi_\infty$.

(3) Split, the Split bundle which has a section whose zero stack is supported on all split maps (Definition 2.2.8).
(4) $\mathrm{L}_{i,\text{not top}}$, the not-top bundle with respect to a distinguished interior marked point i. This bundle has a section whose zero stack is supported on split maps where the ith marked point is not on a top component.
(5) $\mathrm{L}_{i,\text{not bot}}$, the not-bottom bundle with respect to a interior marked point i. This bundle has a section whose zero stack is supported on split maps where the ith marked point is not on a bottom component.

L^0, which is defined in terms of an atlas, has the following intuitive description: given a map to rubber, (C, f) in $\mathcal{MA}$, consider the target of f, ${}_kP$ which has a top component P_k. Let $\widehat{C}$ be the component of C mapping to P_k. There is a $\mathbb{C}^*$ family of maps $\widehat{f} : \widehat{C} \to P$ that occur as the restriction of f; these $\mathbb{C}^*$ families fit together to give a $\mathbb{C}^*$ bundle; the associated $\mathbb{C}$ bundle is L^0. L^∞ is the analogous bundle where we consider the bottom component.

L^0 and L^∞ can be given an interpretation in the stack of rational sausages, the substack of $\mathfrak{M}_{0,2}$ consisting of pre-stable curves so that the two marked points lie on different sides of every node. L^0 and L^∞ are equal to the pullbacks of the cotangent line classes at the two marked points. See [8] for an elaboration.

Theorem 2.5.1. [10] *The line-bundles satisfy the following relations: On $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$,*

$$ev_i^*\mathcal{O}(D) = \mathrm{L}_{i,ext}$$

and on $\mathcal{M}(\mathcal{A}, \Gamma_A)$,

$$\mathrm{L}^0 \otimes \mathrm{L}^\infty = \mathrm{Split} \tag{1}$$

$$\mathrm{L}^0 \otimes ev_i^* L^\vee = \mathrm{L}_{i,not\ top} \tag{2}$$

$$\mathrm{L}^\infty \otimes ev_i^* L = \mathrm{L}_{i,not\ bot} \tag{3}$$

If we consider the stack of rational sausages where L^0 and L^∞ are the restriction of ψ classes on $\mathfrak{M}_{0,2}$, then the (1) is the pullback of the genus 0 recursion relation of Lee and Pandharipande [12]. A proof of the enumerative consequences of (2) and (3) is given in section 7.

3. Generating Functions

An important ideas in Gromov-Witten theory, originating [23], is that of organizing invariants in generating functions. Relations satisfied by the invariants become differential equations for the generating function. In this section, we define generating functions for relative and rubber invariants motivated by Symplectic Field Theory [2].

3.1. Relative Potential.

Let us consider a pair (Z, D) where Z is a projective manifold and $D \subset Z$ is a smooth divisor on Z. The generating function of the relative invariants takes values in a particular graded algebra.

Let us specify the following data: an Euler characteristic, χ; a curve class (up to algebraic equivalence), $d \in B_1(Z)$; a number of interior marked points: m; a number of boundary marked points, r; and a r-tuple of multiplicities to D, $(s_1, s_2, \ldots, s_r)$. Consider the set Ξ of relative graphs Γ so that

(1) $\sum_v d(v) = d$

(2) $\sum_v (2-2g(v)) = \chi$
(3) $|M| = m$
(4) $|R| = r$
(5) $(\mu(1), \mu(2), \ldots, \mu(r)) = (s_1, s_2, \ldots, s_r)$

To each Γ, we associate the moduli space $\mathcal{MZ} = \mathcal{M}(\mathcal{Z}, \Gamma)$ which has an evaluation map at the interior and boundary marked points,

$$\mathrm{Ev} : \mathcal{MZ} \to (Z)^m \times (D)^r.$$

Definition 3.1.1. Given cohomology classes

$$e_1, \ldots, e_n \in H^*(Z)$$

and

$$c_1, \ldots, c_r \in H^*(D),$$

define the *correlator* by

$$< e_1, \ldots, e_n \cdot c_1, \ldots, c_r >_{\chi, A, (s_1, \ldots, s_r)}$$
$$= \sum_{\Gamma \in \Xi} \mathrm{Ev}^*(e_1 \times \cdots \times e_n \times c_1 \times \cdots \times c_r) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{\mathrm{vir}}.$$

By stability considerations, the above sum over relative graphs is finite.

Let $e_1, e_2, \ldots, e_l \in H^*(Z)$ be a homogeneous basis of $H^*(Z)$. Let $c_1, c_2, \ldots, c_k$ be a homogeneous basis of $H^*(D)$. Let $\mathbb{Q}[B_1(Z)]$ be the group algebra on $B_1(Z)$, generated as a vector space by elements of $B_1(Z)$, equipped with multiplication

$$\tilde{z}_1^d \cdot \tilde{z}_2^d = \tilde{z}^{d_1+d_2}$$

where $d_1, d_2 \in B_1(Z)$.

Consider the graded super-commutative algebra over $\mathbb{Q}[B_1(Z)]$ freely generated by $\tilde{\hbar}^{-1}$, $\tilde{\hbar}$, $\theta_1, \theta_2, \ldots, \theta_l$, and, for every positive integer n, elements $\tilde{p}_{n,1}, \tilde{p}_{n,2}, \ldots, \tilde{p}_{n,k}$ with the following degrees

$$\begin{aligned} \deg \tilde{\hbar} &= -2(\dim Z - 3) \\ \deg \tilde{z}^d &= 2 < c_1(TX), d > \\ \deg \theta_i &= 2 - \deg e_i \\ \deg \tilde{p}_{n,j} &= 2 - \deg c_i - 2j \end{aligned}$$

$\hbar$ will be a formal variable corresponding to one half of the Euler characteristic, $\tilde{z}^d$ to degree, θ_i to interior marked points that are mapped to a cycle Poincare-dual to e_i, and $\tilde{p}_{n,j}$ to boundary marked points with multiplicity n and mapped to a cycle on D Poincare-dual to c_j.

We define $\mathcal{F}$ to be a partial completion of the above algebra. We look at Laurent series in $\tilde{\hbar}$ whose coefficients are polynomials in the $\tilde{p}$-variables whose coefficients are power series in the θ variables.

Let $\widetilde{\mathcal{G}}$ be the noncommutative algebra of power series in $\theta_i e_i$. Let $\mathcal{P}$ be the noncommutative algebra of power series in $\tilde{p}_{n,j} c_j$.

Define the $\mathcal{F}$-correlator to be

$$< \theta_{i_1} e_{i_1} \ \ldots \ \theta_{i_n} e_{i_n}, \tilde{p}_{n_1,j_1} c_{j_1} \ \ldots \ \tilde{p}_{n_r,j_r} c_{j_r} >_{\chi, A}$$
$$= \theta_{i_1} \ldots \theta_{i_n} \tilde{p}_{n_1,j_1} \ldots \tilde{p}_{n_r,j_r} < e_{i_1}, \ldots, e_{i_n} \cdot c_{j_1} \ldots c_{j_r} >_{\chi, A, (n_1, n_2, \ldots, n_r)}$$

Extend the $\mathcal{F}$-correlator multi-linearly to a map

$$(,)_{g,A} : \widetilde{\mathcal{G}} \otimes \mathcal{P} \to \mathcal{F}$$

Let $\tilde{\Gamma} \in \widetilde{\mathcal{G}}$, $\tilde{P} \in \mathcal{P}$ be given by

$$\begin{aligned}
\tilde{\Gamma} &= \sum_{l \geq 0} \frac{1}{l!} \left(\sum \theta_i e_i \right)^l, \\
\tilde{P} &= \sum_n \frac{1}{n!} \left(\sum_{k,i} \tilde{p}_{k,i} c_i \right)^n.
\end{aligned}$$

Definition 3.1.2. The *relative potential* of (Z, D) is $F \in \mathcal{F}$ defined by

$$F = \sum_{g \geq 0} \sum_{d \in B_1(Z)} < \tilde{\Gamma}, \tilde{P} >_{g,d} \hbar^{g-1} z^d.$$

Note that F is indeed in $\mathcal{F}$. A given coefficient of z^A is a polynomial in the $\tilde{p}$ variables because the number of $\tilde{p}$ variables is less than $A \cdot D$. By our choice of degrees for the formal variable and by the virtual dimension of $\mathcal{MZ}$, F is a homogeneous element of degree of 0. Our usage of F disagrees with that of [2] because we consider disconnected stable maps.

Example 3.1.3. For the target $(\mathbb{P}^1, \infty)$, the relative potential is

$$F = \exp(\hbar^{-1}(\frac{1}{2!}\theta_0^2\theta_1 + e^{\theta_1} p_{1,1} z) + h^0(-\frac{1}{24}\theta_1))$$

where θ_0 and θ_1 are dual to the classes $[\mathbb{P}^1], [\mathrm{pt}]$. The $\hbar^{-1}\frac{1}{2!}\theta_0^2$ corresponds to contracted rational curves, $\hbar^{-1}e^{\theta_1}p_{1,1}z$ to degree one rational curves with arbitrarily many marked points, and $-\hbar^0\frac{1}{24}\theta_1$ to contracted elliptic curves.

Example 3.1.4. For the target $(\mathbb{P}^2, L)$, the relative potential is

$$F = \exp(F_{d=0} + F_{d \geq 0})$$

where $F_{d=0}$ and $F_{d \geq 0}$ correspond to connected degree 0 and to positive degree maps, respectively:

$$\begin{aligned}
F_{d=0} &= -\hbar^0 \frac{1}{8}\theta_1 + \hbar^0(\frac{1}{2!}\theta_2\theta_0^2 + \frac{1}{2!}\theta_1^2\theta_0) \\
F_{d \geq 0} &= \hbar^{-1}\theta_2 p_{1,1} z + \hbar^{-1}\frac{\theta_2^2}{2!}p_{1,0}z + \hbar^{-1}\frac{\theta_2^3}{3!}(p_{2,1} + \hbar^{-1}\frac{1}{2!}p_{1,1}^2)z^2 \\
&+ \frac{\theta_2^4}{4!}(2p_{2,0} + p_{1,1}p_{1,0})z^2 + \dots
\end{aligned}$$

where $\hbar^{-1}\theta_2 p_{1,1} z$ corresponds to a degree 1 rational map with one interior marked point mapping to a specified point in $\mathbb{P}^2$ and one boundary marked point mapping to a specified point in L; and $\hbar^{-1}\frac{\theta_2^4}{4!}p_{1,1}p_{1,0}z^2$ corresponds to a degree 2 rational map through four specified, generic points in $\mathbb{P}^2$ with two boundary points of contact of order 1 to L, one at a specified point, the other free.

3.2. **The Rubber Potential.** We can write all intersection numbers arising from $\mathcal{MA}$ in terms of a generating function. Let L be a line-bundle over a projective manifold X. Pick an Euler characteristic χ, m interior marked points, $r_0 + r_\infty$ boundary marked points. Fix a curve class $d \in B_1(X)$, a r_0-tuple of multiplicities $(s_1^0, \ldots, s_{r_0}^0)$ to D_0 at the r_0 boundary marked points, and a r_∞-tuple of multiplicities $(s_1^\infty, \ldots, s_{r_\infty}^\infty)$ to D_∞ at the r_∞ boundary marked points. Let Ξ be the set of rubber graphs Γ with no vertices associated to trivial cylinders so that

(1) $\sum_v d(v) = d$
(2) $\sum_v (2 - 2g(v)) = \chi$
(3) $|M| = m$
(4) $|R_0| = r_0$
(5) $(\mu^0(1), \mu^0(2), \ldots, \mu^0(r_0)) = (s_1^0, s_2^0, \ldots, s_{r_0}^0)$
(6) $|R_\infty| = r_\infty$
(7) $(\mu^\infty(1), \mu^\infty(2), \ldots, \mu^\infty(r_\infty)) = (s_1^\infty, s_2^\infty, \ldots, s_{r_\infty}^\infty)$.

