Constructive Semantics for Instantaneous Reactions

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Abstract

This report presents some results towards a game–theoretic account of the constructive semantics of step responses for synchronous languages, providing a coherent semantic framework encompassing both non-deterministic Statecharts (as per Pnueli & Shalev) and deterministic Esterel. In particular, it is shown that Esterel arises from a finiteness condition on strategies whereas Statecharts permits infinite games. Beyond giving a novel and unifying account of these concrete languages the report sketches a general theory for obtaining different notions of constructive responses in terms of winning conditions for finite and infinite games and their characterisation as maximal post-fixed points of functions in directed complete lattices of intensional truth–values.

For conciseness, all proofs have been collected in Appendix B. In Appendix A some relevant auxiliary material on fixed points can be found.

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1 Introduction

The classical theory of games, originally developed in descriptive set theory and long used in economics and engineering control theory, has emerged as a surprisingly versatile mathematical tool also in Logic and Computer Science. The power of the game metaphor rests on its ability to handle combinatorially complex situations, specifically the alternation of quantifiers, in a natural and intuitive fashion [HS95, PS00]. The intensionality of the game model opens up a promisingly wide playground for reconciling the algebraic and operational views in the semantics of proofs and programming languages [Abr97]. The game-theoretic solution of the full-abstraction problem for the functional language PCF [AJM00, HO00], the game-theoretic analysis of proofs in multiplicative linear logic [AJ94, HO92, BDER97, AM99, Gir01] are the most prominent cases in point.

In this report we would like to draw attention to an important aspect of games that deserves to be highlighted more explicitly than it is perhaps currently done. What could become the starting point for many further applications is that games provide a powerful and intuitively rather appealing setting for studying non-monotonic problems with co- and contravariant logical dependencies. Such problems abound in Computer Science. Many of these arise from the need to handle open systems and maintain a compositional system-environment distinction. If the interaction with the environment has both enabling as well as inhibiting effects on the response of a system then the input-output semantics of an individual component necessarily involves non-monotonic functions. When such systems are composed and each component acts both as a system and as (part of) the environment for the others causality cycles can occur, that are not easy to resolve algebraically since for non-monotonic functions the standard least or greatest fixed point approach breaks down. Here, as Hintikka has argued forcefully for logic and set theory [Hin96] game theory, with its strong intensional notion of truth, can come to the rescue.

Game theory handles cyclic systems of non-monotonic behaviours by capturing the system and environment dichotomy through the binary polarity of player and opponent, so that the swapping of roles gives constructive (intensional) meaning to negation. In this report we demonstrate the versatility of this idea for the semantics of step responses in synchronous programming. We describe the reaction of a composite system to stimuli from its environment as a game played by the individual sub-systems in which these negotiate between themselves the final outcome. This negotiation is governed by game rules determining particular notions of constructive response. We show how different forms of winning conditions can generate different response semantics with varying degrees of constructiveness. The games we are using are non-classical in the sense that they are not necessarily determined, i.e. there need not exist winning strategies for either player. This models the non-constructiveness of a system response. Our strategies, unlike those used in type theory, do not have
computational meaning ("proofs", "program") themselves. They are extensional in that they generate constructive truth-values for the interpretation of signals. Our work differs from related work of De Alfaro and Henzinger [dAH01, dAHS02] where the game board (interface automata) represents explicit synchronisation dynamics. In our case of synchronous step responses we are interested only in the stationary behaviour, so the execution sequences and interleaving on the game board (mazes) is abstracted away.

2 Synchronous Programming and Step Responses

Let us begin by discussing the central issues arising from the Synchrony Hypothesis. For a more general introduction the reader is referred to [Hal93, Ber00].

In the synchronous model system execution is thought to be scheduled under the regime of an implicit global clock which marks off a succession of individual reaction instants. In every instant each component delivers a full response to the external stimuli imposed by the environment. According to the Synchrony Hypothesis all the internal signal exchanges needed to produce this reaction are abstracted from in the sense that both the input stimulus and the response are assumed to occur “at the same instant.” This reflects the point of view of an external environment which is always significantly slower than the system it feeds. In this way a reaction instant can be collapsed into a single atomic super-event in which all input and output happens instantaneously. This compactification is the beauty of the synchronous model and makes up much of its algebraic appeal. It adopts the macro-step viewpoint of the environment specified in terms of plain propositional logic and truth values to record only the overall presence or absence of a signal at the respective instant.

To be definite let us fix a concrete system model. Let us assume that components communicate via signals \( S = \{a, b, c, \ldots\} \) each of which can be present or absent for a given reactive instant. A component is built up from individual transitions that emit signals in reaction to certain triggering conditions becoming satisfied. A transition \( t \) of the form

\[
c_1, c_2, \ldots, c_m, \neg b_1, \neg b_2, \ldots, \neg b_n / a_1, a_2, \ldots, a_k
\]

is triggered by the presence of some signals \( \text{pos}(t) = \{c_1, \ldots, c_m\} \subseteq S \) called positive preconditions of \( t \) and the absence of signals \( \text{neg}(t) = \{b_1, b_2, \ldots, b_n\} \subseteq S \) called negative preconditions. The result of its execution is that the action signals \( \text{act}(t) = \{a_1, a_2, \ldots, a_k\} \subseteq S \) are emitted, so that they can be picked up by other transitions to trigger further computations. A reactive component is a structure \( C = (T, \text{act}, \text{trig}) \) where \( T \) is the set of transitions and \( \text{trig} : 2^S \rightarrow 2^T \) a triggering function. For each subset \( E \subseteq S \) the function \( \text{trig} \) picks out the set of transitions \( \text{trig}(E) \subseteq T \) that are triggered by \( E \), assuming all \( a \in E \) are present and all \( b \in S \setminus E \) are absent. If
t \in \text{trig}(E)$ we say $t$ is enabled by $E$. Because of the negative trigger conditions the function $\text{trig}$ is non-monotonic, in general. Increasing the set of signals $E_1 \subseteq E_2$ may both increase or decrease the number of transitions that are enabled.

We are interested in the overall response of a component $C$ in reaction to an initial environment input $E_0$ and determined as the combined effort of all the transitions in $C$. This so-called step response is to be consistent with the abstract model of a transition as an implication

$$(c_1 \land c_2 \land \cdots \land c_m \land \neg b_1 \land \neg b_2 \land \cdots \land \neg b_n) \supset (a_1 \land a_2 \land \cdots \land a_k),$$

specifying the reaction as “if all the $c_i$ are present and all $b_j$ absent, then all of the $a_i$ are emitted”. Parallel composition of transitions would naturally be logical conjunction $t_1 \land t_2$. This propositional reading is appealing but what sort of logic do we get? The simple answer is: It depends. It depends on the properties of the intended operational behaviour that the synchronous abstraction is supposed to model, in particular on how precisely the transitions are scheduled and how the interaction between them is synchronised.

Let us see how one would determine the system response of $C$ operationally for a single reactive instant under a given initial environment stimulus $E \subseteq S$. All transitions in $T$ are assumed to act concurrently with each other. The input $E$, thus, is sensed by all transitions simultaneously but only those in $\text{trig}(E)$ are enabled. Taking the role of a global scheduler we would now select some of these transitions, say $T_1 \subseteq \text{trig}(E)$, and execute them in parallel. How we determine $T_1$ will depend on the operational semantics we have in mind. The two extreme cases are executing only one transition at a time ($|T_1| = 1$) and executing all enabled transitions together in one go ($T_1 = \text{trig}(E)$). Firing $T_1$ emits the action signals $\text{act}(T_1) = \bigcup_{t \in T_1} \text{act}(t)$. These can now trigger further transitions, relative to the extended signal set $E_1 = E \cup \text{act}(T_1)$. Again we schedule a subset $T_2 \subseteq \text{trig}(E_1)$ of transitions enabled by $E_1$, and so on. In this way a chain reaction of transition firings $T_{i+1} \subseteq \text{trig}(E_i)$ and associated signal emissions $E_{i+1} = E_i \cup \text{act}(T_i)$ may ensue. We continue this process ensuring maximal progress for all system parts. When the activation sequence finally stabilises, i.e. $\text{act}(\text{trig}(E_n)) = E_n$, the reactive instant is completed and the step response or macro step of $C$ is the final accumulated signal set $E_n$. For contrast, the individual scheduling stages are sometimes called micro steps.

Within this generic model, which is typical for synchronous declarative languages many variations of scheduling strategies are possible. The crucial point here is how to deal with the potential inconsistencies introduced by the inhibitive effects of negative triggers. Consider the following situation: A transition $t_i = \neg b/a \in \text{trig}(E_i)$ is enabled by the absence of signal $b$ from $E_i$, i.e., because of $b \notin E_i$. Transition $t_i$ is fired and included into $T_i$. This produces action $a \in E_{i+1}$ which sets off further
transitions in subsequent $T_{i+1}, T_{i+2}, \ldots$ and eventually, because of that, signal $b$ is produced, say $b \in \text{act}(t_n)$ where $t_n \in T_n$. Clearly, this is inconsistent with the firing of $t_i$ in the first place which was done under the condition that $b$ is absent. Logically, this is a circular causality glitch: If $b$ is absent then $t_i$ must be executed by maximal progress, but this generates $b$, which means $t_i$ should actually be switched off and not fire at all. But then $b$ is never emitted and thus absent, hence $t_i$ enabled and ready to fire, etc. Physically, this would amount to plain oscillation of signal $b$.

There is not a single canonical way to handle this. Likely, the full range of possibilities have not been explored yet, but already there is a profusion of different solutions adopted in the literature on synchronous languages. Let us mention some of them. One approach, obviously, is not to bother about this inconsistency and simply run the scheduling through, regardless [HGd88]. This is justified by taking the absence of a signal to be a local condition only, that may be overridden in a later micro step. If we wish to maintain global consistency, however, we must drop the execution in order to avoid having both $t_i$ and $t_n$ in the same step response. E.g. we can do this at the late point where we notice that execution of $t_n$ would contradict that of $t_i$ to which we have already committed. So, we simply do not include $t_n$ into $T_n$ by considering it disabled. This is the approach in [LvdBC99, MSPT96]. Alternatively, we can try and avoid the execution of $t_i$ at stage $i$, either by anticipating somehow that its causal consequences will contradict it or by backtracking from stage $n$ to a stage before $i$ when the problem is noticed, trying another schedule in which $t_i$ will not fire. In other words, we search for a consistent scheduling sequence. This latter solution is the essence of Pnueli & Shalev’s semantics of Statecharts [PS91]. Still another approach is not to run the schedule through transitive causal sequences but complete the reaction with a maximal set of transitions that are enabled and not causally dependent. This is what is done in UML and (synchronous) statecharts [HN96, LHHR94] type of languages and quite a drastic solution as it prevents the communication between transitions within a macro step and thus effectively gives up the Synchrony Hypothesis. Somewhere in between lies Saraswat’s language for timed concurrent constraint programming [SJG94]. There, the negation $\neg b$ in the trigger of $t_i$ would be taken to refer to the previous instant, while its action $a$ and all its consequences are emitted in the current step. In this way causal sequences can still be accumulated according to maximal progress, yet the inconsistency is removed as the absence of $b$ in $t_1$ and its emission by $t_n$ are in successive instants. The conflict, thus, is broken through the (implicit) clock tick at the macro level. Finally, Berry’s ESTEREL [Ber00] has full synchrony but rejects any program admitting a schedule like the one above as being non-constructive. This is in line with the view that consistency in computing a response should not depend on any particular cleverness of the scheduler. Scheduling under all possible input should be confluent and produce the same uniquely determined result. Stratified logic programming [ABW88] uses a syntactic restriction to avoid recursions through
negation. Finally, we should not forget to mention the option of not to permit negative event triggers at all like in Modecharts [JM94].