We have evaluation maps at the marked points

$$\mathrm{Ev} : \mathcal{M}(\mathcal{A}, \Gamma) \to X^m \times X^{r_0} \times X^{r_\infty}.$$

We also have the two line bundles on $\mathcal{MA}$, L^0 and L^∞. Given a cohomology class $c \in H^*(X^{m+r_0+r_\infty})$, we consider intersection numbers of the form

$$c_1(\mathrm{L}^\infty)^l \cup \mathrm{Ev}^*(c) \cap [\mathcal{MA}]^{\mathrm{vir}}.$$

Definition 3.2.1. Given a curve class $d \in B_1(X)$, an Euler characteristic χ, non-negative integers l, n, r_0, r_∞, multiplicities $S^0 = (s_1^0, \ldots, s_{r_0}^0)$, $S^\infty = (s_1^\infty, \ldots, s_{r_\infty}^\infty)$,

$$c_1, \ldots, c_m, e_1^0, \ldots, e_{r_0}^0, e_1^\infty, \ldots, e_{r_\infty}^\infty \in H^*(X),$$

a non-negative integer m, define the *correlator*

$$\begin{aligned} & (c_1, \ldots, c_n \cdot e_1^0, \ldots, e_{r_0}^0 \cdot e_1^\infty, \ldots, e_{r_\infty}^\infty)_{\chi, d, S^0, S^\infty, l} \\ = & \sum_{\Gamma \in \Xi} \deg(c_1(\mathrm{L}^\infty)^l \cup \mathrm{Ev}^*(c_1 \times \cdots \times c_n \times e_1^0 \times \cdots \times e_{r_0}^0 \times e_1^\infty \times \cdots \times e_{r_\infty}^\infty) \\ & \cap [\mathcal{M}(\mathcal{A}, \Gamma)]^{\mathrm{vir}}.) \end{aligned}$$

We organize the correlators into a generating function which we call the *rubber potential* which takes values in a particular graded algebra, $\mathcal{R}$. Pick a a homogeneous basis $c_1, \ldots, c_k$ for $H^*(X)$. We define elements z^A for all $A \in B_1(X)$, $\hbar$, λ, $\beta_1, \ldots, \beta_k$, $p_{n,1}, \ldots, p_{n,k}$, $q_{n,1}, \ldots, q_{n,k}$ for all positive integers n. The z^A's obey the relations $z^A \cdot z^B = z^{A+B}$ where $+$ is addition in $B_1(X)$. The elements are graded of the following degrees

$$\begin{aligned} \deg z^d &= 2 < c_1(TX), d > + 2 < c_1(L), d > \\ \deg \hbar &= -2(\dim X - 2) \\ \deg \lambda &= -2 \\ \deg \beta_i &= 2 - \deg c_i \\ \deg p_{n,i} &= 2 - \deg c_i - 2n \\ \deg q_{n,i} &= 2 - \deg c_i + 2n. \end{aligned}$$

Multiplication in the algebra is defined as follows. The $\hbar$-, λ-, β_i-variables are taken to be supercentral while the p- and $q-$variables obey supercommutation relations

$$[p_{n_1,i_1}, p_{n_2,i_2}] = 0,\ [q_{n_1,i_1}, q_{n_2,i_2}] = 0,\ [q_{n_1,i_1}, p_{n_2,i_2}] = n_1\delta_{n_1,n_2} g^{i_1 i_2}\hbar.$$

where $g^{i_1 i_2}$ is the Poincare pairing on $H^*(X)$. Note that this algebra can be realized by writing $q_{n,i}$ as a differential operator

$$q_{n,i} = n\hbar \sum_j g^{ij}\frac{\partial}{\partial p_{n,j}}.$$

The multiplication keeps track of different ways of joining curves. Let us consider an example in a toy model of our algebra. Consider variables p_1, p_2, p_3 which are all of even parity together with q_1, q_2, q_3 so

$$q_i = \hbar\frac{\partial}{\partial p_i}.$$

Then,

$$(\hbar^{-1}p_1q_1q_2) * (\hbar^{-1}p_1p_2p_3) = p_1p_3 + \hbar^{-2}p_1^2p_2p_3q_1q_2 + \hbar^{-1}p_1^2p_3q_1 + \hbar^{-1}p_1p_2p_3q_2.$$

If we see $\hbar$ as a genus marker where a term of Euler coefficient χ is marked with $\hbar^{-\frac{1}{2}\chi}$, this multiplication corresponds to the geometric situation illustrated in figure 1.

The algebra $\mathcal{R}$ consists of Laurent series in $\hbar$ whose coefficients are power series in the p-variables whose coefficients are power series in the β-variables whose coefficients are polynomials in the q- and λ-variables.

Definition 3.2.2. The $\mathcal{R}$-*correlator* is

$$\begin{aligned}
& (\beta_{i_1}c_{i_1}\ldots\beta_{i_m}c_{i_m}\ , p_{m_1,j_1}c_{j_1}\ldots p_{m_{r_0},j_{r_0}}c_{j_{r_0}},\ q_{n_1,k_1}c_{k_1}\ldots q_{n_{r_\infty},k_{r_\infty}}c_{k_{r_\infty}})_{\chi,d,l}\\
= \ & \beta_{i_1}\ldots\beta_{i_m}p_{m_1,j_1}\ldots p_{m_{r_0},j_{r_0}}q_{n_1,k_1}\ldots q_{n_{r_\infty},k_{r_\infty}}\\
& \cdot(c_{i_1},\ldots,c_{i_n}\cdot c_{j_1},\ldots,c_{j_{r_0}}\cdot c_{k_1},\ldots,c_{k_{r_\infty}})_{\chi,d,(m_1,m_2,\ldots,m_{r_0}),(n_1,n_2,\ldots,n_{r_\infty}),l}
\end{aligned}$$

considered as an element of $\mathcal{R}$.

Let $\mathcal{B}, \mathcal{Q}, \mathcal{P}$ be the (noncommutative) power-series algebras freely generated by $\beta_i c_i$, $q_{n,i}c_i$, $p_{n,i}c_i$ respectively.

We can extend the $\mathcal{R}$-correlator by linearity to give a multi-linear function

$$(\ ,\ ,\ )_{\chi,A,m} : \mathcal{B}\otimes\mathcal{P}\otimes\mathcal{Q}\to\mathcal{R}$$

Let $B\in\mathcal{B}$, $P\in\mathcal{P}$, $Q\in\mathcal{Q}$ be given by

$$\begin{aligned}
B &= \sum_{l\geq 0}\frac{1}{l!}\left(\sum_i \beta_i c_i\right)^l,\\
P &= \sum_n \frac{1}{n!}\left(\sum_{k,i} p_{k,i}c_i\right)^n,\\
Q &= \sum_n \frac{1}{n!}\left(\sum_{k,i} q_{k,i}c_i\right)^n.
\end{aligned}$$

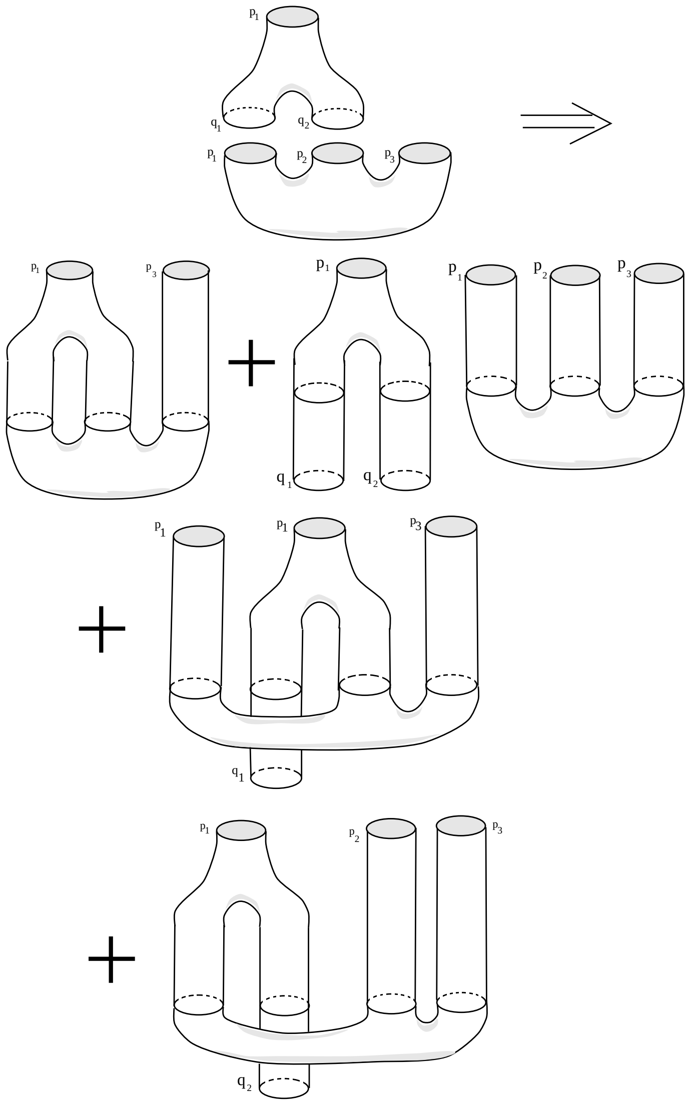


FIGURE 1. Geometric Illustration of Multiplication in $\mathcal{R}$. This figure is borrowed from [2].

Definition 3.2.3. The *rubber potential* A is

$$A = \sum_{\chi} \sum_{d \in B_1(X)} \sum_{l} \hbar^{-\frac{1}{2}\chi} \frac{\lambda^l}{l!} (B, P, Q)_{\chi,d,l} z^d.$$

Note that for a moduli stack $\mathcal{MA}$ to be non-empty, by Lemma 2.2.4, the multiplicities must satisfy

$$s_1^0 + \cdots + s_{r_0}^0 - s_1^\infty - \cdots - s_{r_\infty}^\infty =< c_1(L), A >$$

so

$$r_\infty \leq s_1^0 + \cdots + s_0^{r_0} - < c_1(L), A > .$$

Therefore, the rubber potential satisfies the polynomiality in q condition to lie in $\mathcal{R}$. The rubber potential A is homogeneous of degree 2.

Definition 3.2.4. The *rubber potential without powers of* L^∞ is given by $A_{\lambda=0}$.

Example 3.2.5. The rubber potential of $(X, L) = (\mathrm{pt}, 1_{\mathrm{pt}})$ obeys

$$A_{\lambda=0} = \hbar^{-1}(\frac{1}{3!}\theta_0^3 + \frac{1}{2}\sum_{k,l\geq 1}(p_{k+l}q_kq_l + p_kp_lq_{k+l})) - \hbar^0\frac{1}{24}\theta_0.$$

Note that $\frac{1}{2}\sum_{k,l\geq 1}(p_{k+l}q_kq_l + p_kp_lq_{k+l}))$ are the cut-and-join operators of [6]. The full rubber potential can be related to Hurwitz numbers by use of a localization argument in [17].

Example 3.2.6. As a consequence of Corollary 5.1.9, for $(X, L) = (\mathbb{P}^1, \mathcal{O}(1))$, the terms in the rubber potential without powers of L^∞ corresponding to positive degree maps are

$$\begin{aligned} A_{\lambda=0} &= \hbar^{-1}\frac{1}{2\pi}\int_0^{2\pi}\frac{(\beta_0 + \sum_k p_{k,0}e^{-ikx} + \sum_k q_{k,0}e^{ikx})^2}{2} \\ &\quad \cdot(\beta_2 + \sum_k p_{k,2}e^{-ikx} + \sum_k q_{k,2}e^{ikx})dx \\ &+ \hbar^{-1}\frac{1}{2\pi}\int_0^{2\pi} e^{\beta_2 + \sum_k p_{k,2}e^{-ikx} + \sum_k q_{k,2}e^{ikx}} z e^{ix}dx. \end{aligned}$$

3.3. **Trivial Cylinders.** It was our convention to exclude trivial cylinders from the rubber potential. They will be accounted for by the algebra $\mathcal{R}$. To prove this, it will be advantageous to write down a potential including trivial cylinders and relate it to the rubber potential. Let Γ be some rubber graph. Let $\Gamma_|$ be a rubber graph obtained from Γ by adjoining a degree r trivial cylinder. From [10], we have

Theorem 3.3.1. *There is a natural map*

$$v : \mathcal{M}(\mathcal{A}, \Gamma_|) \to \mathcal{M}(\mathcal{A}, \Gamma) \times X$$

so that

$$v_*[\mathcal{M}(\mathcal{A}, \Gamma_|)]^{vir} = \frac{1}{r}[\mathcal{M}(\mathcal{A}, \Gamma)]^{vir} \times [X]$$

and

$$v^*(\mathrm{L}^\infty) = \mathrm{L}^\infty.$$

Consequently if Γ has m interior marked points and $r_0 + r_\infty$ boundary marked points then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A}, \Gamma_|) & \xrightarrow{\mathrm{Ev}_|} & X^m \times X^{r_0+1} \times X^{r_\infty+1} \\ \downarrow & & \downarrow h \\ \mathcal{M}(\mathcal{A}, \Gamma) \times X & \xrightarrow{\mathrm{Ev}\times\Delta} & X^m \times X^{r_0} \times X^{r_\infty} \times (X \times X) \end{array}$$

where $\Delta : X \to X^2$ is the diagonal and the morphism h reorders the products of X so that the product of X's corresponding to the $r_0 + 1$st and $r_\infty + 1$st boundary marked points are taken to $X \times X$.

For classes $A \in H^*(X^m \times X^{r_0+1} \times X^{r_\infty+1})$, we have

$$\begin{aligned} & \deg(Ev_|^*(A) \cap [\mathcal{M}(\mathcal{A}, \Gamma_|)]^{vir}) \\ = \quad & \frac{1}{r} \deg(((Ev \times \Delta) \circ h^{-1})^*(A) \cap ([\mathcal{M}(\mathcal{A}, \Gamma)]^{vir} \times [X])). \end{aligned}$$

Definition 3.3.2. The *rubber potential with trivial cylinders*, $A_|$ is defined as before except that we allow the set Ξ to contain graphs that have trivial cylinders for vertices.

Define the action of an algebra of power series in infinitely many non-commuting variables $\kappa_1, \kappa_2, \dots$ on monomials $f \in \mathcal{R}$ by

$$\kappa_n \cdot f = \frac{1}{n} \hbar^{-1} \sum_{i_1, i_2} (-1)^{(\deg_{pq}(f))(\deg(p_{n,i_1}))} g^{i_1 i_2} p_{n,i_1} f q_{n,i_2}$$

where $\deg_{pq}(f)$ is the sum of the degrees of the p and q variables in f. Extend the action linearly to $\mathcal{R}$. Let

$$TK = \sum_n \kappa_n$$

and define a map

$$\begin{array}{ccccc} T & : & \mathcal{R} & \to & \mathcal{R} \\ T & : & f & \mapsto & e^{TK} f. \end{array}$$

Lemma 3.3.3. *T takes the rubber potential to the rubber potential with trivial cylinders,*

$$T(A) = A_|$$

Proof. The proof is straightforward. The factorial terms in the exponential come from relabelling the boundary marked points. □

Definition 3.3.4. Let f and h be elements in $\mathcal{R}$. We define a binary operation $f *_| h$ as follows. Introduce a set of auxiliary variables $\tilde{p}_{n,i}$, $\tilde{q}_{n,i} = \sum_j n\hbar g^{ij} \frac{\partial}{\partial \tilde{p}_{n,j}}$. Write $f(p, \tilde{q})$, $h(\tilde{p}, q)$, that is we substitute the tilded variables into the power series. Define

$$f *_| h = f(p, \tilde{q}) h(\tilde{p}, q)|_{\tilde{p}=0}.$$

Note that in the above we treat $\frac{\partial}{\partial \tilde{p}_{n,j}}$ as an element with the same parity as $p_{n,j}$. The operation, $*_|$ is the one that corresponds to stacking curves to form multi-level curves. This will be elaborated in the section on degenerations.