Our claim is that the variety of semantics arising from the different options of handling negation can be described naturally and uniformly using the game-theoretic metaphor. Given the long-standing and sometimes heated debate about the “right” step semantics for synchronous programming it appears to be more than appropriate to search for a convincing unifying setting in which the different dialects can coexist, each having its own characteristic place and application. We present a natural hierarchy of three increasing levels of constructive strength in the interpretation of negation, covering both Pnueli & Shalev’s version of Statecharts and Berry’s Esterel. As the work is still tentative we do not claim to achieve more than outlining this programme here. The guiding idea is to try and characterise the different scheduling disciplines as instances of the ideal propositional view promised by the Synchrony Hypothesis according to which transitions are logical implications and parallel composition is logical conjunction. It is not difficult to see that this requires more than classical two-valued logic (true = presence, false = absence).

Take the four transitions $t_1 := \neg b \supset a$, $t_2 := b \supset a$, $t_3 := true \supset a$, $t_4 := a \supset b$. In classical logic the conjunction $t_1 \land t_2$ is equivalent to $t_3$, so we would expect $t_1 \land t_2$ to be interchangeable with $t_3$. In particular $C_1 := t_1 \land t_2 \land t_4$ should be equivalent to $C_2 := t_3 \land t_4$ and thus represent the same operational behaviour. Now consider the operational semantics of [PS91] and run $C_1$ in the empty environment $E = \emptyset$: Signal $b$ is absent in $E$, so transition $t_1$ fires. This sets off action $a$, triggering $t_4$, which produces signal $b$. This is inconsistent as $b$ was assumed absent when $t_1$ was fired. Hence, we try to find another schedule that is safe. But there is none, so program $C_1$ fails, meaning it does not have a response in the empty environment. On the other hand, $C_2$ happily terminates producing response $b$ in all circumstances. So, $C_1$ and $C_2$ are different, operationally.

It is not surprising that classical logic is not fine enough to model all the intensional aspects of scheduling under inhibiting as well as enabling effects. As seen above the single truth-value false cannot adequately model the meaning of negation $\neg b$ when $b$ is initially absent but occurring later. What is surprising is that for certain coherent scheduling regimes it is possible to maintain the abstract propositional viewpoint, i.e. avoid the complications of modelling scheduling sequences in detail, simply by choosing a constructive logic interpretation. For instance, in [LM02] it was shown that the simple twist of replacing the classical two-valued by an intuitionistic interpretation of signals (specifically, 3-valued Gödel logic) suffices to obtain a fully abstract and compositional model for the original macro-step semantics of Statecharts as given by Pnueli and Shalev [PS91]. Another example in this direction is [ALM03] which explains the constructive semantics of Esterel naturally in terms of winning strategies in finite 2-player games. The present report extends and systematises this work to show how non-classical truth values induced
by logic games can be used to characterise different kinds of constructive single-step semantics for Statecharts-like languages.

3 Synchronous Reactions and Two–Player Symmetric Maze Games

How to Play. A reactive component $C$ can be modelled as a maze $M$ consisting of rooms and directed corridors between them. Every signal in $S$ corresponds to a room in $M$ and the corridors represent the causal relations between signals as given by the transitions. When one is in a room $a$ and there is a corridor $a \rightarrow b$, then it is possible to move into room $b$. In other words, the corridor $a \rightarrow b$ denotes, according to $C$, that the status (present or absent) of $a$ can be justified (computed) in terms of the status of $b$. In the two-player game employed here, this maze is the board and the game figure is a token which is moved from room to room by the players taking turns according to certain rules. In general, the objective of the game is that of defending a set of rooms (region) according to a given winning condition. Thus, this game produces a pair of regions $P \subseteq S$ and $O \subseteq S$ one for each player. These two sets together constitute a possible response of $C$ under a particular constructive semantics such that $P$ and $O$ will contain signals that are present and absent, respectively, when $C$ is executed following the chosen operational model. The two players $P = \{A, B\}$ are the system and the environment denoted respectively by $A$ and $B$, where $A$ plays for region $P$ and $B$ for region $O$. Hence, in all plays from $P$ player $A$ starts while in $O$ his opponent $B$ is the first to play. We will say $A$ defends the front line $(P, O)$ if $A$ has a winning strategy for all plays from $P$ or $O$.

Intuitively, when the token is placed in a room the player whose turn it is, takes that room into his region and moves the token (if possible) to another room, where (depending on the rules) the turn may or may not change. Observe, however, that after a number of moves the token can be returned to a room previously visited, and moreover it can be the case that the turn is now with the opponent of the player who moved the token from this position last time. If this situation is repeated infinitely often and there is no strategy for either player to avoid this then obviously the signal that is represented by this room is oscillating (i.e., this transition is not constructive). Otherwise, it will depend on the semantics chosen (winning condition) if and how the two regions $P$ and $O$ will eventually become stable. If so, moving the token from one region to the other will necessarily imply a change in the turn and moving the token into the same region will not. In the light of this, the rules must be given by two types of valid moves that we shall call visible moves and secret moves, respectively. In our games, we represent this statically in the maze by two types of corridors visible and secret. Every time that the token is moved through a visible corridor the
control is passed to the opponent and if the token is moved through a secret corridor
the turn remains unchanged. In the same vein, it will be useful to distinguish
between visible rooms and secret rooms. Visible rooms, like visible corridors, are
atomic communication points between system and environment where information
is exchanged, while secret rooms represent intermediate or auxiliary positions where
no interaction takes place. This models the difference between atomic signals \(a, b\)
(visible rooms) and composite formulas such as \(a \land \neg b, a \lor \neg b\) (secret rooms). This
distinction will be relevant for certain cases of winning conditions.

Reactive Components as Mazes. Since in our game \(P\) will correspond to the
set of present signals and \(O\) to the set of absent signals the causality expressed in
the transitions \(T\) of \(C\) can be represented in the maze \(M\) as follows.

For any \(a \in \text{act}(t)\), transition \(t \in T\) is expressing the fact that \(a\) is caused to
be in \(P\) if for all \(c \in \text{pos}(t)\), \(c\) is in \(P\) and for all \(b \in \text{neg}(t)\), \(b\) is in \(O\). This
conjunction can be modelled in the maze by means of introducing an intermediate
(secret) room, say \(y\), and by adding a visible corridor between each \(a \in \text{act}(t)\) and
\(y\), a visible corridor between \(y\) and each \(c \in \text{pos}(t)\) and a secret corridor between \(y\)
and each \(b \in \text{neg}(t)\) as seen in Fig. 1, where visible rooms/corridors are marked \(\iota\)
and drawn with solid lines while secret rooms/corridors are drawn with dashed lines
and marked \(\tau\). A transition like \(b/a\) with only one trigger and action can be coded
without the intermediate room as a secret corridor from \(a\) to \(b\) and a transition \(\neg c/a\)
simply as a visible corridor from \(a\) to \(c\).

Note that any environment stimulus can be accounted for as part of a reactive
component. The situation where the environment provides \(x\) and \(z\) to \(C\) can be
expressed as the parallel composition $C \cdot /x, z$, where $\cdot /x, z$ is a transition with an empty precondition which produces signals $x$ and $z$ as required by the environment. A transition $t$ with an empty precondition is reflected in the corresponding maze as a visible corridor from all $a \in act(t)$ to a dungeon, i.e., a room without exits.

**Game Rules.** We now informally present different alternative ways of playing the game. Let us use the maze of Figure 2 assuming that $x$ and $z$ are the only signals that can be used as stimuli by the environment. For the sake of convenience, in what follows whenever $x$ or $y$ are present we will avoid drawing visible corridors to dungeons from these rooms, and instead we will simply assume that $x$ or $y$ are in $P$ when required.

![Maze M]

**Figure 2:** Example maze.

One alternative consists of interpreting each $t \in T$ as an implication in classical logic. The classical interpretation, however, is too weak in general to be compositional. In the synchronous reactive model it is required to produce constructive responses, meaning responses that are supported by some sort of argument that should come from the program. A natural way of obtaining various forms of constructive arguments, would be to assume that at startup and in some pre-established order the players choose a finite amount (discrete quantity) of a particular resource that we may call seeds. During the game, and depending on the rules, a player has to pay with seeds for taking some action (e.g. visiting a room for the first time, reusing a corridor, taking his turn, etc.). A player wins if he can make his opponent get stuck, i.e., force him to play in a dungeon or make him run out of seeds. Note that this metaphor is not only useful for introducing constructive arguments, but also can reflect the underlying resources and features of the physical system (e.g. energy to maintain a signal, time, memory and so on) that the model abstracted.
from. The notion of seeds is an intensional model parameter that can be used to explain extensional but infinite winning conditions in terms of finite processes. In general, different intensional rules, involving seeds as suggested above, may give rise to the same extensional model.

In the coherent maze game, player B (the opponent) is the first to pick up his supply of seeds whereupon player A does the same with the advantage of knowing the resources of his opponent. In this game, both players are required to pay one (unit of) seed every time the token enters a visible room. Under these conditions A can win provided he can keep the play going for an arbitrary number of moves through visible rooms. One possible solution for Fig. 2 is given by \((P, O) = (\{y, r\}, \{z, s\})\). If the initial position of the token is \(r\) then evidently player A can use the secret corridor that forms a self-loop in this room repeatedly without ever getting stuck. If the initial position is \(y\) player A can move the token to room \(s\) and pass the turn to B. From this position, player B will either move to \(r\) and give A the chance to take over forever, or he can place the token back to \(y\) restarting the whole process. In any case, since the number of seeds on both sides is finite the play must eventually finish. Moreover, because A had chosen his number of seeds after B he will have more seeds available than B (assuming A is sensible enough), and so when this process ends it has to be because B runs out of seeds. Observe that although \((P, O) = (\{x, y, r, s\}, \{z\})\) is a classical solution, it is not a solution for this game (coherence) since there is no way in which the player A can keep intact his region \(P\) if the game starts in room \(s\). Since moving the token to either \(r\) or \(y\) will give the turn to B who can take that room.

In the lazy maze game, the rules are quite similar as before except now only the player who actually gets to occupy (receives the turn in) a visible room must pay. Hence, a coherent solution like the one before \((P, O) = (\{y, r\}, \{z, s\})\) is no longer acceptable, because although A can have as many seeds as he wants this number is finite. So if he decides to go in circles from room \(r\) to room \(r\) he will eventually get stuck there without B having lost a single seed. To win, player A must be able to hand over the turn to B arbitrarily often in a visible room. It is not hard to see that a solution for this game is given by \((P, O) = (\{y, s\}, \{x, r, z\})\). Assuming that the presence of signal \(x\) is enforced by the environment we have two solutions \((P, O) = (\{x, y, r\}, \{z, s\})\) and also \((P, O) = (\{x, r, s\}, \{z, y\})\), which illustrates the inherent non-determinism in this type of game.

In the eager maze game everything is as in the previous case but now the order in which the players choose their seeds is inverted, first A chooses and then B. Now loops like the one formed between \(y\) and \(s\) when \(x, r \in P\) can no longer be supported by the fact that A has more seeds than B in such a way that depending on the starting position either \(y \in P, s \in O\) or \(s \in P, y \in O\) is defended by A. It is B who has the advantage of having more seeds, so basically A needs to solve the game
by avoiding circular confrontations and instead try to push \( B \) into a dungeon as soon as possible. In our board the solution for this is given by \( (P, O) = (\{y\}, \{x, z\}) \) which can easily be verified. This game type ensures determinism.