Lemma 3.3.5. *T is a homomorphism from $(\mathcal{R}, *)$ to $(\mathcal{R}, *_|)$.*

Proof.

$$\frac{\partial}{\partial p_{n,i}} (e^{TK} 1) = \hbar^{-1} \frac{1}{n} \sum_j (e^{TK} 1) g^{ij} q_{n,j}$$

which implies for f, a monomial,

$$\frac{\partial}{\partial p_{n,i}} (e^{TK} f) = \hbar^{-1} \frac{1}{n} \sum_j (-1)^{(\deg_{pq} f)(\deg p_{n,i})|} (e^{TK} f) g^{ij} q_{n,j} + e^{TK} \frac{\partial f}{\partial p_{n,i}}.$$

The lemma follows by induction on the number of p and q variables in f. □

Definition 3.3.6. The *rational potential* A is

$$A_{\text{rat}} = \sum_{A \in B_1(X)} (\Gamma, P, Q)^{\bullet}_{g=0,A,m} z^A$$

where $(\ ,\ ,\ )^{\bullet}$ where the sum is taken over moduli spaces $\mathcal{MA}$ involving only connected domains of genus 0.

3.4. **Action of $\mathcal{R}$ on $\mathcal{F}$.**

Lemma 3.4.1. *$\mathcal{F}$ can be given the structure of a graded $\mathcal{R}$-module.*

Proof. Consider the inclusion

$$I : D \to Z$$

and the induced maps

$$\begin{array}{rcccc} I^* & : & H^*(Z) & \to & H(D), \\ I^{*\vee} & : & H^*(D)^\vee & \to & H^*(Z)^\vee. \end{array}$$

We define the action of $\mathcal{R}$ on $f \in \mathcal{F}$ as follows:

$$\begin{array}{rcl} \lambda \cdot f & = & 0 \\ \hbar \cdot f & = & \tilde{\hbar} f \\ p_{n,i} \cdot f & = & \tilde{p}_{n,i} \\ q_{n,i} \cdot f & = & \tilde{\hbar} n \sum_{i'} g^{ii'} \dfrac{\partial}{\partial \tilde{p}_{n,i'}} f \\ \beta_i \cdot f & = & I^{*\vee}(\gamma_i) f \\ z^d \cdot f & = & \tilde{z}^{i_* d} f \end{array}$$

where $g^{ii'}$ is the intersection pairing on $H^*(D)$.

Because

$$\begin{array}{rcccl} \deg \tilde{z}^{i_* d} & = & < c_1(TZ), i_* d > & = & < i^* c_1(TZ), d > \\ & = & < c_1(TD) + c_1(N), d > & = & \deg z^d, \end{array}$$

the action preserves grading. □

Definition 3.4.2. Define a bilinear operation

$$\cdot_| : \mathcal{R} \otimes \mathcal{F} \to \mathcal{F}$$

as follows, for $f \in \mathcal{F}$, $h \in \mathcal{R}$. Given a monomial

$$h = \hbar^{-\frac{1}{2}\chi} \lambda^m \beta_{i_1} \dots \beta_{i_m} p_{m_1,j_1} \dots p_{m_{r_0},j_{r_0}} q_{n_1,k_1} \dots q_{n_{r_\infty},k_{r_\infty}} z^d \in \mathcal{R}$$

define $h \cdot_| f$ by defining the action of $q_{n,i}$ on $\mathcal{F}$ by

$$q_{n,i} = \left(\sum_{i'} g^{ii'} \frac{\partial}{\partial \tilde{p}_{n,i'}} \right)$$

and defining

$$h \cdot_| f = \left((\tilde{\hbar}^{-\frac{1}{2}\chi} \delta_{m0} I^\vee(\beta_{i_1}) \dots I^\vee(\beta_{i_m}) p_{m_1,j_1} \dots p_{m_{r_0},j_{r_0}} q_{n_1,k_1} \dots q_{n_{r_\infty},k_{r_\infty}}) f \right) |_{\tilde{p}=0}$$

and then by substituting $\tilde{p}_{n,j}$ for $p_{n,j}$

This operation corresponds to joining a curve in $\mathcal{MA}$ to one in $\mathcal{MZ}$. Analogously to the multiplication in $\mathcal{R}$, $\cdot_|$ and the module structure $\cdot$ are related as follows:

Lemma 3.4.3. *For $h \in \mathcal{R}$, $f \in \mathcal{F}$, we have*

$$T(h) \cdot_{|} f = h \cdot f.$$

4. Degeneration Formulae

Theorem 2.5.1 gives formulas relating the line-bundles

$$\mathrm{Dil}, \mathrm{Split}, \mathrm{L}_{i,\text{not top}}, \mathrm{L}_{i,\text{not bot}}$$

on $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$ and $\mathcal{M}(\mathcal{A}, \Gamma_A)$. In this section, we will show that the first Chern classes of these line-bundles turn out represent specific geometric situations involving split curves. For example, $c_1(\mathrm{Split})$ is a substack of $\mathcal{M}(\mathcal{A}, \Gamma)$ that is, in a virtual sense, all split curves. $c_1(\mathrm{L}_{i,\text{not top}})$ virtually consists of all split curves in which the ith marked point is not on the topmost component. This allows us to write the cap product of a first Chern class of one of our bundles with the virtual cycle in terms of the virtual cycles of smaller moduli spaces. This provides *degeneration* formulae that can be expressed in the language of generating functions.

We will express the first chern class of various line-bundles geometrically by adapting Li's argument [15]. The argument is in several stages and we state it only in the case $\mathcal{M}(\mathcal{Z}, \Gamma)$ noting that the case for $\mathcal{M}(\mathcal{A}, \Gamma)$ is exactly analogous:

(1) For Γ, consider quadruples $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$ so that the graph join, $\Gamma_Z *_{L,J} \Gamma_A$ is isomorphic to Γ. We can define a line bundle L_Υ on $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$.

(2) We show that

$$c_1(L_\Upsilon) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{\text{vir}} = m(\Upsilon)[\mathcal{M}(\mathcal{Z} \sqcup \mathcal{A}, \Upsilon)]^{\text{vir}}$$

where $[\mathcal{M}(\mathcal{Z} \sqcup \mathcal{A}, \Upsilon)]^{\text{vir}}$ is an appropriately defined virtual cycle.

(3) Given the joining morphism

$$\Phi : \mathcal{M}(\mathcal{A}, \Gamma_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_Z) \to \mathcal{M}(\mathcal{Z}, \Gamma_A \sqcup_{L,J} \Gamma_Z)$$

and the diagram

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A}, \Gamma_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_Z) & \longrightarrow & \mathcal{M}(\mathcal{A}, \Gamma_A) \times \mathcal{M}(\mathcal{Z}, \Gamma_Z) \\ \downarrow & & \downarrow \\ D^r & \xrightarrow[\Delta]{} & D^r \times D^r \end{array}$$

where Δ is the diagonal map. We have

$$\Phi_* \Delta^! ([\mathcal{M}(\mathcal{A}, \Gamma_A)]^{\text{vir}} \times [\mathcal{M}(\mathcal{Z}, \Gamma_Z)]^{\text{vir}}) = [\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)]^{\text{vir}}.$$

(4) Given a line-bundle $L = \mathrm{Dil}$ or $L = \mathrm{L}_{i,\text{ext}}$ (or in the case of $\mathcal{M}\mathcal{A}$, Split,$\mathrm{L}_{i,\text{not top}}$,$\mathrm{L}_{i,\text{not bot}}$), we exhibit a set of join-equivalence classes Ω so that

$$L = \otimes_{[\Upsilon] \in \Omega} L_\Upsilon.$$

(5) Consequently

$$c_1(L) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{\text{vir}} = \sum_{\Upsilon \in \Omega} m(\Upsilon) \Phi_* \Delta^! ([\mathcal{M}(\mathcal{A}, \Gamma_A)]^{\text{vir}} \times [\mathcal{M}(\mathcal{Z}, \Gamma_Z)]^{\text{vir}}).$$

To modify this argument to work for $\mathcal{M}(\mathcal{Z}, \Gamma)$, replace all pairs (Γ_A, Γ_Z) with (Γ_t, Γ_b) and replace $\mathcal{Z}$ with $\mathcal{A}$.

4.1. **Interpretation of Bundles.** Let us rewrite the bundles Dil,$\mathrm{L}_{i,\mathrm{ext}}$, $\mathrm{L}_{i,\text{not top}}$,$\mathrm{L}_{i,\text{not bot}}$ as tensor products of L_Υ's on $\mathcal{M}(\mathcal{Z},\Gamma)$ and $\mathcal{M}(\mathcal{A},\Gamma)$.

On $\mathcal{M}\mathcal{Z}$ where i is the label for an interior marked point,

(1) $\Omega_{\mathrm{Dil}} = \{\Upsilon = (\Gamma_A, \Gamma_Z, L, J)\}$ the set of all join-equivalence classes of quadruples $\Upsilon = (\Gamma_A, \Gamma_Z, L, J)$.
(2) $\Omega_{\mathrm{L}_{i,\mathrm{ext}}} = \{(\Gamma_A, \Gamma_Z, L, J) | i \in J(M_A)\}$.

while on $\mathcal{M}\mathcal{A}$ where i, j are labels for interior marked points,

(1) $\Omega_{\mathrm{Split}} = \{(\Gamma_t, \Gamma_b, L, J)\}$.
(2) $\Omega_{\mathrm{L}_{i,\text{not bot}}} = \{(\Gamma_t, \Gamma_b, L, J) | i \in J(M_t)\}$.
(3) $\Omega_{\mathrm{L}_{i,\text{not top}}} = \{(\Gamma_t, \Gamma_b, L, J) | i \in J(M_b)\}$.
(4) $\Omega_{(i,j)} = \{(\Gamma_t, \Gamma_b, L, J) | i \in J(M_t), j \in J(M_b)\}$.
(5) $\Omega_{(ij,)} = \{(\Gamma_t, \Gamma_b, L, J) | i, j \in J(M_t)\}$.
(6) $\Omega_{(,ij)} = \{(\Gamma_t, \Gamma_b, L, J) | i, j \in J(M_b)\}$.

Theorem 4.1.1. [10] *For* $L = \mathrm{Dil}, \mathrm{L}_{i,ext}, \mathrm{Split}, \mathrm{L}_{i,not\ bot}, \mathrm{L}_{i,not\ top}$,

$$L = \bigotimes_{[\Upsilon]\in\Omega_L} L_\Upsilon.$$

where $[\Upsilon]$ *denotes a join-equivalence class and* Υ *a representative element.*

4.2. **Splitting of Moduli Stacks.** We need to cite a number of results from [15]. These results were proved for a different moduli stack, $\mathcal{M}(\mathcal{W})$, but because of the explicit parallels between that space and the construction of $\mathcal{M}(\mathcal{A},\Gamma)$ and $\mathcal{M}(\mathcal{Z},\Gamma)$, the proofs can be modified in straightforward fashion. We begin by relating the virtual cycle $[\mathcal{M}(\mathcal{A}\sqcup\mathcal{Z},\Upsilon)]^{\mathrm{vir}}$ defined in [15] where $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$ is a graph-join quadruple to other virtual cycles.

Theorem 4.2.1. *We have the following equality among cycle classes*

$$c_1(L_\Upsilon) \cap [\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)]^{vir} = m(\Upsilon)[\mathcal{M}(\mathcal{A}\sqcup\mathcal{Z},\Upsilon)]^{vir}$$

where $m(\Upsilon)$ *is as in Definition 2.4.7.*

Consider the fiber square

$$\begin{array}{ccc} \mathcal{M}(\mathcal{A},\Gamma_A)\times_{D^r}\mathcal{M}(\mathcal{Z},\Gamma_Z) & \longrightarrow & \mathcal{M}(\mathcal{A},\Gamma_A)\times\mathcal{M}(\mathcal{Z},\Gamma_Z) \\ \downarrow & & \downarrow \\ D^r & \xrightarrow[\Delta]{} & D^r\times D^r \end{array}$$

where Δ is the diagonal morphism and the downward maps are induced from evaluation at the boundary marked points of $\mathcal{M}\mathcal{Z}$ and the boundary marked points at D_∞ on $\mathcal{M}\mathcal{A}$. Let the virtual cycle on the fiber product be given by

$$[\mathcal{M}(\mathcal{A},\Gamma_A)\times_{D^r}\mathcal{M}(\mathcal{Z},\Gamma_Z)]^{\mathrm{vir}} = \Delta^!([\mathcal{M}(\mathcal{A},\Gamma_A)]^{\mathrm{vir}}\times[\mathcal{M}(\mathcal{Z},\Gamma_Z)]^{\mathrm{vir}}).$$

Theorem 4.2.2. *If*

$$M_{[\Upsilon]} = \coprod_{(\Gamma'_A,\Gamma'_Z,L,J)\in[\Upsilon]} \mathcal{M}(\mathcal{A},\Gamma'_A)\times_{D^r}\mathcal{M}(\mathcal{Z},\Gamma'_Z)$$

is given the virtual cycle of a disjoint union, then

$$\Phi_{[\Upsilon]} : M_{[\Upsilon]} \to \mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)$$

gives

$$\Phi_{[\Upsilon]*}([M]^{vir}) = |MZ|!|MA|!(|RZ|!)^2[\mathcal{M}(\mathcal{A} \sqcup \mathcal{Z}, \Upsilon)]^{vir}$$

Note that the multiplicity term is natural in light of Proposition 2.4.9.