**Charts and Mazes.** Formally, a maze is a finite labelled transition system \( M = (S_i, S_\tau, \rightarrow^i, \rightarrow^\tau) \), consisting of finite and disjoint sets of visible rooms \( S_i \) and secret rooms \( S_\tau \), together with transition relations \( \rightarrow^\gamma \subseteq S \times S \), where \( \gamma \in \{i, \tau\} \) is a secret or observable action and \( S = S_i \cup S_\tau \) the set of all rooms. The transitions represent valid moves (or corridors) between rooms. A transition \( s \rightarrow^i s' \) corresponds to a visible corridor connecting \( s \) with \( s' \) and \( s \rightarrow^\tau s' \) to a secret corridor. For technical convenience we assume that rooms without exiting corridors (dungeons) are visible. In this way it is always observable when a player gets stuck. Moreover, we assume that no pair of secret rooms is connected by a secret corridor, which ensures that each move involves at least one visible interaction. We denote the opponent of player \( U \) by \( \overline{U} \). More generally, for \( \gamma \in \{i, \tau\} \) and \( U \in P = \{A, B\} \) we define \( U^\gamma \in P \) to be the unique \( U' \) such that \( U = U' \) iff \( \gamma = \tau \). We will assume throughout that \( M \) is a fixed, finitely branching, maze.

**Configurations and Plays.** A (game) configuration is a pair \( c = (\text{pos}(c), \text{turn}(c)) \in S \times P \).

The first part \( \text{pos}(c) \) is a position in \( M \) and \( \text{turn}(c) \) denotes the player that has the turn at this point. A play is a (possibly empty) finite or infinite sequence of configurations

\[
\pi = (m_0, t_0) \cdot (m_1, t_1) \cdot (m_2, t_2) \cdots
\]

consistent with the game rules, i.e., each step follows some corridor in \( M \) and the player’s turn changes exactly if this corridor is visible. Formally, for all \( (m_i, t_i) \) that have a successor \( (m_{i+1}, t_{i+1}) \) in \( \pi \) we must have \( m_i \rightarrow^i m_{i+1} \) if \( t_i = t_{i+1} \) or \( m_i \rightarrow^\tau m_{i+1} \) if \( t_i \neq t_{i+1} \). The domain \( \text{dom}(\pi) \in \omega + 1 \) of a game path is the set of indices, i.e. \( \text{dom}(\pi) = \omega \) if the path is infinite and \( \text{dom}(\pi) \in \omega \) if it is finite. In handling such indices it is expedient to consider the domain as an ordinal number. Specifically, \( \omega = \{0, 1, 2, \ldots\} \) and each natural number \( n \in \omega \) is identified with the set of its predecessors, i.e., \( n = \{0, 1, 2, \ldots, n - 1\} \). We can then present a path as a function \( \pi : \text{dom}(\pi) \to S \times P \). The empty play is denoted \( \epsilon \). From now on, let \( \Pi \) denote the set of plays in \( M \). As usual we write \( \subseteq \) for the prefix ordering on \( \Pi \), i.e., \( \pi_1 \subseteq \pi_2 \) if there exists a play \( \sigma \) such that \( \pi_2 = \pi_1 \cdot \sigma \). A play \( \pi_1 \) is a suffix of another play \( \pi_2 \) if \( \pi_2 = \sigma \cdot \pi_1 \) for some finite play \( \sigma \). 

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4 Strategies and Defensible Front Lines

As usual a strategy for a player $U$ is a subset of plays in which $U$’s moves are uniquely determined at each stage of a play where he holds the turn, while keeping unconstrained the decisions of his opponent.

**Definition 4.1** A strategy for $U$, or $U$-strategy, is a non-empty $\sqsubseteq$-closed subset $\Sigma \subseteq \Pi$ of plays that is continuous, $U$-deterministic and $\overline{U}$-closed, where $\Sigma$ is

- continuous, if $\pi \notin \Sigma$ then there is a finite prefix $\pi' \sqsubseteq \pi$ with $\pi' \notin \Sigma$.
- $U$-deterministic, if $\pi \cdot (m, U) \cdot (m_i, U_i) \cdot \sigma_i \in \Sigma$ for $i = 1, 2$, then $m_1 = m_2$ and $U_1 = U_2$.
- $\overline{U}$-closed, if $\pi \cdot (m, \overline{U}) \cdot \sigma \in \Sigma$ then for all corridors $m \xrightarrow{\gamma} m'$ in the maze there is an extension $\pi \cdot (m, \overline{U}) \cdot (m', \overline{U'} \cdot \sigma' \in \Sigma$.

Two features of $U$-strategies are noteworthy. Firstly, they are partial, i.e. player $U$ is not forced to make a move. He may decide to stop the play for good, even if there is an outgoing corridor. Secondly, a strategy in general has many initial configurations generating several independent threads of plays from different initial positions with different starting players.

Throughout the report we will fix $U = A$ and simply talk about a strategy when we mean $A$-strategy. Also, we will be interested only in consistent and positional strategies. A strategy is consistent if no position is occupied by more than one player, i.e., $\pi_i \cdot (m, U_i) \cdot \sigma_i \in \Sigma$ for $i = 1, 2$ and fixed $m \in S$ implies $U_1 = U_2$. A strategy is positional if a player’s every move only depends on the current position, not on the history of the play. Formally, $\pi \cdot (m, A) \cdot \sigma \in \Sigma$ iff $(m, A) \cdot \sigma \in \Sigma$. By default all strategies are assumed to be positional and consistent $A$-strategies.

Let us look at some consequences of consistency. For an $A$-strategy $\Sigma$ let $P_\Sigma$ and $O_\Sigma$ be the sets of positions in $\Sigma$ in which player $A$ and $B$ receive the turn, respectively. Formally, $P_\Sigma := \{m \mid \exists \pi, \sigma \cdot (m, A) \cdot \sigma \in \Sigma\}$ and dually $O_\Sigma := \{m \mid \exists \pi, \sigma \cdot (m, B) \cdot \sigma \in \Sigma\}$. Consistency of strategies, then, is equivalent to the condition $P_\Sigma \cap O_\Sigma = \emptyset$. We call pairs $(P, O)$ of regions with the property that $P \cap O = \emptyset$ front lines and $(P_\Sigma, O_\Sigma)$ the front line defended by strategy $\Sigma$. More generally, a front-line $(P, O)$ is called defensible if there exists a strategy $\Sigma$ such that $P = P_\Sigma$ and $O = O_\Sigma$, in which case we say that $\Sigma$ defends $(P, O)$. A front line $(P, O)$ where $P$ is the complement of $O$ is called binary or two-valued.

The pair $(P_\Sigma, O_\Sigma)$ is an emergent property of $\Sigma$ in the following sense: We may imagine that when the play enters a room $m$ in which player $A$ receives the turn, then $m$ is conquered and thus becomes part of $A$’s territory. The same room $m$ may later be revisited in the play and depending on who has the turn then, $m$ may either
fall to the other player \( B \), or possession of \( m \) is perpetuated by \( A \). In a consistent strategy \( \Sigma \), then, all the positions ever occupied by \( A \) are never lost to \( B \) and \( A \) never enters a room earmarked for \( B \). A consistent strategy keeps player \( A \) safely within the region \( P_\Sigma \) while at the same time it makes sure that the opponent is confined to \( O_\Sigma \) from which he cannot escape.

The positional nature admits of a simple concrete representation of strategies. It restricts its player (\( A \), by default) to play consistently, so he uses the same corridor every time he moves out of the same room. Given a positional strategy \( \Sigma \) we can extract a transition relation \( \alpha_\Sigma \subseteq S \times \{ \iota, \tau \} \times S \) such that \( \alpha_\Sigma := \{(m, \gamma, m') \mid (m, A) \cdot (m', A^\gamma) \cdot \sigma \in \Sigma \} \). Since \( \Sigma \) is \( A \)-deterministic \( \alpha_\Sigma \) is in fact a partial function \( \alpha_\Sigma : S \rightarrow \{ \iota, \tau \} \times S \), the transition strategy underlying \( \Sigma \). From the front-line \((P_\Sigma, O_\Sigma)\) and transition strategy \( \alpha \) the strategy \( \Sigma \) can be reconstructed. Every transition strategy \( \alpha \) and configuration \((m, U)\) induces a set of plays \( \Pi^\alpha(m, U) \) starting in \((m, U)\) in which \( A \) follows \( \alpha \):

\[
\Pi^\alpha(m, U) = \{ \pi \in \Pi \mid \forall n < \text{dom}(\pi). \forall \gamma \in \{ \tau, \iota \}. \forall x \in S.
\]

\[
(i) \quad n = 0 \Rightarrow \pi(n) = (m, U) \&
(ii) \quad \text{turn}(\pi(n)) = A \& n + 1 \in \text{dom}(\pi) \Rightarrow \alpha(\text{pos}(\pi(n))) = (\gamma, x) \iff \pi(n + 1) = (x, A^\gamma)
\]

It is easy to see that \( \Pi^\alpha(m, U) \) is non-empty and prefix-closed. Then, given a pair of regions \((P, O)\) and transition strategy \( \alpha \) we define

\[
\Pi^\alpha(P, O) := \bigcup_{m \in P} \Pi^\alpha(m, A) \cup \bigcup_{m \in O} \Pi^\alpha(m, B)
\]

which is the set of “free” plays starting in any \( m \in P \) by player \( A \) or in \( m \in O \) by player \( B \) such that \( A \) moves according to \( \alpha \). By construction, \( \Pi^\alpha(P, O) \) is continuous, \( A \)-deterministic, \( B \)-closed, and positional. In general, however, it may not be consistent.

**Lemma 4.2** For every strategy \( \Sigma \) we have \( \Sigma = \Pi^\alpha(P_\Sigma, O_\Sigma) \).

We will be interested in maximal front-lines that are defensible using particular types of strategies and characterise them in terms of post-fixed points. To understand the connection it is useful to view the maze \( M \) as a Kripke transition structure in labels \( \gamma \in \{ \iota, \tau \} \) in which regions may be specified using formulas of propositional modal \( \mu \)-calculus. We assume the reader is familiar with this language and its semantics (see e.g. [Sti01, GTW02]). In this language we have the standard modalities \( \langle \gamma \rangle, [\gamma] \) each of which is of type \( 2^S \rightarrow 2^S \) on sets of rooms, defined in the usual way:

\[
\langle \gamma \rangle(R) := \{ m \mid \exists m' \in R. m \xrightarrow{\gamma} m' \}
\]

\[
[\gamma](R) := \{ m \mid \forall m' \in R. m \xrightarrow{\gamma} m' \}
\]

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for arbitrary $R \subseteq S$. The logical connectives $\lor$, $\land$ correspond to union and intersection on $2^S$, respectively, and $\lnot$ is complementation, i.e., $R \lor S := R \cup S$, $R \land S := R \cap S$, $\lnot R := 2^S \setminus R$. Further, there are least fixed point $\mu X. \phi$ and greatest fixed point operators $\nu X. \phi$ with the usual interpretation assuming that the formula scheme $\phi$ is monotonic in the recursion variable $X$.

Let us observe that defending a front-line with an arbitrary strategy does not require much cleverness, since as long as from $O$ there is no secret corridor into $P$ and no visible corridor into $O$ the trivial empty transition strategy $\alpha = \emptyset$ will do, in which $A$ does not make any move at all. The following Proposition 4.3 characterises defensible front-lines, without reference to strategies:

**Proposition 4.3** Let $(P, O)$ be a front-line. Then $(P, O)$ is defensible iff $O \subseteq [\tau]O \land [\iota]P$. Further, $(P, O)$ is maximal defensible iff additionally $P = \lnot O$.

Using Proposition 4.3 it is possible to make a connection between maximal defensible front-lines and classical logic. Read each visible corridor $a \mapsto \iota b$ as the logical implication $\lnot b \supset a$ and each secret corridor $a \mapsto \tau b$ as $b \supset a$. In this manner a finite maze $M$ corresponds to a formula $\phi_M$ given by the conjunction of all its corridor implications. Taking the classical truth-value interpretation and identifying binary valuations $V : S \to \mathbb{B}$ with subsets $V \subseteq S$ we find that $(P, \lnot P)$ is a maximal defensible front line iff $P$ as a binary truth valuation $(P(x) = 1$ iff $x \in P)$ satisfies $\phi_M$. In other words,

**Proposition 4.4** The maximal defensible front lines (without further conditions on strategies) coincide with the classical binary truth-value models of $\phi_M$.

Thus, the notion of defensibility just coincides with classical logic, the weakest notion of synchronous response. In the next Sec. 5 we shall study specific winning conditions making the defence of front-lines constructive. Depending on which winning conditions we consider we will get different types of games on the same maze, implementing different degrees of causal justifications.

Before we study winning conditions a couple of general observations are in order on the algebraic nature of our problem. Let $\text{FL}_M$ be the set of front lines of $M$, i.e., the set of $(P, O) \subseteq (2^S)^2$ such that $P \cap O = \emptyset$. Then, first observe that $(P, O) \in \text{FL}_M$ iff $(P, O)$ is a post-fixed point (pfp) of the “De-Morgan” function $\text{dm} : (2^S)^2 \to (2^S)^2$ defined as $\text{dm}(X, Y) := (\lnot Y, \lnot X)$, which interchanges the role of the players and complements their set of positions. From Prop. 4.3 we gather that defensible front lines can be described as the pfps of $\text{dfl}-M : \text{FL}_M \to \text{FL}_M$, where

$$\text{dfl}-M(X, Y) := (X \lor (\tau) X \lor (\iota) Y) \land (\tau) Y \land (\iota) X),$$

taken as a function on $\text{FL}_M$. It is easy to verify that $\text{dfl}-M$ indeed preserves front lines, i.e. if $(P, O) \subseteq \text{dm}(P, O)$ then $\text{dfl}-M(P, O) \subseteq \text{dm}(\text{dfl}-M(P, O))$. 

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Having pfps the next question is what are the greatest fixed points? If $f$ is a monotone function on lattice $(2^S)^2$ the greatest fixed point of $f$ is the union of all pfps. However, for functions such as $dm$ and $dfl-M$ this does not work, in the former case since $dm$ is not monotone and in the latter since $FL_M$ is not a lattice. Obviously, two front lines $(P_1, O_1)$ and $(P_2, O_2)$ cannot be joined consistently if $P_i \cap O_j \neq \emptyset$. This also applies to defensible front lines (pfps of $dfl-M$). It may happen that player $A$ can defend a room $x \in P_1$ by leaving some other room $y \in O_1$ to the opponent, yet he could equally well set up another front line in which he wins room $y \in P_2$ but this time hands over the first room $x \in O_2$ to the opponent. In such a case the union $(P_1 \cup P_2, O_1 \cup O_2)$ is not only plainly inconsistent, it may in fact not be possible to defend both $x, y$ at the same time, in any front line $(P_1 \cup P_2, O)$ regardless $O$.

The sub-domains $FL_M$ of front lines (generated by $dm$) and $DFL_M$ of defensible front lines (generated by $dfl-M$) are not completely without structure, though. It is easy to see that they are directed complete semi-lattices.\footnote{A set is directed if any two elements have an upper bound. In a directed complete semi-lattice every non-empty directed set has a least upper bound.} By Zorn’s Lemma such domains have maximal elements. So, we may not have unique greatest pfps but we can try and identify the maximal pfps instead. The maximal pfps of $dm$ are the two-valued front lines of the form $(P, \neg P)$ and Prop. 4.3 implies that the maximal defensible front lines are the two-valued fixed points of $dfl-M$. Indeed it is not difficult to check that $(P, O)$ is a maximal pfp (and maximal fixed point) of $dfl-M$ iff $P = \neg O$ and $O \subseteq [\tau]O \land [\iota]P$.

In the next Sec. 5 we will identify a series of winning conditions of increasing strength that generate non-classical game semantics. In each case we give a presentation of the associated notion of front line in terms of the pfps of a characteristic system function $f : FL_M \rightarrow FL_M$. In general, $f$ may have many maximal pfps of which some are two-valued and others are not, reflecting non-deterministic and non-classical system responses. For reasons of conciseness we do not expand further on fixed points here. Relevant material can be found in Appendix A. Instead, we shall go straight into the matter and feel free to make forward reference to Appendix A where needed.

## 5 Winning Conditions

A winning condition, for our purposes, is a property of plays that is time invariant, i.e., invariant under shifting of the initial position. In other words, a play $\pi$ that is considered winning for some player is also won if the play had started in any of the configurations reached during $\pi$. The intuition is that winning should not depend on a finite initial part of a play, but only on the long-term stationary behaviour. After all, our games are played between a system and its environment to determine
a single synchronous response. Only the final (i.e., stationary) sets of present and absent signals matter, not the order in which these signals are generated (Synchrony Hypothesis).

A rich source for winning conditions is provided by the different acceptance conditions for $\omega$-regular languages over the alphabet $C$. All usual acceptance criteria for infinite paths such as the Muller, Rabin, Streett, Parity, Büchi conditions (see [Tho95, GTW02] for a comprehensive introduction) are time invariant, simply because they all are based on the set of configurations that occur infinitely often in a play. Obviously, this set does not change under shift of initial position. Though it would be an intriguing project to explore the full range of possibilities, we shall only consider some special cases, here, related to the two well-known synchronous languages, Esterel and Statecharts. A more comprehensive study is left to future work. The winning conditions to be discussed here are the following:

**Definition 5.1** Let $\pi$ be a play and $U$ a player. We say that $\pi$ is

- **$U$-live** if $U$ always enables another visible move, i.e., for all $i \in \text{dom}(\pi)$ such that $\text{turn}(\pi(i)) = U$ there exists $i < j < \text{dom}(\pi)$ such that $\text{pos}(\pi(j)) \in S$.

- **$U$-reactive** if $U$ always eventually hands over to $U$ in a visible room. Formally, for all $i \in \text{dom}(\pi)$ if $\text{turn}(\pi(i)) = U$ then there exists $i < j < \text{dom}(\pi)$ such that $\text{pos}(\pi(j)) \in S$ and $\text{turn}(\pi(j)) = \overline{U}$.

- **$U$-terminating** if all observable actions eventually stop and $U$ is the last player, i.e., there exists $i < \text{dom}(\pi)$ such that $\text{turn}(\pi(i)) = \overline{U}$ and for all $i < j < \text{dom}(\pi)$, we have $\text{pos}(\pi(j)) \in S$ and $\text{turn}(\pi(j)) = \overline{U}$.

The conditions of Def. 5.1 reflect the different intensional ways — discussed in Sec. 3 — of using seeds as additional (finite) resources. One can easily see that these conditions are general winning conditions according to the above definition, i.e., that they are time-invariant. On finite plays all three notions agree, i.e. if $\text{dom}(\pi) < \omega$ then $\pi$ is $U$-live iff it is $U$-reactive iff it is $U$-terminating. They simply state that player $U$ is the last to hold the execution token and thus responsible of “stopping” the computation. On infinite plays reactiveness is a proper strengthening of liveness, and termination a proper strengthening of reactiveness. A play may be $U$-live but not $U$-reactive, e.g., if player $U$ keeps going forward indefinitely along secret corridors between visible rooms without ever again passing the turn to his opponent. This corresponds to “divergence” on the side of $U$. A play may be $U$-reactive but not $U$-terminating, e.g., if $U$ infinitely passes the turn to his opponent.

---

2This intensional interpretation of the winning conditions depends on the restriction that dungeons are always visible and secret rooms cannot be connected by secret corridors, which may be dropped by adjusting the rules for paying seeds.
(in visible rooms) who keeps challenging him over and over again without the play ever stopping. An $U$-terminating play must eventually leave the onus on the side of $\overline{U}$ forever.

To keep matters simple we will discuss only a special case of the above winning conditions, namely when the maze does not have any secret rooms. So, unless specified otherwise, $S = S_\iota$. This will suffice to convey the basic ideas. The general case can be developed without difficulties by refinement from this simplified setting.

To assume that all rooms are visible essentially amounts to the special case of synchronous programs in which all transitions have single triggers and actions, i.e. are of the form $b/a$ or $\neg b/a$. As indicated before, these can be represented directly by secret or visible corridors, respectively, connecting the visible rooms $a$ and $b$. As far as terminating and live winning strategies are concerned this is without loss of generality as all transitions can be broken down into these special cases: E.g., one can show that $c, \neg b/a, d$ has the same semantics as the set of transitions $\neg c/x, b/x, \neg x/a, \neg x/d$ together. This does not hold for reactiveness, though, as we will indicate later. Under the assumption $S = S_\iota$, or $S_\tau = \emptyset$, the winning conditions specialise as follows: A play $\pi$ is

- **$U$-live** if $U$ always makes another move when he gets the turn, i.e., for all $i \in \text{dom}(\pi)$ if $\text{turn}(\pi(i)) = U$ then $i + 1 \in \text{dom}(\pi)$
- **$U$-reactive** if $U$ always eventually hands over to $\overline{U}$. Formally, for all $i < \text{dom}(\pi)$ there exists $i \leq j < \text{dom}(\pi)$ with $\text{turn}(\pi(j)) = \overline{U}$
- **$U$-terminating** if it is finite and $\overline{U}$ is the last player, i.e., $\text{dom}(\pi) < \omega$ and $\text{turn}(\pi(\text{dom}(\pi) - 1)) = \overline{U}$.

Given a winning condition $\text{Win}$ we say that a strategy $\Sigma$ is a $\text{Win}$-strategy if all prefix-maximal plays in $\Sigma$ satisfy $\text{Win}$, and a front-line $(P, O)$ is called $\text{Win}$-defensible if there exists a $\text{Win}$-strategy $\Sigma$ that defends it, i.e. for which $(P, O) = (P_\Sigma, O_\Sigma)$.

### 5.1 Coherent Responses

The first constructive strengthening of strategies is $A$-liveness. In an $A$-live strategy player $A$ must make sure he is never blocked, i.e. he always makes a move when he receives the turn. This means for every “present” signal in $P$ the player must be able to offer a justifying transition. Of course, like for ordinary defence strategies, he must play consistently, i.e., always move from a (“safe”) $P$-position in his territory while confining the opponent to region $O$ and preventing him from conquering any position in $P$. Let us say a front-line $(P, O)$ is coherent if it is defensible by an $A$-live strategy. In other words, a front line is defended coherently if player $A$ is consistent and can avoid ever getting trapped in a dungeon. Note that coherent front-lines in particular admit infinite plays. Let $\text{CFL}_M \subseteq 2^S \times 2^S$ be the collection of all coherent
front lines. Like for general defensibility a simple characterisation can be given that does not refer to strategies.

**Proposition 5.2** A front line \((P, O)\) is coherent iff both \(O \subseteq [\tau]O \land [\iota]P\) and \(P \subseteq \langle \tau \rangle P \lor \langle \iota \rangle O\).