Corollary 4.2.3.

$$c_1(L_\Upsilon) \cap [\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)]^{vir} = \frac{m(\Upsilon)}{|MZ|!|MA|!(|RZ!)^2} \Delta^!(M_\Upsilon).$$

L together with $i : X \to Z$ induces a morphism

$$\Lambda : (X^{|MA|} \times X^{|RA_0|}) \times Z^{|MZ|} \to Z^M \times X^R$$

where $M = |MZ| + |MA|$ and $R = |RA_0|$ are the number of interior and boundary marked points in $\Gamma_A *_{L,J} \Gamma_Z$

We have morphisms

$$\begin{array}{ccc} X^{|MA|+|RA_0|} \times X^{|RZ|} \times Z^{|MZ|} & \xrightarrow{\tilde{\Delta}} & X^{|MA|+|RA_0|} \times X^{|RA_\infty|} \times Z^{|MZ|} \times X^{|RZ|} \\ \downarrow{\scriptstyle p} & & \\ (X^{|MA|} \times X^{|RA_0|}) \times Z^{|MZ|} & & \end{array}$$

where $\tilde{\Delta}$ is induced by $\Delta : X^{|RZ|} \to X^{|RA_\infty|} \times X^{|RZ|}$ and p is the projection.

Therefore, for $c \in H^*(Z^{|MZ|} \times X^{|RZ|}$,

$$\deg((\mathrm{Ev}^*(c) \cup c_1(L_\Upsilon)) \cap [\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)]^{\mathrm{vir}})$$

$$= \frac{m(\Upsilon)}{\mathrm{Aut}_{\Gamma_Z,\Gamma_A,L}(RZ, RA_\infty)} \deg(\mathrm{Ev}^*(\tilde{\Delta}_!(p^*\Lambda^*c) \cap ([\mathcal{M}(\mathcal{A}, \Gamma_A)]^{\mathrm{vir}} \times [\mathcal{M}(\mathcal{Z}, \Gamma_Z)]^{\mathrm{vir}})$$

If Ω is one of the sets of join-equivalence classes, from

$$c_1(L_\Omega) = \sum_{[\Upsilon]\in\Omega} c_1(L_\Upsilon)$$

we have

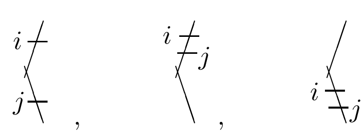
Theorem 4.2.4.

$$\deg((Ev^*(c) \cup c_1(L_\Omega)) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{vir}) =$$

$$\sum_{[\Upsilon]\in\Omega} \frac{m(\Upsilon)}{|MZ|!|MA|!|RZ|!^2} \deg(Ev^*(\tilde{\Delta}_!(p^*\Lambda^*c)) \cap [M_\Upsilon]^{vir})$$

and analogously for $\mathcal{MA}$.

Definition 4.2.5. Let the symbols



denote the cohomology classes

$$\sum_{[\Upsilon]\in\Omega_{(i,j)}} c_1(L_\Upsilon), \sum_{[\Upsilon]\in\Omega_{(ij,)}} c_1(L_\Upsilon), \sum_{[\Upsilon]\in\Omega_{(,ij)}} c_1(L_\Upsilon)$$

respectively.

These cohomology classes are dual to the cycles in $\mathcal{MA}$ representing split curves with i and j specified on top and bottom component as specified in the symbol, counted with the appropriate weight.

4.3. **Normal Bundle to Split Curves.** The following is useful for localization computations.

Let $(\Gamma_Z, \Gamma_A, L, J)$ be graph-join quadruple. Consider L_Υ for the quadruple $\Upsilon = (\Gamma_Z, \Gamma_A, L, J)$. Then $c_1(L_\Upsilon)$ is a substack of $\mathcal{M}(\mathcal{Z}, \Gamma_Z *_{L,J} \Gamma_A)$.

Consider the moduli stacks $\mathcal{M}(\mathcal{Z}, \Gamma_Z)$, $\mathcal{M}(\mathcal{A}, \Gamma_A)$, $\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)$, and the inclusion

$$\Phi : \mathcal{M}(\mathcal{A}, \Gamma_A) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_Z) \to \mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z).$$

$\mathcal{M}(\mathcal{A}, \Gamma_Z) \times_{D^r} \mathcal{M}(\mathcal{Z}, \Gamma_A)$ has projections p_A, p_Z to its $\mathcal{MA}$ and $\mathcal{MZ}$ factors.

Theorem 4.3.1. *On* $\mathcal{M}(\mathcal{Z}, \Gamma_A *_{L,J} \Gamma_Z)$, $\Phi^* L_\Upsilon = p_Z^* \mathrm{Dil}^\vee \otimes p_A^* \mathrm{L}^{\infty\vee}$.

Similarly, we have

Theorem 4.3.2. *On* $\mathcal{M}(\mathcal{A}, \Gamma_{A_t} *_{L,J} \Gamma_{A_b})$, $\Phi^* L_\Upsilon = p_{Ab}^* \mathrm{L}^{0^\vee} \otimes p_{At}^* \mathrm{L}^{\infty\vee}$.

4.4. **Degeneration Formulae.** The above degeneration formulas can be written in terms of generating functions.

For $L = \mathrm{Dil}$, we can write down a potential F_{Dil} which is defined by a formula similar to that of the relative potential except that instead of evaluating all possible cohomology classes on $[\mathcal{MZ}]^{\mathrm{vir}}$, we evaluate them on $c_1(\mathrm{Dil}) \cap [\mathcal{MZ}]^{\mathrm{vir}}$. That is, we define the Dil correlator by for $a_1, \ldots, a_n \in H^*(Z)$, $b_1, \ldots, b_r \in H^*(X)$

$$< a_1, \ldots, a_m \cdot b_1, \ldots, b_r >_{\mathrm{Dil},\chi,A,(s_1,\ldots,s_r)}$$
$$= \sum_{\Gamma\in\Xi} (\mathrm{Ev}^*(a_1 \times \cdots \times a_m \times b_1 \times \cdots \times b_r) \cup c_1(\mathrm{Dil})) \cap [\mathcal{M}(\mathcal{Z}, \Gamma)]^{\mathrm{vir}}$$

and define F_{Dil} as in the previous section. Define the rubber potential with trivial cylinders without powers of $c_1(\mathrm{L}^\infty)$ by

$$A_{|\lambda=0} = (A_|)_{\lambda=0}.$$

Then we have by Lemma 3.4.3, Theorem 4.1.1, and Theorem 4.2.4

Theorem 4.4.1.

$$F_{\mathrm{Dil}} = (A_|)_{\lambda=0} \cdot_| F$$
$$F_{\mathrm{Dil}} = A_{\lambda=0} \cdot F.$$

To study insertions of $c_1(\mathrm{L}_{i,\mathrm{ext}})$, we choose an element e_j of our basis for $H^*(Z)$. Because $c_1(\mathrm{L}_{i,\mathrm{ext}})$ is dependent on the choice of marked point, we add a *distinguished* marked point to all of the relative graphs that contribute to our potential. At this marked point, we evaluate $c_1(\mathrm{L}_{i,\mathrm{ext}}) \cup \mathrm{ev}_i^* e_j$. More formally, given a graph Γ with m marked points, consider the set $D(\Gamma)$ consisting of all graphs Γ' with $m+1$ marked points such that when we forget $m+1$st marked point on

Γ', we obtain Γ. Consider the (ex, e_j) correlator given by $a_1, \ldots, a_m \in H^*(Z)$, $b_1, \ldots, b_r \in H^*(X = D)$

$$\begin{aligned} & < a_1, \ldots, a_m \cdot c_1, \ldots, c_r >_{(\mathrm{ex}, e_j), \chi, A, (s_1, \ldots, s_r)} \\ = \quad & \textstyle\sum_{(\Gamma,k)\in\Xi}(\sum_{\Gamma'\in D(\Gamma)}(\mathrm{Ev}^*(a_1 \times \ldots \cdots \times a_m \times e_j \times b_1 \times \cdots \times b_r) \\ & \cup c_1(\mathrm{L}_{m+1,\mathrm{ext}}) \cap [\mathcal{M}(\mathcal{Z}, \Gamma')]^{\mathrm{vir}})) \end{aligned}$$

We write down $F_{\mathrm{ex},j}$ by using the modified correlator. Then, for $i : X = D \to Z$, we write

$$i^* e_j = \sum_l M_{jl} c_l$$

for $M_{jl} \in \mathbb{Q}$ which gives

$$\begin{aligned} F_{\mathrm{ex},j} &= \sum_l M_{jl} \frac{\partial (A_|)_{\lambda=0}}{\partial \beta_l} \cdot_| F \\ &= \sum_l M_{jl} \frac{\partial A_{\lambda=0}}{\partial \beta_l} \cdot F. \end{aligned}$$

Now, since $\mathrm{L}_{k,\mathrm{ext}} = \mathrm{ev}_k^*([D])$, we have

Theorem 4.4.2. *Let e_j be an element of our basis for $H^*(Z)$. Let N be a matrix defined by*

$$e_j \cup [D] = \sum_l N_{jl} e_l$$

then

$$\sum_l N_{jl} \frac{\partial F}{\partial \theta_l} = \sum_l M_{jl} \frac{\partial A_{\lambda=0}}{\partial \beta_l} \cdot F$$

Proof. $\mathrm{ev}_k^* e_j \cup c_1(\mathrm{L}_{k,\mathrm{ext}}) = (\mathrm{ev}_k^*(e_j \cup [D]))$ implies

$$F_{\mathrm{ex},j} = \sum_l N_{jl} \frac{\partial F}{\partial \theta_l}.$$

□

Likewise, we can write down a rubber potential with $c_1(\mathrm{Split})$ inserted in the correlator. This is analogous to Dil in the relative case.

Theorem 4.4.3. $A_{\mathrm{Split}} = A_{\lambda=0} * A$.

We can write down a potential involving insertions of $c_1(\mathrm{L}_{i,\text{not bot}})$. This is analogous to $\mathrm{L}_{i,\mathrm{ext}}$ in the relative case. Again, we have to single out a cohomology class $c_j \in H^*(X)$ where at some marked point i, we will evaluate $\mathrm{ev}_i^*(c_j)$ and $\mathrm{L}_{i,\text{not bot}}$.

Theorem 4.4.4. $A_{\mathrm{L}_{i,\textit{not bot}}, c_j} = \frac{\partial A_{\lambda=0}}{\partial \beta_j} * A$.

Similarly, for $\mathrm{L}_{i,\text{not top}}$, we have

Theorem 4.4.5. $A_{\mathrm{L}_{i,\textit{not top}}, c_j} = A_{\lambda=0} * \frac{\partial A}{\partial \beta_j}$.

From Theorem 2.5.1(3) one can obtain a degeneration formula for the rubber potential by inserting

$$c_1(\mathrm{L}^\infty \otimes \mathrm{ev}_i^* L)\mathrm{ev}_i^*(c_j) = (c_1(\mathrm{L}^\infty) + ev_i^* c_1(L))\mathrm{ev}_i^*(c_j)$$

at a distinguished point in the rubber potential to compute $A_{\mathrm{L}_{i,\text{not top}},c_j}$. Note that $\frac{\partial A}{\partial \lambda}$ is the rubber potential with an extra insertion of $c_1(\mathrm{L}^\infty)$.

Theorem 4.4.6. *Define the matrix N_{ij} by*

$$c_1(L) \cup c_i = \sum_j N_{ij} c_j.$$

Then for each i, we have

$$\frac{\partial}{\partial \lambda}\frac{\partial A}{\partial \beta_i} + \sum_j N_{ij}\frac{\partial A}{\partial \beta_j} = \frac{\partial A_{\lambda=0}}{\partial \beta_i} * A.$$

4.5. **Reconstruction of the Relative Potential.** Let us look at the relative case with (Z, D). Let N be the normal bundle to D in Z. We can use the degeneration formula for $\mathrm{L}_{i,\text{ext}}$ to reconstruct the relative Gromov-Witten invariants of (Z, D) from the rubber potential (without powers of L^∞) of (D, N) and from seed values of the relative Gromov-Witten invariants of (Z, D). This is a formal consequence of the fact that $\mathrm{ev}_i^* D = \mathrm{L}_{i,\text{ext}}$ and Theorem 4.4.2.

We need to pick a particular basis for $H^*(Z)$. Let $V \subseteq H^*(Z)$ be the subspace

$$V = \mathrm{im}(\cup[D] : H^{*-2}(Z) \to H^*(Z)).$$

Let $e_1, \ldots, e_v$ be a homogeneous basis for V, ordered by degree. Extend this to a basis $\{e_{v+1}, \ldots, e_{v+w}\}$ of $H^*(Z)$.