Using Prop. 5.2 one can show that for arbitrary \(P\) there exists \(O\) such that \((P, O)\) is a coherent front line iff \(P \subseteq \text{is\_present}(P)\) where \(\text{is\_present}(X) := \langle \tau \rangle X \lor \langle \iota \rangle (\nu Y. (\neg X \land [\tau]Y \land [\iota]X))\). Dually, a set \(O\) can be extended to a coherent front line \((P, O)\) iff \(O \subseteq \text{is\_absent}(O)\) for \(\text{is\_absent}(Y) := [\tau]Y \land [\iota](\nu X. (\neg Y \land (\langle \tau \rangle X \lor \langle \iota \rangle Y)))\). These characterisations describe the verification procedures to check that particular signals are coherently present (absent) in the system response without specifying the full expected response up front. It is important to point out that the negations appearing in \(\text{is\_present}\) and \(\text{is\_absent}\) are crucial to make sure the front lines extracted are consistent. As a consequence of this the arguments \(P\) and \(O\), respectively, appear simultaneously co- and contravariantly in the right-hand sides of both inequations \(P \subseteq \text{is\_present}(P)\) and \(O \subseteq \text{is\_absent}(O)\). Therefore, we cannot hope to obtain unique maximal solutions. The co-contravariance problem also shows up in our expanded setting of the doubled-up lattice \(2^5 \times 2^5\). Proposition 5.2 says that coherent front-lines \((P, O)\) coincide with the consistent\(^3\) pfps of the set function \(\text{cfl}\_M : (2^5 \times 2^5) \rightarrow (2^5 \times 2^5)\) defined as

\[
\text{cfl}\_M(X, Y) = (\text{cfl}\_M_1(X, Y), \text{cfl}\_M_2(X, Y)) := ((\langle \tau \rangle X \lor \langle \iota \rangle Y, [\tau]Y \land [\iota]X).$
\]

The fact that \(\text{cfl}\_M\) is monotone does not help us since we must take \(\text{cfl}\_M\) in the sub-domain \(\text{FL}_M := \{ (P, O) \mid P \cap O = \emptyset \} \subseteq 2^5 \times 2^5\) of front lines which is not a complete lattice but only directed complete. In a directed complete lattice monotone functions need not have a greatest fixed point, only maximal ones. The pfps of the system function \(\text{cfl}\_M\) identify the coherent front lines and the maximal ones among them may count as the possible coherent responses of \(M\).

**Proposition 5.3** \((P, O)\) is a maximal coherent front line iff it is a maximal fixed point of \(\text{cfl}\_M : \text{FL}_M \rightarrow \text{FL}_M\). Moreover, \((P, \neg P)\) is coherent iff \(P = \langle \tau \rangle P \lor \langle \iota \rangle \neg P\).

Obviously, two-valued coherent front lines \((P, \neg P)\) are a special case of two-valued defensible front lines which must only satisfy (Prop. 4.3) the inequation \(\neg P \subseteq [\tau] \neg P \land [\iota]P\), which is the same as \(P \supseteq \langle \tau \rangle P \lor \langle \iota \rangle \neg P\).

What do coherent front lines mean for synchronous programming? Let us look at Esterel. Considering a corridor \(x \xrightarrow{\tau} y\) as a statement \texttt{present} \(y\) \texttt{then} \(x\) and \(x \xrightarrow{\iota} y\) as \texttt{present} \(y\) \texttt{else} \(x\) associates an Esterel program \texttt{esterel}(\(M\)) with our maze \(M\). A front line \((P, O)\) of \(M\) then corresponds to a potential response of \texttt{esterel}(\(M\)) where

\(^3\)Recall that a pair \((P, O)\) is consistent if \(P\) and \(O\) are disjoint.
all signals in $P$ are considered present and all in $O$ absent. Now, $(P, \neg P)$ is coherent if a signal is present $x \in P$ in the response iff there is a statement in esterel$(M)$ that emits it, i.e. there is present $y_1$ then $x$ and $y_1 \in P$ or present $y_2$ else $x$ and $y_2 \in \neg P$; and a signal is absent $x \in \neg P$ exactly if all statements that can emit $x$ are switched off. Two-valued coherent responses have been called logically coherent by Berry [Ber99], hence our terminology. An Esterel program is logically reactive (logically deterministic) if it has at least (at most) one logically coherent response.

What is constructive about coherence? Well, coherence is strongly related to the notion of inertiality. If a transition, say $c, \neg a/b$ represents a response function with inertial delay then it has the following extra property: If the input trigger $c \land \neg a$ becomes satisfied and then false again before output $b$ could be produced, then the transition being “inertial” completely forgets its previous excitation. Another way to put this is: if $c \land \neg a$ holds during some non-empty interval $[s, t) \subset \mathbb{R}$ of time and the output $b$ indeed reacts while it is on, say at $t_b \in [s, t)$, then $b$ must remain (at least) present strictly beyond $t$, i.e. during some interval $[t_b, t + \epsilon)$, for $\epsilon > 0$. Inertiality is an important assumption in hardware design [BS95] and the key to implementing memory.

It is not difficult to verify, then, that a component (represented in Fig. 3) such as $(c, \neg a/b) \ | \ (b, \neg a/c) \ | \ (\neg d, b, c/d)$ can exhibit a real-time waveform in which both $b, c$ are constant true mutually supporting each other by inertiality when $a$ is absent. This corresponds to the coherent front-line $(P_1, O_1) = (\{b, c\}, \{a, y, z\})$. Observe that $(P_1, O_1)$ can not be extended to a two-valued solution. For even with inertial delays signal $d$ cannot be guaranteed to stabilise when $b, c$ are present. The example also has the coherent and two-valued solution $(P_3, O_3) = (\{x, y, z\}, \{a, b, c, d\})$, which means our example is non-deterministic. In contrast, the defensible front-line $(P_2, O_2) = (\{a, b, c, y, z\}, \emptyset)$ is not coherent since $a, b, c$ would have to be maintained by the environment, which is not an autonomous and constructive response.

Figure 3: Maze for component $(c, \neg a/b) \ | \ (b, \neg a/c) \ | \ (\neg d, b, c/d)$ with front-line $(P_1, O_1)$. 

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5.2 Lazy Responses

Coherent strategies, though stronger than classical responses, still constitute a weak notion of winning. A player can defend his positions simply by avoiding ever to get stuck in a dungeon while maintaining consistency. This includes the possibility that \( A \) cycles along an infinite path of secret corridors where he keeps the turn forever. In this section we will strengthen the winning conditions so as to eliminate such behaviour. We still permit infinite plays but require the winning player to be reactive in the sense that he is never embarrassed about a move when challenged and always generates a proper response (hands over to opponent in a visible room) in finite time, though we do not insist he can stop the opponent from ever challenging him again. We shall call these defence strategies lazy and write \( \text{LFL}_M \) for the set of lazy defensible front lines. For the example of Fig. 3 we find that the coherent valuation \((P_1, O_1) = (\{b, c\}, \{a, y, z\})\) is not lazy and thus ruled out. To defend room \( b \in P_1 \), say, player \( A \) would indefinitely send his opponent around the intermediate secret rooms \( y, z \) (corresponding to the composite trigger conjunctions \( c \land \neg a \) and \( b \land \neg a \)) which violates reactiveness. Only \((P_3, O_3) = (\{x, y, z\}, \{a, b, c, d\})\) remains as a lazy front-line.

The terminology is inspired by the computational intuition that in the game the system player \( A \) produces a result to a request\(^4\) from the environment player \( B \). In a lazy computation it suffices that the system respond to every request from the environment in finite time, without necessarily being able to make the environment stop all further requests. Notice again, the central feature of these games is their use of infinite plays.

We will show how lazy front lines \( \text{LFL}_M \) are determined as the post-fixed points of a suitable monotone function \( \text{lf}-M : \text{FL}_M \to \text{FL}_M \) such that all pfps of \( \text{lf}-M \) are pfps of \( \text{cfl}-M \), which we abbreviate as \( \text{lf}-M \leq \text{cfl}-M \). This implies that \( \text{LFL}_M \subseteq \text{CFL}_M \) as expected, i.e., all lazy front lines are necessarily coherent. Moreover, we show that lazy defence strategies are intimately related with Pnueli & Shalev’s interpretation of Statecharts \([PS91]\) in the sense that the maximal elements in \( \text{LFL}_M \) correspond to the step responses of \( M \) viewed as a Statechart. For the following we assume that all rooms are visible. For arbitrary mazes both Props. 5.4 and 5.5 need to be generalised slightly, by another least fixed point that implements finite skipping over secret rooms.

**Proposition 5.4** A front-line \((P, O)\) is lazy iff \( P \subseteq \mu X.((\langle \tau \rangle(P \land X) \lor \langle \nu \rangle O) \land \langle \tau \rangle O \land [\nu]P) \).

Using the characterisation of Prop. 5.4 we can eliminate the negative part \( O \) from the definition of lazy front lines as before: A set \( P \) can be lazily defended by the

\(^4\)As in the standard type-theoretic setting we imagine environment \( B \) making the first move with a choice of a secret corridor to any room in \( O \) or a visible corridor to any of the \( P \) positions. Logically this is the conjunction of all \( P \) with the negation of all \( O \) positions.
starting player, i.e. is part of a lazy front line \((P, O)\) for some \(O\) iff \(P \subseteq \mu X.((\tau)(P \land X) \lor (\iota)O^*)\) for the fixed ("loosest upper approximation") \(O^* := \nu Y. \neg P \land [\tau]Y \land [\iota]P.\)

A similar statement can be made for the opponent part: A set \(O\) can be defended by the second player, i.e. is part of a lazy front line \((P, O)\) for some \(P\) iff \(O \subseteq [\tau]O \lor [\iota]P^*\) for the fixed set \(P^* := \nu Y. (\neg O \land \mu X.((\tau)(Y \land X) \lor (\iota)O)).\) Note again how these "one-sided" formulations bring up the combined co- and contravariance inherent in our setting.

Prop. 5.4 states that \((P, O)\) is a lazy front line if it is a consistent pfp of the set function \(lfl-M : (2^S \times 2^S) \to (2^S \times 2^S)\) defined as

\[
lfl-M(P, O) = (lfl-M_1(P, O), lfl-M_2(P, O)) \\
:= (\mu X.((\tau)(P \land X) \lor (\iota)O), [\tau]O \land [\iota]P) \\
= (\mu X.cfl-M_1(P \land X, O), cfl-M_2(P, O)).
\]

Clearly, \(lfl-M\) is monotone and \(lfl-M \leq cfl-M\). This implies that \(lfl-M\), too, preserves front lines, whence it is a monotone function on \(LFL_M\). We may now rephrase Prop. 5.4, thus: A front line is lazy if it is a pfp of \(lfl-M : FL_M \to FL_M\). The fact that \(lfl-M \leq cfl-M\) then explains why \(LFL_M \subseteq CFL_M\). Again, from general results (see Appendix A) it follows that \(LFL_M\) is closed under directed union and preserved by the game function \(cfl\).

As an aside we note that \(lfl-M\) is not just an arbitrary “sub-function” of \(cfl-M\), but one that preserves fixed points. Every fixed point of \(lfl-M\) is also a fixed point of \(cfl-M\). For suppose that \((P, O) = lfl-M(P, O) = (\mu X.cfl-M_1(P \land X, O), cfl-M_2(P, O))\) then we compute

\[
P = \mu X.cfl-M_1(P \land X, O) \\
= cfl-M_1(P \land X, O) \land [\tau]O \\
= cfl-M_1(P \land P, O) \\
= cfl-M_1(P, O)
\]

and so, \((P, O) = cfl-M(P, O)\) as claimed. The operational intuition behind this is that the complete lazy response of any sub-system of \(M\) (a fixed point of \(lfl-M\)) remains a complete response according to the coherent semantics (fixed point of \(cfl-M\)). By relaxing the game semantics more parts of \(M\) may produce a response, while stronger responses are preserved.

**Proposition 5.5** Let \((P, O)\) be a front line. Then, \((P, O)\) is maximal lazy if it is a maximal fixed point of \(lfl-M\). Specifically, \((P, \neg P)\) is lazy iff \(P = \mu X.((\tau)(P \land X) \lor (\iota)\neg P)\).