Theorem 4.5.1. *The relative potential F of (Z, D) can be reconstructed from the rubber potential of (D, N) together with the relative potential involving only the classes $\{e_{v+1}, \ldots, e_{v+w}\}$, that is, from*

$$F|_{\theta_1=\theta_2=\cdots=\theta_v=0}.$$

Proof. We add in one θ_i variable at a time. So, suppose we have determined

$$F|_{\theta_j=\cdots=\theta_v=0}.$$

Since $e_j \in V$,

$$e_j = \sum_k a_k e_k \cup [D]$$

for $a_k \in \mathbb{Q}$ where $a_l = 0$ for $l \in \{j, \ldots, v\}$ for degree reasons. By the above, we have

$$\frac{\partial F}{\partial \theta_j} = \sum_{k,l} a_k M_{kl} \frac{\partial A_{\lambda=0}}{\partial \beta_l} \cdot F$$

which allows us to solve for

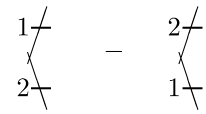
$$F|_{\theta_{j+1}=\cdots=\theta_v=0}.$$

□

4.6. Transferring Classes between Split Curves.

Theorem 4.6.1. *On $\mathcal{MA}$ with two distinguished interior marked points, the following equation holds among divisors.*

$$ev_2^*(c_1(L)) - ev_1^*(c_1(L)) \quad = $$



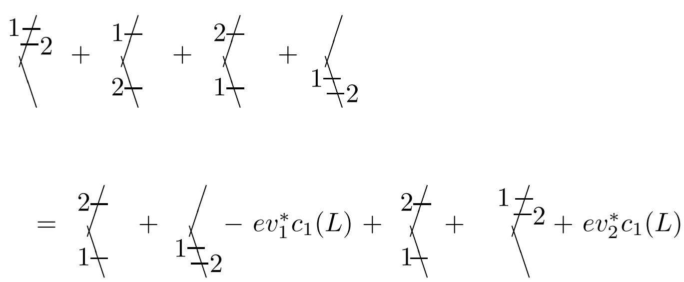
Proof. Recall the following facts:

$$\begin{aligned} c_1(\mathrm{L}^0) + c_1(\mathrm{L}^\infty) &= c_1(\mathrm{Split}), \\ c_1(\mathrm{L}^0) &= c_1(\mathrm{L}_{i,\text{not top}}) + \mathrm{ev}_i^* c_1(L), \\ c_1(\mathrm{L}^\infty) &= c_1(\mathrm{L}_{i,\text{not bot}}) - \mathrm{ev}_i^* c_1(L).; \end{aligned}$$

Putting these together, we see

$$c_1(\mathrm{Split}) = c_1(\mathrm{L}_{1,\text{not top}}) - ev_1^* c(L) + c_1(\mathrm{L}_{2,\text{not top}}) + ev_2^* c_1(L).$$

Diagrammatically, this equation can be expressed as



The result follows. ∎

One can use this result to reconstruct all rubber invariants from those where all powers of $c_1(L)$ are at a single interior marked point.

5. Examples

5.1. Computation of the rational rubber potential. We compute the rational rubber potential of $L = \mathcal{O}(m)$ over $\mathbb{P}^r$. We do this by comparing the rubber potential to the rational Gromov-Witten invariants of $\mathbb{P}^r$. There is a clear geometric reason why this should be possible. Given a smooth rational curve $\mathbb{P}^1$ with interior and boundary marked points, $\{x_1, \ldots, x_k, p_1^0, \ldots, p_{s^0}^0, p_1^\infty, \ldots, p_{s^\infty}^\infty\}$ with the multiplicities $(m_1^0, \ldots, m_{s^0}^0, m_1^\infty, \ldots, m_{s^\infty}^\infty)$, specifying a rubber map to L is equivalent to finding a degree d map of $f : \mathbb{P}^1 \to \mathbb{P}^r$ and a nowhere zero section (defined up to $\mathbb{C}^*$-action) of

$$f^* L \otimes \mathcal{O}(-m_1^0 p_1^0 - \cdots - m_{s^0}^0 p_{s^0}^0 + m_1^\infty p_1^\infty + \cdots + m_{s^\infty}^\infty).$$

The numerical condition in Lemma 2.2.4 for multiplicities implies that this bundle must be trivial. Therefore, there is only one section up to multiplication by an element of $\mathbb{C}^*$. The automorphism group of the map to rubber is equal to that of the stable map, so the rubber invariant should equal the Gromov-Witten invariant.

Unfortunately, this intuitive picture may not be true for singular curves. One, however is able to prove these results when the target is $\mathbb{P}^r$ in which case rubber invariants count maps of smooth curves.

Let $L = \mathcal{O}(m)$ be a line-bundle over $\mathbb{P}^r$. Let $P = \mathbb{P}_{\mathbb{P}^r}(L \oplus 1_{\mathbb{P}^r})$ be the projective completion of L. We consider a stack $\mathcal{MA}$ of rubber maps to (X, L).

To prove that intersections on $\mathcal{MA}$ occur away from singular curves, we will use a Kleiman-Bertini theorem argument. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{MA} & \xrightarrow{\quad \mathrm{Ev}_{\mathcal{MA}} \quad} & \\ \downarrow{\scriptstyle \mathrm{ft}} & \searrow & \\ \overline{\mathcal{M}}_{0,n+r_0+r_\infty}(\mathbb{P}^r, d) & \xrightarrow[\mathrm{Ev}_{\overline{\mathcal{M}}}]{} & (\mathbb{P}^r)^{n+r_0+r_\infty} \end{array}$$

where ft is the map that takes a rubber map to $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(n) \oplus \mathcal{O})$ to its projection to $\mathbb{P}^r$, contracts unstable components, and sees boundary and interior marked points as marked points.

We need the following well-known:

Theorem 5.1.1. *$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is non-singular and $ev_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d) \to \mathbb{P}^r$ is a smooth morphism.*

Theorem 5.1.2. *The locus of maps with singular domain in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is the finite union of sub-stacks M' so that*

1. *M' is non-singular and of positive codimension.*
2. *The evaluation map $Ev : M' \to X^{n+r_0+r_\infty}$ is smooth.*

Now, we will use the Kleiman-Bertini theorem in the following form.

Theorem 5.1.3. *Let $f : X \to (\mathbb{P}^r)^p$ be a morphism from X, a non-singular stack of dimension d. Let $c = [H]^{a_1} \times \cdots \times [H]^{a_p} \in H^l((\mathbb{P}^r)^p)$ where $[H]$ is a hyperplane class. Then there exists an open set $U \subseteq ((\mathbb{P}^r)^\vee)^l$ each point of which corresponds to a product of linear subspaces $K = V_1 \times \cdots \times V_p$, Poincare-dual to c, such that $f^{-1}(K)$ is either empty or non-singular of dimension $\dim X - l$.*

Let us first show that there is no contribution coming from rational curves with disconnected domain.

Theorem 5.1.4. *Let $\mathcal{MA}$ be some rubber moduli space consisting of maps from the disjoint union of $q \geq 2$ rational curves, none of which are trivial cylinders. Let $l = \dim \mathcal{MA}$. If $c \in H^l((\mathbb{P}^r)^{n+r_0+r_\infty})$, then there exists K, a product of linear subspaces in $(\mathbb{P}^r)^{n+r_0+r_\infty}$, Poincare-dual to c so that $Ev^{-1}_{\mathcal{MA}}(K)$ is empty and therefore,*

$$\deg(Ev^*_{\mathcal{MA}}(c) \cap [\mathcal{MA}]^{vir}) = 0.$$

Proof. Let us write $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{\circ}_{\chi,n+r_\infty+r_0}(\mathbb{P}^r, d)$ for the moduli space of stable maps of the disjoint union of q rational curves to $\mathbb{P}^r$ of degree d. $\chi = 2q$ is the Euler characteristic of the domain. Then, by the dimension formula,

$$\dim \mathcal{MA} - \dim \overline{\mathcal{M}} = \frac{1}{2}\chi - 1 = q - 1 > 0.$$

Therefore, the codimension of the class $c \in H^l((\mathbb{P}^r)^{n+r_0+r_\infty})$ exceeds the dimension of $\overline{\mathcal{M}}$, so its Poincare-dual can be represented by a product of hyper-planes, C so that that $\mathrm{Ev}_{\overline{\mathcal{M}}}^{-1}(C)$ is empty. □

Now, let us compute the rational rubber invariants coming from curves with connected domains. We begin with the straight-forward combinatorial lemma

Lemma 5.1.5. *A map in $\mathcal{MA}$ with connected rational singular domain is mapped by ft to a map with singular domain in $\overline{\mathcal{M}}(\mathbb{P}^r, d)$*

Putting everything together,

Theorem 5.1.6. *Let $\mathcal{MA}$ be some rubber moduli space with target (X, L) and evaluation map*

$$Ev : \mathcal{MA} \to (\mathbb{P}^r)^n \times (\mathbb{P}^r)^{r_0} \times (\mathbb{P}^r)^{r_\infty}.$$

Let $l = \dim[\mathcal{MA}]^{vir}$. Then for $c \in H^l((\mathbb{P}^r)^{n+r_0+r_\infty})$, there exists a product of linear subspaces K, Poincare-dual to c so that $Ev^{-1}(K)$ is a finite union of reduced points, supported away from curves with singular domain.

Now, we specify the rubber moduli space we are considering. Let genus be 0. Fix a degree $dL \in H_2(\mathbb{P}^r) > 0$, a number of interior marked points n and boundary marked points with tangencies to D_0 and D_∞, r_0 and r_∞ respectively. Choose multiplicities $m_1^0 \dots m_{r_0}^0$ and $m_1^\infty \dots m_{r_\infty}^\infty$ so that $(m_1^0 + \cdots + m_{r_0}^0) - (m_1^\infty + \cdots + m_{r_\infty}^\infty) = md$. Then, given a stable map $(f, C) \in \overline{\mathcal{M}}_{0,n+r_0+r_\infty}(\mathbb{P}^r, d)$ with smooth domain and marked points,

$$\{x_1, \dots, x_n, p_1^0, \dots, p_{r_0}^0, p_1^\infty, \dots, p_{r_\infty}^\infty\}$$

the invertible sheaf

$$\mathcal{L} = f^*L \otimes \mathcal{O}(-(m_1^0 p_1^0 + \cdots + m_{r_0}^0 p_{r_0}^0) + (m_1^\infty p_1^\infty + \cdots + m_{r_\infty}^\infty p_{r_\infty}^\infty))$$

has degree 0. $\mathcal{L}$'s nonzero section, defined up to $\mathbb{C}^*$ induces a map

$$C \to P = \mathbb{P}_{\mathbb{P}^r}(L \oplus 1_{\mathbb{P}^r})$$

giving a point in $\mathcal{MA}$. Moreover, the automorphisms of the map in $\overline{\mathcal{M}}_{0,n+r_0+r_\infty}(\mathbb{P}^r, d)$ are in bijective correspondence with the automorphisms of map in $\mathcal{MA}$.

Theorem 5.1.7. *With the data on $\mathcal{MA}$ as above, $l = \dim[\mathcal{MA}]^{vir}$, $c \in H^l((\mathbb{P}^r)^{n+r_0+r_\infty})$, we have the following equality of rubber and Gromov-Witten invariants*

$$\deg(Ev_{\overline{\mathcal{M}}}^*(c) \cap \overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)) = \deg(Ev_{\mathcal{MA}}^*(c) \cap [\mathcal{MA}]^{vir}).$$

Proof. Pick a representative cycle K as above. Then $\mathrm{Ev}_{\overline{\mathcal{M}}}^{-1}(K) \subseteq \overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)$ is a zero dimensional reduced substack corresponding to maps with smooth domains. $\mathrm{Ev}_{\mathcal{MA}}^{-1}(K) = \mathrm{ft}^{-1}\mathrm{Ev}_{\overline{\mathcal{M}}}^{-1}(K)$. By the above consideration, given an integral zero dimensional substack, x in $\mathrm{Ev}_{\overline{\mathcal{M}}}^{-1}(K)$, $\mathrm{ft}^{-1}(x)$ is an integral zero dimensional substack with the same automorphism group as x.

Consider the fiber square

$$\begin{array}{ccc} D & \xrightarrow{i'} & \overline{\mathcal{M}}_{0,n+r_0+r_\infty}(\mathbb{P}^r, d) \\ \downarrow & & \downarrow \\ K & \xrightarrow[i]{} & (\mathbb{P}^r)^{n+r_0+r_\infty} \end{array}$$

Now, the refined Gysin map $i^! : A_j(\overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)) \to A_{j-l}(D)$ satisfies

$$\deg(i^![\overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)]^{\mathrm{vir}}) = \deg(\mathrm{Ev}^*(c) \cap [\overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)]^{\mathrm{vir}}).$$

But since K is a regularly embedded substack, $i^! = (i')^*$.

Now, we need the following fact that ensures the compatibility of the Gysin map with the virtual cycle construction ([16], 3.9).

Lemma 5.1.8. *Let $\xi : X_0 \to X$ be a regular embedding of codimension d, W, a scheme, W_0, a scheme defined by the following fiber square*

$$\begin{array}{ccc} W_0 & \longrightarrow & W \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[\xi]{} & X \end{array}$$

If the tangent obstruction complexes $\mathcal{T}^\bullet_{W_0}$ and $\mathcal{T}^\bullet_W$ are compatible in the sense of ([16]*, 3.8) and certain technical assumptions are satisfied, then*

$$\xi^![W]^{vir} = [W_0]^{vir}$$

It can be checked that the induced virtual cycle on D is just $[D]^{\mathrm{vir}} = [D]$. Now recall ([22], 1.17) for a map from a integral zero-dimensional stack F to a point, pt (all stacks over $\mathbb{C}$),

$$\deg(F/\mathrm{pt}) = \frac{1}{\delta(F)}$$

where $\delta(F)$ is the order of the automorphism group of F.

Therefore, the Gromov-Witten invariant is

$$\deg((i')^*[\overline{\mathcal{M}}_{0,N}(\mathbb{P}^r, d)]^{\mathrm{vir}}) = \deg(D/\mathrm{pt}) = \sum_{F \in D} \frac{1}{\delta(F)}.$$

Likewise, if E is defined by

$$\begin{array}{ccc} E & \to & \mathcal{MA} \\ \downarrow & & \downarrow \\ C & \underset{i}{\to} & (\mathbb{P}^r)^N \end{array}$$

the rubber invariant is

$$\deg((i')^*[\mathcal{MA}]^{\mathrm{vir}}) = \deg(E/\mathrm{pt}) = \sum_{F \in E} \frac{1}{\delta(F)}.$$

□

This result can be stated with beautiful succinctness following ([2] 2.9.2). Consider rubber invariants into $P = \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(m) \oplus \mathcal{O})$. The numerical condition for multiplicities implies

$$(m^0_1 + \cdots + m^0_{r_0}) - (m^\infty_1 + \cdots + m^\infty_{r_\infty}) = md.$$

Pick a homogeneous basis $\{a_1, \ldots, a_v\}$ of $H^*(X)$. Pick variables β_i dual to a_i as in Definition 3.2.3. Let $p_{k,i}$ be the variables corresponding to contact to D_0 with multiplicity k, dual to a_i. $q_{k,i}$ corresponds analogously to contact to D_∞. Consider a real variable x. Let

$$P_j = \sum_{k \geq 1} p_{k,j} e^{-ikx}$$

$$Q_j = \sum_{k\geq 1} q_{k,j} e^{ikx}$$

Let f be the rational Gromov-Witten potential of $\mathbb{P}^r$, that is

$$\begin{aligned} f(t_1,\ldots,t_v,z) &= \sum_d \sum_{n_i} \frac{1}{n_1!\ldots n_v!} \\ &\quad ((\mathrm{Ev}^*_{\mathcal{M}}((t_1 a_1)^{n_1} \times \cdots \times (t_v a_v)^{n_v})) \\ &\quad \cap [\overline{\mathcal{M}}_{0,\sum n_i}(\mathbb{P}^r, d)]^{\mathrm{vir}}). \end{aligned}$$

Corollary 5.1.9. *The rational rubber potential is given by*

$$A = \frac{1}{2\pi}\int_0^{2\pi} f(\beta_1 + P_1 + Q_1, \beta_2 + P_2 + Q_2, \ldots, \beta_v + P_v + Q_v, ze^{imx})\, dx$$

where within the above formula, we treat $p_{k,i}$, $q_{k',i'}$ as super-commuting variables of degree

$$\deg p_{k,i} = 2 - 2\deg a_i - 2k$$

$$\deg q_{k',i'} = 2 - 2\deg a_{i'} + 2k$$

and the p-variables are to be written before the q-variables.

Proof. The operation

$$\frac{1}{2\pi}\int_0^{2\pi} \quad dx$$

has the effect neglecting all coefficient of e^{mix} for $m \neq 0$ which ensures that numerical condition for multiplicities is satisfied. It is clear then that the rubber invariant is equal to the corresponding Gromov-Witten invariant where interior and boundary marked points are treated as marked points. □

This Fourier series formalism is similar to the residue formalism of [5].

5.2. **Caporaso-Harris formula.** Here we examine rubber invariants of $\mathcal{O}(1)$ over $\mathbb{P}^1$ without powers of L^0 or L^∞.

Lemma 5.2.1. *All higher genus ($g \geq 1$) rubber invariants of $L = \mathcal{O}(1)$ over $X = \mathbb{P}^1$ with at least one point of tangency to D_∞ and one interior marked point vanish.*

Proof. This is a virtual dimension count. □

Now, following [2], let us apply Corollary 5.1.9 to compute the rational rubber potential. Let us change notation slightly and write a basis for $H^*(\mathbb{P}^1)$ as

$$a_0 \in H^0(\mathbb{P}^1)$$

$$a_2 \in H^2(\mathbb{P}^1)$$

and write t_0, t_2 for variables dual to a_0, a_2. The rational Gromov-Witten potential for $\mathbb{P}^1$ is

$$f_{\mathbb{P}^1}(t_0, t_2, z) = \frac{t_0^2 t_2}{2} + e^{t_2} z.$$

Therefore, the rubber potential, A satisfies

$$\begin{aligned} A_2 &\equiv \frac{\partial A}{\partial \beta_2}|_{\beta_2=0} \\ &= \hbar^{-1}\left(\beta_0^2+\sum_k p_{k,0}q_{k,0}+\frac{1}{2\pi}\int_0^{2\pi} e^{\sum_k p_{k,2}e^{-ikx}+\sum_k q_{k,2}e^{ikx}+ix}z\ dx\right). \end{aligned}$$

Let us write down the relative potential of $(\mathbb{P}^2, L)$, that is the projective plane relative a line. Let us choose $\{H^2, H^1, 1\}$ as a basis of $H^*(\mathbb{P}^2)$. We restrict ourselves to the potential involving only cohomology of the form H^2 and at least one p variable. Let us use θ_1 to express the element of $\mathcal{F}$ dual to H^2. Let us use $p_{n,0}$ and $p_{n,2}$ to express nth order multiplicities to H at 1 and $[\mathrm{pt}] \in H^*(L)$ respectively. Let us write the degree as z^d where d denotes the class of $dL \in H_2(\mathbb{P}^2)$. Therefore, F is an expression in $\tilde{\hbar}$, θ_1, $\tilde{p}_{n,0}$, $\tilde{p}_{n,2}$, and z^d.

By dimensional considerations, $F|_{\theta_1=0} = 0$ and the differential equation of Theorem 4.5.1 becomes

$$\frac{\partial F}{\partial \theta_1} = A_2 \cdot F.$$

Unwinding the action of $\mathcal{R}$ on $\mathcal{F}$, we see that this becomes

$$\frac{\partial F}{\partial \theta_1} = (\sum_k k\tilde{p}_{k,0}\frac{\partial}{\partial \tilde{p}_{k,2}} + \hbar^{-1}\frac{1}{2\pi}\int_0^{2\pi} e^{\sum_k \tilde{p}_{k,2}e^{-ikx}+\sum_k k\tilde{\hbar}\frac{\partial}{\partial \tilde{p}_{k,0}}e^{ikx}+ix}z\ dx)\ F.$$

This is the expression of the Caporaso-Harris formula as written in [5].

5.3. **Ruled Surfaces $\mathbb{F}_n$.** We can apply the rubber formalism to derive the inductive formula for the relative Gromov-Witten invariants on Hirzebruch surfaces from [20]. Let $\mathbb{F}_n$ be the ruled surface

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(n) \oplus \mathcal{O})$$

where $n \geq 0$. Let π be the projection $\pi : \mathbb{F}_n \to \mathbb{P}^1$

Let $D \subset \mathbb{F}_n$ be the infinity section of $\mathcal{O}_{\mathbb{P}^1}(n)$ The second homology of $\mathbb{F}_m$, $H_2(\mathbb{F}_m)$ is generated by $C_0 = D$, and f, a fiber of π.

Again, let us consider the terms in F with at least one p-variable and no point classes at interior marked points. By dimensional reasons, the only non-vanishing invariant comes from degree 1 maps to a fiber. In this case, the virtual cycle of the moduli space agrees with the usual fundamental cycle, yielding

$$F|_{\theta_1=0} = \tilde{p}_{1,2}z^f.$$

Now, we can compute the rubber invariants of $X = \mathbb{P}^1$, $L = N_{D/\mathbb{F}_m} = \mathcal{O}(-m)$. By the same arguments as above,

$$A_2 = \hbar^{-1}\left(\beta_0^2+\sum_k p_{k,0}q_{k,0}+\frac{1}{2\pi}\int_0^{2\pi} e^{\sum_k p_{k,2}e^{-ikx}+\sum_k q_{k,2}e^{ikx}-imx}z\ dx\right).$$

We get the differential equation

$$\frac{\partial F}{\partial \theta_1} = (\sum_k k\tilde{p}_{k,0}\frac{\partial}{\partial \tilde{p}_{k,2}} + \hbar^{-1}\frac{1}{2\pi}\int_0^{2\pi} e^{\sum_k \tilde{p}_{k,2}e^{-ikx}+\sum_k k\tilde{\hbar}\frac{\partial}{\partial \tilde{p}_{k,0}}e^{ikx}-imx}z^{C_0}\ dx)\ F.$$

Under the identification

$$\mathrm{Bl}_0\mathbb{P}^2 = \mathbb{F}_1$$

the above recursion formula reduces to Ran's [19].

5.4. **Rational Gromov-Witten Invariants for $\mathbb{P}^n$.** Here we consider the rational relative Gromov-Witten invariants of $(\mathbb{P}^n, H)$ where H is a hyperplane in $\mathbb{P}^n$ and $n \geq 3$. We follow the computation of [2] which gives the formula of [21]. Let us compute the potential F where we count positive degree curves and at interior marked points we pull back cohomology classes from $H^k(\mathbb{P}^n)$ where $k \geq 4$. Consider cohomology classes

$$e_4, e_6, \ldots, e_{2n}$$

where $e_{2i} = [H]^i$. Let θ_{2i} be dual to e_{2i} On $H^*(H) = H^*(\mathbb{P}^{n-1})$, we look at cohomology classes

$$c_0, c_2, \ldots, c_{2n-2}$$

where c_{2i} generates $H^{2i}(\mathbb{P}^r)$. Let $\tilde{p}_{k,i}$ be dual to c_{2i}.

By dimensional considerations,

$$F|_{\theta_4=\cdots=\theta_{2n}=0} = 0.$$

Let

$$f_{\mathbb{P}^{n-1}}(t_0, t_2, \ldots, t_{2n-2}, z)$$

be the rational Gromov-Witten potential of $\mathbb{P}^{n-1}$ where t_{2i} is dual to a cohomology class in $H^{2i}(\mathbb{P}^{n-1})$ and z is the degree marker. Then, we can use Corollary 5.1.9 to write the rubber potential,

$$A_{2i} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f}{\partial t_{2i}}(\beta_1 + P_1 + Q_1, \beta_2 + P_2 + Q_2, \ldots, \beta_v + P_v + Q_v, ze^{ix})\, dx.$$

Our differential equations become

$$\frac{\partial F}{\partial \theta_{2i}} = A_{2i-2} \cdot F.$$

6. Hamiltonian Formalism

In this section, we study a formalism for relative Gromov-Witten invariants that duplicates the structure of the Symplectic Field Theory [2] of Eliashberg, Givental, and Hofer. This formalism has the added advantage that it takes into account some of the redundancies of rubber invariants given by Theorem 4.6.1. The rubber invariants are encoded in a certain generating function called the *Hamiltonian*

Let X be a projective manifold and let $P = \mathbb{P}(L \oplus 1_X)$ be the projective completion in a line-bundle L over X. We considered $\mathcal{MA}$, a moduli space of curves in P relative the zero and infinity sections and quotiented by a $\mathbb{C}^*$-action that dilates the fibers. This moduli space possesses an evaluation map

$$\mathrm{Ev} : \mathcal{MA} \to X^m \times X^{r_0} \times X^{r_\infty}$$

where the three factors in the target denote the image of n interior marked points, r_0 boundary marked points evaluating to the zero section and r_∞ boundary marked points evaluating to the infinity section.

In Symplectic Field Theory, there is a similar moduli space, $\mathcal{M}$. The construction and compactification of this moduli space are markedly different and the evaluation map is

$$\mathrm{Ev} : \mathcal{M} \to (S(L))^m \times X^{r_0} \times X^{r_\infty}$$

where $S(L)$ is the unit circle bundle in L. Consequently, the classes that are pulled back at interior marked points are from $H^*(S(L))$. Now the cohomology of $S(L)$ is related to that of X by the following Gysin sequence.

$$\begin{array}{ccccccccc}
H^*(L,L_0) & \longrightarrow & H^*(L) & \longrightarrow & H^*(L_0) & \longrightarrow & H^{*+1}(L,L_0) & \longrightarrow & H^{*+1}(L) \\
\| & & \| & & \| & & \| & & \| \\
H^{*-2}(X) & \xrightarrow[\cup c_1(L)]{} & H^*(X) & \xrightarrow[\pi^*]{} & H^*(S^1(L)) & \xrightarrow[\pi_*]{} & H^{*-1}(X) & \xrightarrow[\cup c_1(L)]{} & H^{*+1}(X)
\end{array}$$

where $\pi : S(L) \to X$.

Classes in $H^*(S^1(L))$ are non-canonically decomposable into classes of two kinds, those in the image of π^* and those in the cokernel of π^*. Classes in the image of π^* do not fix the S^1 symmetry of the fibers in $S^1(L)$ while classes in the cokernel of π^* do. Classes in the cokernel, we consider *phase-fixing* marked points while those in the image we consider *non-phase-fixing*. There is a subset of Symplectic Field Theory invariants that involve pulling back phase-fixing classes at exactly one marked point. It is this theory that we translate into our context. As an aside, we note that we believe that the analog of SFT in which we use a number of phase-fixing classes is the invariants as below, but enriched by powers of $c_1(\mathrm{L}^\infty)$.

Consider two types of marked point depending on the type of cohomology classes that is pulled back along their evaluation maps.

1) Phase-fixing marked points ϕ that involve a class of the form $\mathrm{ev}_\phi^*(\pi_* a)$ or alternatively by a class $\mathrm{ev}_\phi^* b$ where $b \in \ker(\cup c_1(L) : H^*(X) \to H^{*+2}(X))$.
2) Non-phase fixing marked points i that involve $ev_i^*(a)$ where a is arbitrary but can be considered to be a representative of a class in $\mathrm{coker}(\cup c_1(L) : H^{*-2}(X) \to H^*(X))$.

6.1. **Definition of Hamiltonian.** The Hamiltonian will be dependent on the choice of a particular kind of basis.

Definition 6.1.1. A *phase fixing basis* is a homogeneous basis $\{a_1, \ldots, a_m\}$ for $\ker(\cup c_1(L) : H^*(X) \to H^{*+2}(X))$

Definition 6.1.2. A *non-phase fixing basis* is a set of homogeneous elements $\{b_1, \ldots, b_n\}$ of $H^*(X)$ that descend to a basis of $\mathrm{coker}(\cup c_1(L) : H^{*-2}(X) \to H^*(X))$.