As before with coherence, maximal lazy front line \((P, O)\) need neither be uniquely defined nor two-valued. For the component \((-c/b) \mid (-b/c) \mid (c, \neg a, \neg b/a)\) we find
that there are two lazy responses (maximal lazy front lines) \((P_1, O_1) = (\{b\}, \{a, c\})\) and \((P_2, O_2) = (\{c\}, \{b\})\). Of those only the former is two-valued. In the latter signal \(a\) remains in untenable no-man’s-land. It cannot be defended consistently neither as part of \(P_2\) nor of \(O_2\).

\[
\begin{align*}
M_1, M'_1 & \text{ representing } a/a \\
M_2, M'_2 & \text{ representing } (\neg a/x) \mid (\neg x/a)
\end{align*}
\]  

Figure 4: The role of secret rooms.

Note that in this example it does not matter if the intermediate room associated with the trigger conjunction \(c \land \neg a \land \neg b\) of transition \(c, \neg a, \neg b/a\) is visible or secret. We obtain the same lazy front-lines. This is not in general so. Consider the mazes \(M_1\), \(M'_1\) and \(M_2\), \(M'_2\) in Fig. 4 corresponding to components \(a/a\) and \((\neg a/x) \mid (\neg x/a)\), respectively, where \(M'_1\) is a short-hand for \(M_1\) and \(M_2\) a short-hand for \(M'_2\). In \(M_1\) the room \(x\) is secret which means that \((\{x\}, \{a\})\) is the only non-trivial lazy front-line, while in \(M_2\) both \((\{x\}, \{a\})\) and \((\{a\}, \{x\})\) are lazily defensible.

We can now state the connection between lazy front lines and Pnueli & Shalev Statecharts responses. We read each secret corridor \(x \xrightarrow{\tau} y\) as a transition \(y/x\) and each visible corridor \(x \xrightarrow{\iota} y\) as the transition \(y/x\) of Statecharts and maze \(M\) as the parallel composition \text{statecharts}(M) of all these transitions, which may thought of as the flat and normalised encoding of a complex hierarchical Statecharts automaton \cite{LM01}.

**Theorem 5.6** Let \(M\) be a finite maze and \text{statecharts}(M) the Statecharts program associated with \(M\). Then, \(P\) is a Pnueli & Shalev response of \text{statecharts}(M) iff \((P, \neg P)\) is a lazy front line in \(M\).

Thm. 5.6 is a consequence of Prop. 5.7 below and the results presented in \cite{LM02} on the connection between Pnueli & Shalev step responses and intuitionistic Kripke models. It will be instructive to sketch this here. A linear two-world intuitionistic model over atoms \(S\) is a structure \(K = (W, \leq)\) of worlds \(W = \{0, 1\}\) with accessibility relation \(0 \leq 1\), together with a monotone valuation \(V\) of worlds \(w \in W\) by subsets \(V(w) \subseteq S\) of propositional atoms, viz those that are forced true at this world. We
can then validate arbitrary propositional formulas over atoms $a \in S$ on worlds in $K$ in the intuitionistic fashion by stipulating $K, w \models a$ iff $a \in V(w)$, $K, w \models \phi \land \psi$ iff $K, w \models \phi$ and $K, w \models \psi$, and $K, w \models \phi \supset \psi$ iff for all $w \subseteq u$ and $K, u \models \phi$ we have $K, u \models \psi$. A formula is valid in $K$, written $K \models \phi$, if it is valid globally, i.e., in all worlds of $K$. Classical logic is the special case where the valuation $V$ is constant over $W$, so $K$ essentially is a single point. In the following we identify models $K$ with their valuations $V$.

We introduce a partial ordering on models so that $V_1 \leq V_2$ iff $V_1(0) \subseteq V_2(0)$ and $V_1(1) = V_2(1)$. A model is called a response model of $\phi_M$ if it is a minimal intuitionistic model of $\phi_M$ in this sense.

Proposition 5.7 $(P, \neg P)$ is a lazy front line of $M$ iff $P$ is a response model of $\phi_M$.

In [LM02] it has been shown that response models of $\phi_M$ coincide with the Pnueli & Shalev step responses of statecharts ($M$). It is interesting to note that our semantics of maximal lazy front lines amounts to a refinement of that of Pnueli and Shalev, dealing also with situations, such as exhibited by the example above where the lazy response of a system is not two-valued and Pnueli & Shalev’s step construction procedure would result in failure.

5.3 Eager Responses

Finally, we banish infinite plays and decree that player $A$ must terminate his response in a finite number of steps by pushing the (control, execution) token back to the environment, which in turn must become satisfied and stop eventually. This is the winning condition of $A$-termination. We say a front line $(P, O)$ is eager if it is defensible by an $A$-terminating strategy. Let $EFL_M$ be the set of eager front lines. Eager defence is a finite process: A front line $(P, O)$ can only be defended if player $A$ manages to drive the opponent into a dungeon. In this case all rooms in $P$ are

5More compactly speaking, models $K$ may be identified with 3-valued interpretations $V : S \rightarrow \{0, \frac{1}{2}, 1\}$ where $V(a) = 1$ iff $K \models a$, $V(a) = 0$ iff $K \models \neg a$ and $V(a) = \frac{1}{2}$ otherwise. The truth table then is Gödel’s 3-valued logic [Göd32].
Lemma 5.8 Let \( \{P, O\} \) for \( i \in I \) be an arbitrary family of eagerly defensible front lines. Then, their union \( \bigcup_{i \in I} P_i \cup \bigcup_{i \in I} O_i \) is eagerly defensible, too.

By Lem. 5.8 the pair \( (P^*, O^*) := \bigcup \{(P, O) \mid (P, O) \text{ is eager} \} \) is eagerly defensible. Obviously, \( (P^*, O^*) \) is the greatest eager front line and as such uniquely defined.

Theorem 5.9 Let \( M \) be a finite maze. Then, the greatest eager front line \( (P^*, O^*) \) coincides with the constructive response of \( \text{esterel}(M) \), i.e., a signal \( a \in S \) is present in \( \text{esterel}(M) \) iff \( a \in P^* \) and \( a \) is absent in \( \text{esterel}(M) \) iff \( a \in O^* \). The program \( \text{esterel}(M) \) is constructive in the sense of Esterel iff \( (P^*, O^*) \) is two-valued, i.e., \( O^* = \neg P^* \).

To prove Thm. 5.9 it is useful to analyse the structure of eagerly defensible front lines. They have a particularly simple structure in that they are just collections of individual winning and losing positions. First, we introduce some notation. Let \( \text{Term}_U := \{ \pi \mid \text{dom}(\pi) < \omega \land \pi_2(\text{dom}(\pi) - 1) = \overline{U} \} \) be the set of \( U \)-terminating plays. Given two strategies \( \alpha \) and \( \beta \) for players \( A \) and \( B \), respectively, a starting position \( m \) and starting player \( U \) there is a unique play \( \text{play}_{\alpha, \beta}(m, U) \) in which player \( U \) starts off the game in position \( m \), player \( A \) plays strategy \( \alpha \) and player \( B \) strategy \( \beta \).
Then \( m \) is a winning position if there exists \( \alpha \) such that \( \text{play}_{(\alpha, \beta)}(m, A) \in \text{Term}_A \) for all opponent strategies \( \beta \), in which case \( \alpha \) is referred to as the winning strategy (for \( m \)). Dually, \( m \) is a losing position if there exists \( \beta \) such that \( \text{play}_{(\alpha, \beta)}(m, A) \in \text{Term}_B \) for all \( \alpha \). Let \( \text{Win} \) and \( \text{Lose} \) be the sets of all winning and losing positions, respectively. In [ALM03] it is shown that \( \text{Win} \) coincides with the set signals present in the constructive Esterel response of \( \text{esterel}(M) \) and that \( \text{Lose} \) are precisely the absent signals. Theorem 5.9 can thus be established by verifying that \( (\text{Win}, \text{Lose}) \) is the greatest eager front line of \( M \). This is the content of Proposition 5.10 below, which are going to tackle next. To begin with we define the sets

\[
\text{Win}(\alpha) := \{ m \in S | \forall \beta. \text{play}_{(\alpha, \beta)}(m, A) \in \text{Term}_A \}
\]

\[
\text{Lose}(\beta) := \{ m \in S | \forall \alpha. \text{play}_{(\alpha, \beta)}(m, A) \in \text{Term}_B \}
\]

of winning and losing positions, relative to given strategies, respectively. Hence, \( \text{Win} = \bigcup_\alpha \text{Win}(\alpha) \) and \( \text{Lose} = \bigcup_\beta \text{Lose}(\beta) \). The next proposition will become obvious from the following discussions:

**Proposition 5.10** \((\text{Win, Lose})\) is the greatest eager front-line.

There is also a fixed point characterisation of eager front lines. It is known from the theory of Esterel (see [Ber99, ALM03]) that \((\text{Win}, \text{Lose})\) can be obtained as the (2-dimensional) least fixed point of \( \text{cfl-M} : \text{FL}_M \rightarrow \text{FL}_M \), i.e., \((\text{Win}, \text{Lose}) = \mu(X, Y) \cdot \text{cfl-M}(X, Y)\), where \( \text{cfl-M}(X, Y) = (\langle \tau \rangle X \lor \langle i \rangle Y, [\tau] Y \land [i] X) \) as in Sec. 5.1. Assuming \( \text{cfl-M} \) is continuous (always for finite or finite branching mazes, in particular those generated from pure Esterel programs) this fixed point can be computed in the standard fashion by iteration, viz. as \( \bigcup_{i \in \omega} \text{cfl-M}^i(\emptyset, \emptyset) \). The successive approximations \( \text{cfl-M}^i(\emptyset, \emptyset) \) not only accumulate the front line \((\text{Win, Lose})\) but also construct an eager defense strategy for it, which essentially corresponds to the Must/Can analysis of Esterel [Ber99].

It is instructive to take a closer look at this fixed point iteration. The two components \((\text{Win, Lose})\) of the least fixed point can be computed in one big two-dimensional iteration as suggested above, or component-wise as follows:

\[
\text{Win} := \mu X. ((\langle \tau \rangle X \lor \langle i \rangle \text{must}(X)) \quad \text{must}(X) := \mu Y. ([\tau] Y \land [i] X)
\]

\[
\text{Lose} := \mu Y. ([\tau] Y \land [i] \text{can}(Y)) \quad \text{can}(Y) := \mu X. ((\langle \tau \rangle X \lor \langle i \rangle Y).
\]

Let \( \mu Z^n \) denote the \( n \)-th approximation of the fixed point, i.e., \( \mu Z^n. F(Z) = F^n(\emptyset) \). Then, we can approximate \( \text{Win} \) and \( \text{Lose} \) in the two dimensions independently, viz. \( \text{Win} = \bigcup_{n,m} P(n, m) \) and \( \text{Lose} = \bigcup_{n,m} O(n, m) \), where

\[
P(n, m) := \mu X^n. ((\langle \tau \rangle X \lor \langle i \rangle \text{must}^n(X))
\]

\[
\text{must}^n(X) := \mu Y^n. ([\tau] Y \land [i] X)
\]

\[
O(n, m) := \mu Y^n. ([\tau] Y \land [i] \text{can}^n(Y))
\]

\[
\text{can}^n(Y) := \mu X^n. ((\langle \tau \rangle X \lor \langle i \rangle Y).
\]

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There are simple intuitive game interpretations of these approximants. First, we observe that for any set \( X \), the approximant \( \text{must}^m(X) \) corresponds to those positions from which the starting player must necessarily hand over the turn to the opponent in region \( X \), after at most \( m + 1 \) own moves, or gets stuck in a dungeon before, no matter what strategy he plays. From this we can see that the approximant \( P(n, m) \) describes those positions from which the starting player has a strategy to make the opponent get stuck in a dungeon within \( n + 1 \) own moves, and until this happens the opponent must pass back after at most \( m + 1 \) of his moves each time he has the turn. So, the opponent can survive at most \( n(m + 1) + m \) moves overall. In a finite or finite branching maze, a position is a winning position (in the eager sense) \( x \in \text{Win} \) iff there exist \( n, m \) such that \( x \in P(n, m) \). On the other side of the approximation, the expression \( \text{can}^n(Y) \) in \( O(n, m) \) are the rooms from which the starting player has a (survival) strategy eventually to give the turn to his opponent in \( Y \) within \( n + 1 \) moves (of his). Hence, the dual approximant \( O(n, m) \) represents the rooms from which the starting player must loose the play in finite time, where he can make at most \( m \) moves while the winner does not need to expend more than \( n + 1 \) moves each time. Again, in a finite or finite branching maze, we have \( x \in \text{Lose} \) iff there exist \( n, m \) such that \( x \in O(n, m) \).