We define $\mathcal{H}$, a graded super-Weyl algebra over $\mathbb{Q}[H_2(X)]$. Pick a phase-fixing basis $\{a_1, \ldots, a_m\}$, a non-phase fixing basis $\{b_1, \ldots, b_n\}$, and a homogeneous basis $\{c_1, \ldots, c_s\}$ of $H^*(X)$. We will consider variables σ_i, τ_j corresponding to a_i, b_j respectively. For every positive integer n, we will have $p_{n,i}, q_{n,i}$ corresponding to c_i. Let us consider a graded algebra generated by the following elements

$$\begin{aligned}
\deg \hbar &= -2(\dim X - 2) \\
\deg z^A &= 2 < c_1(TX), A > + < c_1(L, A) > \\
\deg \sigma_i &= -1 - \deg a_i \\
\deg \tau_i &= 2 - \deg b_i \\
\deg p_{n,i} &= 2 - \deg c_i - 2n \\
\deg q_{n,i} &= 2 - \deg c_i + 2n.
\end{aligned}$$

Note that we have changed the grading of σ_i from what we would have expected in the definition of the rubber potential. In $\mathcal{H}$, $\hbar$, z^A, σ_i, τ_i are supercentral while the p- and q-variables satisfy

$$[p_{n_1,i_1}, p_{n_2,i_2}] = 0,\ [q_{n_1,i_1}, q_{n_2,i_2}] = 0,\ [q_{n_1,i_1}, p_{n_2,i_2}] = n_1\delta_{n_1,n_2} g^{i_1 i_2}\hbar.$$

As in $\mathcal{R}$, the algebra $\mathcal{H}$ consists of Laurent series in $\hbar$ whose coefficients are power series in the p-variables whose coefficients are power series in the τ- variables whose coefficients are polynomials in the σ and q-variables.

To define the Hamiltonian, we define the following formal sums

$$\begin{aligned} S &= \sum \sigma_i a_i, \\ T &= \sum_n \frac{1}{n!}(\sum_i \tau_i b_i)^n, \\ P &= \sum_n \frac{1}{n!}\left(\sum_{k,i} p_{k,i}\right)^n, \\ Q &= \sum_n \frac{1}{n!}\left(\sum_{k,i} q_{k,i}\right).^n \end{aligned}$$

Definition 6.1.3. The *Hamiltonian* H is

$$H = \sum_\chi \sum_{A\in H_2(X)} \hbar^{-\frac{1}{2}\chi}(S\cdot T, P, Q)_{\chi,A,0} z^A$$

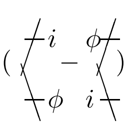
where $S\cdot T$ denotes multiplication in the algebra $\mathcal{H}$.

H is linear in the σ-variables. The σ variable keeps track of the phase-fixing classes which are evaluated at a single, distinguished marked point. H is homogeneous of degree -1.

6.2. **Dependence on Representatives.** We will show how the Hamiltonian depends on the choice of representative classes in $\mathrm{coker}(\cup c_1(L) : H^{*-2}(X) \to H^*(X))$.

Lemma 6.2.1. *Consider $\mathcal{MA}$ with two distinguished marked points, ϕ and i. Let $a \in H^*(X)$ satisfy $a \cup c_1(L) = 0$. Then*

$$ev^*_\phi(a)ev^*_i(c_1(L)) = ev^*_\phi(a)(\quad - \quad)$$



Proof. Multiply the formula from Theorem 4.6.1 with $\mathrm{ev}^*_\phi(c_1(L))$. □

If we view ϕ as the phase-fixing marked point, every term in the Hamiltonian will involve a factor of the form $\mathrm{ev}^*_\phi a$ where $a\cup c_1(L) = 0$

Let us pick a phase-fixing basis,

$$\{a_1, \dots, a_m\} \subset H^*(X)$$

and a non-phase fixing basis

$$\{b_1, \dots, b_n\} \subset H^*(X).$$

From the construction of our algebra $\mathcal{H}$, the Hamiltonian is invariant under change of basis of the form

$$b_i' = M_{ij} b_j.$$

Therefore, we need only determine how H varies when we change $b_i \in H^*(X)$ representing a class $[b_i] \in \operatorname{coker}(\cup c_1)$. Let us change $\{b_1, \ldots, b_n\}$ one element at a time. Write

$$b_1' - b_1 = dc_1(L).$$

Define $\tilde{b_t} = (1-t)b_1 + tb_1'$. Let H_t be H with $\tilde{b_t}$ substituted for b_1 so H_0 is the Hamiltonian with b_1 in place, H_1 is the Hamiltonian with b_1' in place.

$$H_t = \sum_\Gamma \frac{1}{\operatorname{Aut}_\Gamma(R_0)\operatorname{Aut}_\Gamma(R_\infty) k_1! \ldots k_n!} < a_j, \tilde{b_t}^{k_1} b_2^{k_2} \ldots b_n^{k_n} >$$

$$\cdot \hbar^{\frac{\chi}{2}} \sigma_i \tau_1^{k_1} \ldots \tau_n^{k_n} p^\Gamma q^\Delta z^d$$

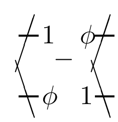
so

$$\frac{\partial H_t}{\partial t} = \sum_\Gamma \frac{1}{\operatorname{Aut}_\Gamma(R_0)\operatorname{Aut}_\Gamma(R_\infty) k_1! \ldots k_n!}$$

$$\cdot < a_j, k_1(dc_1(L)) \tilde{b_t}^{k_1 - 1} b_2^{k_2} \ldots b_n^{k_n} > \hbar^{-\frac{1}{2}\chi} \sigma_i \tau_1^{k_1} \ldots \tau_n^{k_n} p^\Gamma q^\Delta z^d.$$

But

$$< a_j, (dc_1(L)) \tilde{b_t}^{k_1 - 1} b_2^{k_2} \ldots b_n^{k_n} >=$$

$$\operatorname{Ev}^*(a_j \times d \times \tilde{b_t}^{k_1 - 1} b_2^{k_2} \ldots b_n^{k_n}) \cdot ($$ 

$$).$$

where the first two entries in Ev corresponds to the marked points denoted by ϕ and 1 respectively.

Let us define the generating function K_t by

$$K_t = \sum_\Gamma \frac{1}{\operatorname{Aut}_\Gamma(R_0)\operatorname{Aut}_\Gamma(R_\infty) k_1! \ldots k_n!}$$

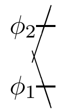
$$\cdot < d_1 \tilde{b_t}^{k_1} b_2^{k_2} \ldots b_n^{k_n} > \hbar^{\frac{\chi}{2}} \tau_1^{k_1 + 1} \ldots \tau_n^{k_n} p^\Gamma q^\Delta z^d.$$

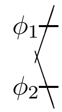
Note that K is of even degree and so

$$\frac{\partial H_t}{\partial t} = [K_t, H_t].$$

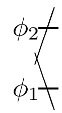
6.3. **The Hamiltonian as a Differential.** We may define a differential, D^H, on the algebra $\mathcal{H}$. One can compute the homology with respect to this differential. This homology will be an invariant of (X, L).

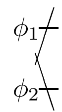
Lemma 6.3.1. *Consider $\mathcal{MA}$ with at least two marked points including ϕ_1 and ϕ_2. Let $a_1, a_2 \in \ker(\cup c_1(L) : H^*(X) \to H^{*+2}(X))$. Then*

$ev^*_{\phi_1}(a_1)ev^*_{\phi_2}(a_2)$ 

$= ev^*_{\phi_1}(a_1)ev^*_{\phi_2}(a_2)$ 

Proof. We intersect

$\mathrm{ev}^*_{\phi_1}(c_1(L)) - \mathrm{ev}^*_{\phi_2}(c_1(L)) \quad = $ 

$-$ 

with $\mathrm{ev}^*_{\phi_1}(a_1)\mathrm{ev}^*_{\phi_2}(a_2)$. □

Theorem 6.3.2. *H is nilpotent of order 2 in $\mathcal{H}$.*

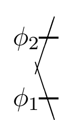
Proof. Write

$$H = \sum_i \sigma_i H^i.$$

Notice that H^i is an element of degree $1 - \deg \sigma_i = 2 + \deg a_i$.

$$\begin{aligned} H^2 &= (\sum_i \sigma_i H^i)^2 \\ &= \sum_i \sigma_i H^i \sigma_i H^i + \sum_{i<j} (\sigma_i H^i \sigma_j H^j + \sigma_j H^j \sigma_i H^i) \\ &= \sum_i \sigma_i^2 (H^i)^2 + \sum_{i<j} (\sigma_i \sigma_j ((-1)^{(\deg a_i)(\deg a_j - 1)} H^i H^j \\ &\quad +(-1)^{(\deg a_i - 1)(\deg a_j) + (\deg a_i - 1)(\deg a_j - 1)} H_j H_i)). \end{aligned}$$

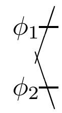
Note that if $\deg a_i$ is odd, then σ_i is even, so $\sigma_i^2 \neq 0$, so we must show $(H^i)^2 = 0$. But by arguments analogous to the proof of the degeneration formulae, $(H^i)^2$ is given by

$H_i^2 = (\mathrm{ev}^*_{\phi_1}(a_1)\mathrm{ev}^*_{\phi_2}(a_2)(\text{sum over terms})\cup$ 

$) \cap \ [\mathcal{MA}]^{\mathrm{vir}}.$

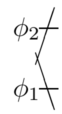
where "sum over terms" indicate a sum over all the cohomology classes and formal variables that enter into Definition 6.1.3. But by the above lemma, this equals

$$(\mathrm{ev}^*_{\phi_1}(a_i)\mathrm{ev}^*_{\phi_2}(a_i)(\text{sum over terms})\cup$$



$$) \cap\ [\mathcal{MA}]^{\mathrm{vir}}$$

$$= -(\mathrm{ev}^*_{\phi_2}(a_i)\mathrm{ev}^*_{\phi_1}(a_i)(\text{sum over terms})\cup$$



$$) \cap\ [\mathcal{MA}]^{\mathrm{vir}} = -H_i^2.$$

Since $\sigma_i\sigma_j \neq 0$, we must show

$$\begin{aligned} 0 &= (-1)^{(\deg a_i)(\deg a_j - 1)} H^i H^j + (-1)^{(\deg a_i - 1)(\deg a_j) + (\deg a_i - 1)(\deg a_j - 1)} H^j H^i \\ &= (-1)^{\deg a_i - 1}(H^j H^i - (-1)^{(\deg a_i)(\deg a_j)} H^i H^j). \end{aligned}$$

The proof is analogous to the one above. □

Definition 6.3.3. The operator D^H, $D^H : \mathcal{H} \to \mathcal{H}$ is defined on homogeneous elements of $\mathcal{H}$ by $D^H : f \mapsto Hf - (-1)^{\deg f} fH$.

Corollary 6.3.4. *D^H is a differential on the graded algebra $\mathcal{H}$*

Proof. It is a simple computation to show $(D^H)^2 = 0$ and $D^H(fg) = D^H(f)g + (-1)^{\deg f} f D^H(g)$. □

6.4. **Hamiltonian Homology.** The chain complex $(\mathcal{H}, D^H)$ is invariant under changing the choice of representatives of $\operatorname{coker}(\cup c_1(L))$ in the non-phase fixing basis.

Theorem 6.4.1. *Let $\tilde{b_t} = (1-t)b_1 + tb_1'$. Let H_t be H with $\tilde{b_t}$ substituted for b_1. Then there is a family of isomorphisms $Q_t : \mathcal{H} \to \mathcal{H}$ so that*

$$D^{H_t} Q_t(f) = Q_t(D^H f).$$

Proof. Let K_t be as above. Define A_t by

$$A_t = \int_0^t K_s\ ds,$$

and let Q_t be given by

$$Q_t(f) = e^{A_t} f e^{-A_t}.$$

Write $f_t = Q_t(f)$ and let

$$E_t = D^{H_t}(f_t) - Q_t(D^{H_0} f).$$

Therefore,

$$E_t = [H_t, f_t] - Q_t([H_0, f],$$

$$\begin{aligned} \frac{\partial E_t}{\partial t} &= [[K_t, H_t], f_t] + [H_t, [K_t, f_t]] - [K_t, Q_t([H, f])] \\ &= [K_t, E_t]. \end{aligned}$$

Since $E_0 = 0$, it follows that $E_t = 0$. □

By changing representatives of $\operatorname{coker}(\cup c_1(L))$ one-by-one, we see that the differential graded algebra and hence the Hamiltonian homology is invariant.

There are other versions of the Hamiltonian Homology, rational Hamiltonian homology and contact Hamiltonian homology. They bear the same relation to Hamiltonian Homology as their analogs do to Symplectic Field Theory Homology. We refer the reader to [2] for details.

7. Localization Proof of Degeneration Formula

In this section, we give a proof of the degeneration formula in Theorem 4.4.6. That degeneration formula encodes in a generating function relation (3) among line-bundles on $\mathcal{MA}$:

$$\mathrm{L}^{\infty} \otimes \mathrm{ev}_i^* L = \mathrm{L}_{i,\text{not bot}}\,.$$

A proof is outlined above, but here we give a more direct proof using the virtual localization technique from [11] and [7] and adapted for the relative case in [8].

Our strategy is to evaluate the equivariant cap product,

$$\hbar \cup \mathrm{Ev}^* c \cup \mathrm{ev}_i^*(c_1(\mathcal{O}(1)) + c_1(L))) \cap [\mathcal{MY}]^{\mathrm{vir}}.$$

which we know to be zero on a particular stack, $\mathcal{MY}$ by localization. The localization formula will give a relation among cycle classes which when intersected with cohomology classes will give Theorem 4.4.6.

7.1. **Target Schemes.** We need to construct a stack $\mathcal{MY}$ that is closely related to $\mathcal{MA}$. Let X be a projective manifold and L be a line-bundle over X. Let $P = \mathbb{P}_X(L \oplus 1_X)$. Let $p : P \to X$ be the projection. Let $i_0 : D_0 \to P$, $i_\infty : D_\infty \to P$ be the inclusions of the zero and infinity sections respectively.

We want to consider stable maps into P relative D_0 and D_∞. The stack of stable maps we construct will differ from $\mathcal{MA}$ in that we do not have a $\mathbb{C}^*$-action that dilates the fiber of P. The construction, however, is analogous to that of $\mathcal{MZ}$ and $\mathcal{MA}$.