The following Proposition 5.11 identifies a monotone function \( \text{efl}-M : \text{FL}_M \rightarrow \text{FL}_M \) satisfying \( \text{efl}-M \leq \text{lfl}-M \) (i.e., a strengthening of the lazy semantics) such that eager front lines coincide with the pfps of \( \text{efl}-M \).

**Proposition 5.11** A front line \((P, O)\) is eager iff it is a pfp of the function \( \text{efl}-M \) defined as \( \text{efl}-M(P, O) := \mu(X, Y). \text{cfl}-M(P \land X, O \land Y) \).

Thus, eager front lines, too, are nothing but pfps of particular monotone functions on front lines, this time of \( \text{efl}-M(P, O) \). Again, the maximal pfps coincide with the maximal fixed-points. From general fixed point theory it follows that these are uniquely defined and indeed the least fixed point of \( \text{cfl}-M \). Although eager responses may not be two-valued they are always deterministic.

### 6 Conclusions

In this report we have identified four natural levels of semantics for synchronous (instantaneous) response in a game-theoretic setting as defensible front lines according to increasing restrictions on winning conditions. The levels \( \text{DFL}_M \supset \text{CFL}_M \supset \text{LFL}_M \supset \text{EFL}_M \) correspond to classical, coherent, lazy, and eager valuations, respectively. Each level is associated with a particular degree of computational constructiveness, \( \text{DFL}_M \) being the weakest and \( \text{EFL}_M \) the strongest, reflecting a characteristic operational interpretation of system execution. At \( \text{DFL}_M \) there is no constructiveness requirement, \( \text{CFL}_M \) is intimately linked with inertiala, \( \text{LFL}_M \) is Statecharts,
and \( EFL_M \) corresponds to Esterel. The game theory gives a coherent interpretation for non-determinism and partiality of the non-classical semantics. We have shown that these semantics can be obtained algebraically as (maximal) post-fixed points of a decreasing sequence of monotone functions \( dfl-M > cfl-M > lfl-M > efl-M \) on the directed complete partial ordering \( FL_M \subset (2^S)^2 \) of front lines. In this way the game semantics turns algebraic and induces suitable truth-values for presence and absence of signals at a given level. The explicit presentation of these truth-values, however, is only known for \( dfl-M \), viz. classical Boolean logic, and for \( lfl-M \) where we get Gödel's 3-valued intuitionism. The truth-value interpretation of \( cfl-M \) and \( efl-M \) is left open, in particular it is not clear if these are finite-valued.

The levels of constructiveness characterised here using games and winning conditions also play an important role in Normal Logic Programming (NLP), which extends standard definite Horn clause programming by permitting negative literals in clause bodies and queries. Various types of models based on three- and many-valued interpretations have been developed in the literature for normal logic programs. We refer the reader to [She91] for a survey of the classic results.

There are two important methodological differences between logic programming and synchronous languages: First, algebraic and logic models of NLP are judged according to their ability to reconstruct a fixed (standard) operational execution model, viz. negation as finite failure also known as SLDNF resolution. Where a many-valued semantics does not fit SLDNF completely, one seeks to identify restricted classes of programs for which they coincide. In synchronous programming, in contrast, one is not dealing with just a single operational model but with many of them, each accommodating different scheduling principles and implementation platforms. Second, since synchronous programs often model embedded and reactive systems with some degree of (low-level) asynchrony it is essential that non-determinism and concurrency is represented adequately. NLP, on the other hand is based on a strong sequential execution model, which even constrains the order in which clauses and literals are executed. In this sense the work presented here aims at a rather more general setting than what is considered in NLP. In another sense, though, our scope is more restricted here, viz., in considering only propositional programs. We believe that generating constructive models of Horn clauses from different types of winning conditions may provide further insights into the relationship between operational and denotational semantics of Logic Programming. At the propositional level eager front-lines are related to the 3-valued models of Fitting, lazy front-lines to the stable models and the supported models of NLP are the binary coherent front-lines (see [She91] for definitions of these types of models and references). To appreciate the versality of the games model suppose, e.g., we wanted to capture the standard sequential execution in literals \( a, b \) in a Prolog clause \( c :- a, b \) such that the whole clause loops if \( a \) loops even if the second clause \( b \) has a finite failure. Under a symmetric interpretation of conjunction (or parallel models such as those considered in
the clause would not hang up in executing $a$ but fail instead on the grounds that a conjunction is false if any of the conjuncts is.

![Figure 5: Maze for the Prolog clause $c:=-a, b$ where $a$ is evaluated strictly before $b$.](image)

We can model the asymmetric form of conjunction in terms of mazes as seen in Fig. 5, say under the eager\(^6\) winning condition: Suppose that $a$ is undefined, i.e. it cannot be won by the starting nor by the second player. Then, as we check easily, room $c$ is undefined, too. The first player $U$ cannot win: He can only go into the intermediate room $x$, from where $\overline{U}$ puts him into $a$ which is undefined by assumption. Similarly, the first player will not lose: If $\overline{U}$ takes him to $a$ he will not lose by assumption, if $\overline{U}$ takes him to $b$ then he can avoid losing by going to intermediate room $y$. There $\overline{U}$ has two options, either to move down to $b$ again, from where $U$ can repeat, or to go up to $a$, either handing over the turn to $U$ or not, depending on which of the two parallel corridors he choses. Yet, in either case $U$ will not lose by assumption on $a$. Further, one can show that if $a$ is decided, i.e. the first or the second player has a winning strategy from $a$, then no player will ever move from $b$ into the intermediate room $y$ since he would then give his opponent the option to move up into $a$ under full control of who gets the turn in $a$. Since $a$ is determined the opponent will win as first or second player, accordingly. But if room $y$ is never used we may as well remove it together with all corridors connecting it with $b$ and $a$. This yields the same maze as the one we get from the symmetric translation of $a, b/c$ (see Fig. 1). Thus, if $a$ is determined our coding in Fig. 5 coincides with the standard symmetric conjunction. It would be interesting to explore this further and to undertake a systematic study of three-valued semantics of NLP in terms of games and winning conditions.

Regarding future work it should also be noted that we have not discussed the algorithmic computation of front lines. For classical semantics this is well-known and for Statecharts and Esterel these can be derived directly from the respective original work [PS91, Ber99]. The algorithmic construction of (binary) lazy front lines has been described in [PS91] in terms of a non-deterministic fixed point search.

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\(^6\)This winning condition corresponds directly to the standard least fixed-point interpretation employed in Prolog.
with backtracking and in [Fri02] deterministically using BDD-techniques. If the immediate benefits of the game-theoretic setting for improvements on the algorithmic side may be modest it can at least guide the search profitably as a new reference point. Also, not having considered any composition operations for mazes we must leave open questions of compositionality and full abstraction. This will be addressed in future work. From [LM02] is known already that the game semantics from $lf\!l$-$M$ yields a fully abstract model for Statecharts under parallel composition, a problem that had been open for a long time.

The main contribution of this report, so we believe, is to identify game theory as an expressive framework for studying seemingly disparate step semantics for synchronous languages (implementing the Synchrony Hypothesis) from a single vantage point. The field of Statecharts semantics (see e.g. [von94]) in particular has been notoriously incoherent and controversial. We believe this is partly due to the lack of an adequate semantic framework to manage the subtleties of causal cycles. Game theory surely has a lot to offer here.

References


A Some Post-Fixed Points

The various notions of strategies and defensible front lines are intimately linked with different kinds of post-fixed points. We state some general technical devices to do with post-fixed points. This section does not intend to be more than providing a minimum of auxiliary material in support of the results discussed in this report and to suggest a domain-theoretic backdrop for further developing this line of research. As the work is still tentative, we will not wear the tightest belt necessarily to phrase all results in their most general form. We recommend [DP02] for details regarding standard terminology. Proofs can be found in appendix B.

Definition A.1 Let \( f : W \rightarrow W \) be a monotone function on a meet semi-lattice \((W, \sqcap)\). An element \( r \in W \) is called stable for \( f \) if \( r \sqsubseteq f(r) \) and inseparable for \( f \) if \( r \sqcap f(r \sqcap y) \sqsubseteq y \) implies \( r \sqsubseteq y \) for all \( y \in W \).

Stable elements are simply the post-fixed points of \( f \). Inseparability\(^7\) expresses an induction principle. For instance, consider the power-set lattice \((2^D, \subseteq)\) for some set \( D \). Read \( x \subseteq y \) as the statement “all elements of \( x \) have property \( y \)”. Then inseparability says that if one can show that all elements of \( r \) that can be generated by \( f \) from elements of \( r \) which are already known to have property \( y \), also have property \( y \), then all elements of \( r \) must have property \( y \). If this holds for all properties \( y \) then this means that \( r \) is inductively generated from \( f \). This does not require that \( r \) has any elements at all and in fact \( r = \emptyset \) is trivially inseparable. In a sense inseparability is like induction without the base case.

It is easy to see that inseparability implies stability. For, by monotonicity of \( f \), we always have \( r \sqcap f(r \sqcap f(r)) \sqsubseteq r \sqcap f(r) \sqsubseteq f(r) \). So, if \( r \) is inseparable we can infer \( r \sqsubseteq f(r) \) from this. Hence \( r \) is stable. Below (Prop. A.5) we will see that in complete Boolean lattices inseparable elements of \( f \) can be seen as stable elements (post-fixed points) of specific functions \( f' \leq f \). First we explore some useful alternative definitions of inseparability.

Lemma A.2 Let \( f : W \rightarrow W \) be a monotone function on a Boolean lattice \((W, \sqsubseteq)\). For \( r \in W \) the following are equivalent:

(i) \( r \) is inseparable

(ii) For all \( x \subseteq r \), if \( r \sqcap \neg x \sqcap f(x) = \bot \) then \( x = r \)

(iii) For all \( x \subseteq r \), if \( r \sqcap f(x) \sqsubseteq x \), then \( x = r \)

(iv) For all \( x \subseteq r \), if \( r \sqcap f(r \sqcap \neg x) = \bot \), then \( x = \bot \)

\(^7\)This term has been taken from [PS91]

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The equivalence between (i) and (iii) in Lem. A.2 actually holds for arbitrary meet semi-lattices. Stability and inseparability enjoy nice closure properties.

**Lemma A.3** Let \( f : W \to W \) be a monotone function on the complete distributive lattice \( W \). Then, the set of stable (inseparable) elements is closed under \( f \) and arbitrary joins.

One can show that on complete lattices (such as the powerset lattice) every monotonic function has a unique inseparable fixed point which is the least fixed point.

**Lemma A.4** A fixed point of a monotone function \( f : W \to W \) on a complete lattice \( W \) is inseparable for \( f \) iff it is the least fixed point of \( f \).

The next Lemma states that in certain situations the inseparable elements of \( f \) are nothing but post-fixed points of the function \( f'(r) := \mu x. f(r \sqcap x) \).

**Lemma A.5** An element \( r \in W \) is inseparable for monotone function \( f : W \to W \) on a complete Boolean lattice \( W \) iff \( r \sqsubseteq \mu x. f(r \sqcap x) \).