Define the scheme ${}_{k,l}Y$ as the union of $k + l + 1$ copies of P,

$${}_{k,l}Y = P_{-k} \sqcup_X \cdots \sqcup_X P_{-1} \sqcup_X P_0 \sqcup_X P_1 \sqcup_X \cdots \sqcup_X P_l$$

where $X_0 \subset P_i$ is identified with $X_\infty \subset P_{i+1}$. Let the automorphism group of ${}_{k,l}Y$ be $(\mathbb{C}^*)^k \times (\mathbb{C}^*)^l$ where the first k copies of $\mathbb{C}^*$ dilate the fibers of $P_{-k}, \ldots, P_{-1}$ and the last l copies of $\mathbb{C}^*$ dilate the fibers of $P_1, \ldots, P_l$. Note that there is no $\mathbb{C}^*$-factor dilating P_0. $\mathcal{MY}$ is the stack of stable pre-deformable maps to ${}_{k,l}Y$ with data given by a rubber graph.

The rigorous definition of $\mathcal{MY}$ is analogous to those of $\mathcal{MA}$ and $\mathcal{MZ}$. Begin by defining triples $(Y[k,l], D_0[k,l], D_\infty[k,l])$ indexed by a pair of non-negative integers where $Y[k,l]$ is a projective manifold with a $G[m] \times G[n]$-action, and $D_0[k,l]$ and $D_\infty[k,l]$ are smooth divisors. Let

$$\begin{aligned} Y[0,0] &= P \\ D_0[0,0] &= D_0 \\ D_\infty[0,0] &= D_\infty \end{aligned}$$

where D_0 and D_∞ are the zero and infinity sections in P. We define $Y[k,l]$ inductively,

$$Y[k+1,l] = \mathrm{Bl}_{D_\infty[k,l]\times\{0\}}(Y[k,l] \times \mathbb{A}^1).$$

$D_\infty[k+1,l]$ is the proper transform of $D_\infty[k,l]\times\mathbb{A}^1$, $D_0[k+1,l]$ is the inverse image of $D_0[k,l]\times\mathbb{A}^1$.

$$Y[k,l+1]=\mathrm{Bl}_{D_0[k,l]\times\{0\}}(Y[k,l]\times\mathbb{A}^1).$$

$D_0[k+1,l]$ is the proper transform of $D_0[k,l]\times\mathbb{A}^1$, $D_\infty[k+1,l]$ is the inverse image of $D_\infty[k+1,l]\times\mathbb{A}^1$. The $G[k]\times G[l]$ actions dilates the fibers in the tails.

Given a rubber graph Γ, we construct $\mathcal{MY}=\mathcal{M}(\mathcal{Y},\Gamma)$ by mimicking the construction of $\mathcal{MZ}$ and $\mathcal{MA}$. We consider families of pre-deformable relative maps described by Γ that are stable under the $G[m]\times G[n]$-action. We glue these families together into a stack and then quotient by the $G[m]\times G[n]$-action. $\mathcal{MY}$ carries a virtual cycle.

Note that $\mathcal{MY}$ is different from $\mathcal{MZ}(P,D_0\sqcup D_\infty)$ since in the construction of $Z[n]$, D_0 and D_∞ are blown up simultaneously and the $\mathbb{C}^*$-action dilates the fibers of their exceptional divisors simultaneously.

7.2. **Equivariant Data.** We will perform a virtual localization computation on $\mathcal{MY}$. We first define a $\mathbb{C}^*$ action on P. Specify points of $P=\mathbb{P}_X(L\oplus 1)$ by $[l:t]$. For $\lambda\in\mathbb{C}^*$, define the group action by

$$\lambda\cdot[l:t]=[\lambda l:t].$$

This $\mathbb{C}^*$ action induces a $\mathbb{C}^*$-action on $\mathcal{MY}$ so that

$$\mathrm{Ev}:\mathcal{MY}\to P^m\times X^{r_0}\times X^{r_\infty}$$

is equivariant.

Now, write $H^*_{\mathbb{C}^*}(\mathrm{pt})=\mathbb{C}[\hbar]$ where $\hbar$ is the Euler class of the equivariant line bundle on pt under the group action

$$\lambda\cdot t=\lambda t.$$

Now, let Γ be a rubber graph where there is at least one vertex that does not correspond to a trivial cylinder. The proof of the virtual localization theorem holds for $\mathcal{M}(\mathcal{Y},\Gamma)$ with trivial modifications.

Let $\pi:P\to X$. Consider the composition

$$\mathrm{Ev}_X=(\pi^m\times\mathrm{id}_{X^{r_0}}\times\mathrm{id}_{X^{r}_{\infty}}):\mathcal{MY}\to P^m\times X^{r_0}\times X^{r_\infty}\to X^m\times X^{r_0}\times X^{r_\infty},$$

and for i, an interior marked point, consider the evaluation map

$$\mathrm{ev}_i:\mathcal{MY}\to P.$$

Let $\mathcal{O}(1)$ be the equivariant line bundle over P that is dual to $\mathcal{O}(-1)$ equipped with the linearization

$$\begin{array}{ccc}(l,t)&\mapsto&(l,\lambda^{-1}t)\\ \downarrow&&\downarrow\\ [l:t]&\mapsto&[\lambda l:t]\end{array}$$

Note that $i_\infty^*(\mathcal{O}(1)\otimes p^*L)$ is the equivariant trivial bundle so

$$i_0^*(c_1(\mathcal{O}(1))+c_1(L))=\hbar+c_1(L)\in H^*_{\mathbb{C}^*}(D_0)=H^*(D_0)[\hbar]$$

while

$$i_\infty^*(c_1(\mathcal{O}(1)+c_1(L))=0\in H^*_{\mathbb{C}^*}(D_\infty)=H^*(D_\infty)[\hbar].$$

Note also that there is also a natural map $\mathrm{pt}^*:\mathbb{C}[\hbar]=H^*_{\mathbb{C}^*}(\mathrm{pt})\to H^*_{\mathbb{C}^*}(\mathcal{MY})$ and that $\mathrm{pt}^*\hbar$ (which we will denote by $\hbar$) is an equivariant extension of $0\in H^2(\mathcal{MY})$.

Let $n=\mathrm{vdim}\,\mathcal{MY}$ and $c\in H^{n-4}(X^m\times X^{r_0}\times X^{r_\infty})$

Note that

$$\deg\left((\hbar \cup \mathrm{Ev}^* c \cup \mathrm{ev}_i^*(c_1(\mathcal{O}(1)) + c_1(L))) \cap [\mathcal{MY}]^{\mathrm{vir}}\right)|_{\hbar=0} = 0$$

because the cohomology class is an equivariant extension of 0. We will prove Theorem 4.4.6 by computing this degree by virtual localization.

7.3. **Fixed Loci.** We identify the $\mathbb{C}^*$ fixed loci in $\mathcal{MY}$. For a map f in a $\mathbb{C}^*$ fixed locus, we have the composition

$$C \to \ {}_{k,l}Y \to .P$$

The irreducible components of C are of three types: (a) those mapping into D_0; (b) those mapping into D_∞; (c) those mapping into a fiber of $P \to X$ totally ramified over two points. There are, therefore, three types of fixed loci:

(1) Those whose generic element only has components of type (a) and (c)
(2) Those whose generic elements only has components of type (b) and (c)
(3) Those whose generic element has components of all three types.

A fixed locus of the first type is parameterized by $T : \mathcal{M}(\mathcal{A}, \Gamma) \to F$ where the morphism T attaches components of type (c) of degrees $\{\mu^\infty(1), \ldots, \mu^\infty(|R_\infty|)\}$ to the boundary marked points. We have $T_*([\mathcal{MA}]^{\mathrm{vir}}) = [F]^{\mathrm{vir}}$. The evaluation maps fit into the following commutative diagram

$$\begin{array}{ccc} \mathcal{MA} & \longrightarrow & X^m \times X^{r_0} \times X^{r_\infty} \\ \downarrow & & \downarrow \\ \mathcal{MY} & \longrightarrow & P^m \times X^{r_0} \times X^{r_\infty} \end{array}$$

where the vertical map $X \to P$ is given by including X as $D_0 \subset P$. The virtual normal bundle to this fixed locus has Euler class

$$e(N) = \hbar - c_1(\mathrm{L}^\infty).$$

A fixed locus of the second type is parameterized by $B : \mathcal{M}(\mathcal{A}, \Gamma) \to F$ where the morphism B attaches components of degrees $\{\mu^0(1), \ldots, \mu^0(|R_0|)\}$. The evaluation maps fit together as before except that the inclusion is now $D = D_\infty \hookrightarrow P$. $B_*([\mathcal{MA}]^{\mathrm{vir}}) = [F]^{\mathrm{vir}}$ and

$$e(N) = -\hbar - c_1(\mathrm{L}^0).$$

A fixed locus of the third type is parameterized by stacks of the form $I : \mathcal{M}(\mathcal{A}, \Gamma_{A_t}) \times_{D^r} \mathcal{M}(\mathcal{A}, \Gamma_{A_b}) \to F$ corresponding to a graph join quadruple $(\Gamma_{A_t}, \Gamma_{A_b}, L, J)$ such that $\Gamma_{A_t} *_{L,J} \Gamma_{A_b} = \Gamma$. The morphism I inserts components of type (c) of degrees

$$\{\mu_t^\infty(1), \ldots, \mu_t^\infty(|R_{t\infty|})\} = \{\mu_b^0(1), \ldots, \mu_b^0(|R_{b0}|)\}$$

between the components coming from each factor of $\mathcal{MA}$. The evaluation map takes the interior marked points on $\mathcal{M}(\mathcal{A}, \Gamma_{A_t})$ and $\mathcal{M}(\mathcal{A}, \Gamma_{A_b})$ to $X = D_0 \subset P$ and $X = D_\infty \subset P$, respectively. We have

$$I_*([\mathcal{M}(\mathcal{A}, \Gamma_{A_t}) \times_{D^r} \mathcal{M}(\mathcal{A}, \Gamma_{A_b})]^{\mathrm{vir}}) = |\operatorname{Aut}_{\Gamma_{A_b}, \Gamma_{A_t}, L}(RA_{b0}, RA_{t\infty})|.$$

If p_t, p_b are the projections of $\mathcal{M}(\mathcal{A}, \Gamma_{A_t}) \times_{D^r} \mathcal{M}(\mathcal{A}, \Gamma_{A_b})$ onto each factor, then the normal bundle to the fixed locus has Euler class

$$e(N) = (\hbar - p_t^* c_1(\mathrm{L}^\infty))(-\hbar - p_b^* c_1(\mathrm{L}^0)).$$

7.4. **Localization Computation.** We now compute the contribution from each fixed locus. The virtual localization formula [7] states that given a top-dimensional class $b \in H^*(\mathcal{MY})$, we have

$$\deg(b \cap [\mathcal{MY}]^{\mathrm{vir}}) = \sum_{I:F\to\mathcal{MY}} \frac{1}{\deg(I)} \deg\left(\frac{I^*b}{e(N_F)} \cap [F]^{\mathrm{vir}}\right).$$

The fixed locus of the first type contributes

$$\begin{aligned} & \deg\left(\frac{T^*(\hbar \mathrm{ev}_i^*(c_1(\mathcal{O}(1)) + c_1(L))\mathrm{Ev}^*c)}{e(N)} \cap [\mathcal{MA}]^{\mathrm{vir}})\right) \\ = \quad & \deg\left(\frac{\hbar(\hbar + \mathrm{ev}_i^* c_1(L)}{\hbar - c_1(\mathrm{L}^\infty)} \mathrm{Ev}^*c \cap [\mathcal{MA}]^{\mathrm{vir}})\right) \\ = \quad & \deg((\mathrm{ev}_i^* c_1(L) + c_1(\mathrm{L}^\infty))\mathrm{Ev}^*c \cap [\mathcal{MA}]^{\mathrm{vir}}). \end{aligned}$$

Fixed loci of the second type do not contribute to the localization formula because $\mathrm{ev}_i : F \to P$ factors as

$$\mathrm{ev}_i : F = \mathcal{MA} \to X = D_\infty \hookrightarrow P$$

and $i_\infty^*(c_1(\mathcal{O}(1)) + c_1(L)) = 0$.

The only fixed loci of the third type that contribute are those in which the ith marked point is mapped to D_0. Such a fixed locus contributes

$$\begin{aligned} & \frac{1}{\mathrm{Aut}} \deg\left(\frac{I^*(\hbar \mathrm{ev}_i^*(c_1(\mathcal{O}(1)) + c_1(L))\mathrm{Ev}^*c)}{e(N)} \cap [\mathcal{MA} \times_{D^r} \mathcal{MA}]^{\mathrm{vir}})\right) \\ = \quad & -\frac{1}{\mathrm{Aut}} \deg(Ev^*c \cap [\mathcal{MA} \times_{D^r} \mathcal{MA}]^{\mathrm{vir}}) \end{aligned}$$

where $\mathrm{Aut} = |\operatorname{Aut}_{\Gamma_{A_b}, \Gamma_{A_t}, L}(RA_{b0}, RA_{t\infty})|$.

Putting everything together we get

$$\begin{aligned} 0 \quad = \quad & \deg((\mathrm{ev}_i^* c_1(L) + c_1(\mathrm{L}^\infty))\mathrm{Ev}^*c \cap [\mathcal{MA}]^{\mathrm{vir}}) \\ & - \frac{1}{|MA_b|!|MA_t|!(|RA_{b0}|)^2} \sum_{\Upsilon \in \Omega_{\mathrm{L}_{i,\text{not bot}}}} \deg(Ev^*c \cap [\mathcal{MA} \times_{D^r} \mathcal{MA}]^{\mathrm{vir}}) \end{aligned}$$

where the sum is over all quadruples $\Upsilon = (\Gamma_b, \Gamma_t, L, J) \in \Omega_{\mathrm{L}_{i,\text{not bot}}}$ as in Theorem 4.1.1. By choosing c to be a formal sum of variables as in the definition of the correlators, we get Theorem 4.4.6.

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Department of Mathematics, Duke University
E-mail address: eekatz@math.duke.edu