In situations where there are no greatest post-fixed points the structure of maximal post-fixed points becomes interesting. Here is what we have used in this report:

**Lemma A.6** Let \( f : W \to W \) be a monotone function on a chain–complete partial ordering \((W, \sqsubseteq)\). Then an element \( x \in W \) is a maximal post-fixed point of \( f \) iff it is a maximal fixed point.

When it comes to front lines we are dealing with post-fixed points in two dimensions, viz. one degree of freedom for each one of the players. From now on we assume that our lattice \((W, \sqsubseteq)\) is the Cartesian product \( W = W_1 \times W_2 \) and \( \sqsubseteq = \sqsubseteq_1 \times \sqsubseteq_2 \) of two lattices \((W_i, \sqsubseteq_i)\). Of course, all operations in \( W \) are taken component-wise. Note that if both \((W_i, \sqsubseteq_i)\) are complete (distributive) lattices then \( W \) is complete (distributive), too. By \( \top \) we abbreviate the top element of the lattice.

**Definition A.7** Let \( f = (f_1, f_2) : W \to W \) be a monotone function on \( W = W_1 \times W_2 \). An element \( r \in W \) is called left inseparable (left stable) if \( r \) is inseparable (stable) for the function \((\text{Id} \times \top) \circ f \) and, symmetrically, right inseparable (right stable) if \( r \) is inseparable (stable) for the function \((\top \times \text{Id}) \circ f \). Here, \( \circ, \times \) are function composition and product, respectively, \( \top, \text{Id} \) the functions \( \top(x) = \top, \text{Id}(x) = x \) in the appropriate domains.
To explain the terminology we observe that left inseparability of \((r_1, r_2)\) is equivalent to the condition (see also the proof of Lem. A.8)

\[
\forall x. (x \sqsubseteq r_1 \& r_1 \cap f_1(x, r_2) \subseteq x) \Rightarrow x = r_1
\]

which has the same form as inseparability except that only the first (=left) dimension \(f_1\) of the function \(f\) is involved, while the second argument \(r_2\) remains fixed. The same applies to left stability which comes down to the condition \(r_1 \sqsubseteq f_1(r_1, r_2)\). Symmetrically, for right inseparability and right stability only the second dimension \(f_2\) is relevant.

Clearly, from the definition of left (right) inseparability for \(f\) as inseparability for the modified functions \((\text{Id} \times \top) \circ f \ ((\top \times \text{Id}) \circ f)\) and the fact that inseparability implies stability we find that left (right) inseparability are strengthenings of left (right) stability. Moreover, by Lem. A.3, one-sided inseparability and one-sided stability are closed under arbitrary unions and the function \(f\). The following Proposition is not so obvious:

**Lemma A.8** Let \(f = (f_1, f_2) : W \rightarrow W\) be a monotonous function on a complete lattice \(W = W_1 \times W_2\). Then, every inseparable element of \(f\) is both left and right inseparable.

The converse of Lem. A.8 does not hold in general. One can show that inseparability is not the conjunction of left and right inseparability. For instance, for mazes \(M\) without secret corridors all fixed points of \(\text{cfl-M}\) are trivially left and right inseparable. On the other hand, the inseparable fixed points of \(\text{cfl-M}\) are precisely its least fixed points.

Just like for inseparability proper its one-sided cousins, too, can be characterised in terms of post-fixed and least fixed points.

**Lemma A.9** Let \((f_1, f_2) : W \rightarrow W\) by a monotone function on a complete distributive lattice \(W = W_1 \times W_2\). An element \((r_1, r_2) \in W\) is left inseparable iff \(r_1 \sqsubseteq \mu x.f_1(r_1 \cap x, r_2)\) and right inseparable iff \(r_2 \sqsubseteq \mu y.f_2(r_1, r_2 \cap y)\). Further, \((r_1, r_2)\) is left stable iff \(r_1 \sqsubseteq f_1(r_1, r_2)\) and right stable iff \(r_2 \sqsubseteq f_2(r_1, r_2)\).

**B Proofs**

**B.1 Proof of Lemma 4.2**

Proof: It has been noted already that \(\Sigma \subseteq \Pi^\alpha(P_\Sigma, O_\Sigma)\). For the other direction, assume a finite play \(\pi \in \Pi^\alpha(m, A)\) for \(m \in P_\Sigma\) or \(\pi \in \Pi^\alpha(m, B)\) for \(m \in O_\Sigma\). We
prove $\pi \in \Sigma$ by induction on $\text{dom}(\pi)$. If $\pi = \epsilon$ then $\pi \in \Sigma$, trivially, since strategies are non-empty and $\Sigma$-closed.

If $\text{dom}(\pi) = 1$ and $\pi = (m, A)$ then $m \in P_\Sigma$ implies there exists a play $\pi' \cdot (m, A) \cdot \sigma \in \Sigma$, whence by closure under prefixes and possibility, $(m, A) \in \Sigma$ as desired. If $\pi = (m, B)$ we obtain the same conclusion from $m \in O_\Sigma$.

Now assume $\text{dom}(\pi) \geq 2$, say, $\pi = \pi' \cdot (m_1, U_1) \cdot (m_2, U_2)$. Since $\pi \in \Pi$ there must exist a corridor $m_1 \xrightarrow{\gamma} m_2$ with $U_2 = U_1^\gamma$. Also, by prefix closure of $\Pi^{\alpha}(m, U)$ and the induction hypothesis, $\pi' \cdot (m_1, U_1) \in \Sigma$. We distinguish the two possible cases of $U_1 \in P$.

If $U_1 = A$ then we must also have $\alpha_\Sigma(m_1) = (\gamma, m_2)$ and $U_2 = A^\gamma$ by definition of $\Pi^{\alpha}(m, U)$. The former implies (by construction of $\alpha_\sigma$) there exists a play $(m_1, U_1) \cdot (m_2, U_2) \cdot \sigma \in \Sigma$ so that because of the positional nature of $\Sigma$ we have $\pi' \cdot (m_1, U_1) \cdot (m_2, U_2) \in \Sigma$, too.

If $U_1 = B$ then $B$-closure gives us $\pi' \cdot (m_1, U_1) \cdot (m_2, U_2) \cdot \sigma \in \Sigma$ for some $\sigma$, whence $\pi' \cdot (m_1, U_1) \cdot (m_2, U_2) \in \Sigma$ by prefix-closure. In either case, thus, $\pi \in \Sigma$ as claimed.

So, we have shown that all finite plays of $\Pi^{\alpha}(P_\Sigma, O_\Sigma)$ are contained in $\Sigma$. It is easy to see that for infinite plays $\pi \in \Pi^{\alpha}(P_\Sigma, O_\Sigma)$ all its finite prefixes $\pi' \subseteq \pi$ are in this set, too, whence they are all in $\Sigma$. But then, by continuity of $\Sigma$, $\pi \in \Sigma$. This completes the proof that $\Pi^{\alpha}(P_\Sigma, O_\Sigma) = \Sigma$.

\section*{B.2 Proof of Proposition 4.3}

\textbf{Proof:} Suppose $(P, O)$ is defensible and $\Sigma$ is the defence strategy, with $P = P_\Sigma$ and $O = O_\Sigma$. Since $(P, O)$ is $\Sigma$-closed, all $\sigma$-corridors from $O$ must lead into $P$ and all $\tau$-corridors from $O$ back to $O$, i.e., $O \subseteq [i] P \land [\tau] O$.

Consider those $m \in \neg(O \lor P)$, which by definition do not appear in any play $\pi \in \Sigma$. In particular, since we have $m \not\in P$ the underlying transition strategy $\alpha_\Sigma$ is undefined on $m$. We can add them as new configurations to $\Sigma$, in which $A$ starts but does not not play, to form $\Sigma' := \Sigma \cup \{(m, A) \mid m \in \neg(O \lor P)\}$ which again is a strategy. But $\Sigma'$ obviously now defends $P_{\Sigma'} = P \lor \neg(O \lor P) \supseteq P$ and $O_{\Sigma'} = O$. Now, if $(P, O)$ was already maximal then we must have in fact $P \lor \neg(O \lor P) = P$, which implies $P \land \neg O = P$ and thus $\neg O \subseteq P$. Together with $P \subseteq \neg O$ (property of front-line) we obtain $P = \neg O$ as desired.

Vice versa, suppose front-line $(P, O)$ is such that $O \subseteq [i] P \land [\tau] O$. We claim that $\Sigma := \Pi^{\alpha}(P, O)$ — for the trivial transition function $\alpha = \emptyset$ — is a strategy. All we need to show is that $\Sigma$ is consistent. This is not difficult. For every $\pi \in \Sigma$ one shows by induction on $n \in \text{dom}(\pi)$ that whenever $\text{turn}(\pi(n)) = A$ then $\text{pos}(\pi(n)) \in P$. 

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and if $\text{turn}(\pi(n)) = B$ then $\text{pos}(\pi(n)) \in O$. This holds in the initial configuration $\pi(0)$ by definition of $\Pi^a(P, O)$ and is preserved trivially throughout $\pi$ as player A never makes any move ($\alpha = \emptyset$) and player B is restrained to preserve the invariant because of $O \subseteq [i]P \wedge [\tau]O$. So, $(P, O)$ is defensible. Further if $\neg O = P$ then $(P, O) = (\neg O, O) = (P, \neg P)$ is trivially maximal since there cannot be a larger front-line $(P', O')$ without violating disjointness $P' \cap O' = \emptyset$. Every room is already contained in either $P$ or $O$. \hfill \Box

B.3 Proof of Proposition 4.4

Proof: Let $(P, \neg P)$ be a front line such that $P(\phi_M) = 1$. We show that $(P, \neg P)$ is maximal defensible. By Prop. 4.3 it suffices to verify that $\neg P \subseteq [\tau]\neg P \wedge [i]P$. So, given any $x \in \neg P$, i.e., $P(x) = 0$, and corridor $x \xrightarrow{\gamma} y$ in $M$. Then, the implication $y \supset x$ is a conjunct of $\phi_M$, whence $P(y \supset x) = 1$, i.e., $P(y) = 0$ or $P(x) = 1$. The second case is impossible by assumption, so we must have $P(y) = 0$, i.e., $y \in \neg P$. This shows that $\neg P \subseteq [\tau]\neg P$. Next, consider a corridor $x \xrightarrow{\gamma} y$ in $M$, i.e., $\neg y \supset x$ as a conjunct of $\phi_M$. Since $P(x) = 0$ and $P(\neg y \supset x) = 1$ we conclude $P(y) = 1$ and, equivalently, $y \in P$. This proves $\neg P \subseteq [i]P$, so overall $\neg P \subseteq [\tau]\neg P \wedge [i]P$ as desired.

Vice versa, suppose $(P, \neg P)$ is a maximal defensible front line, i.e., $\neg P \subseteq [\tau]\neg P \wedge [i]P$ by Prop. 4.3. This implies $\langle\tau\rangle P \vee \langle i\rangle \neg P \subseteq P$. We must prove that $P(\phi_M) = 1$. Consider an implication $\neg y \supset x$ appearing among the conjuncts of $\phi_M$, corresponding to a corridor $x \xrightarrow{\gamma} y$. If $P(y) = 0$, then $y \in \neg P$. This means $x \in \langle i\rangle \neg P \subseteq P$ from which we conclude $P(x) = 1$. This proves that $P(\neg y \supset x) = 1$. Next, if $y \supset x$ represents a corridor $x \xrightarrow{\gamma} y$ in $M$ and $P(y) = 1$, i.e., $y \in P$, then $x \in \langle\tau\rangle P \subseteq P$ whence again $P(x) = 1$. So, $P(y \supset x) = 1$. This completes the proof that $P(\phi_M) = 1$. \hfill \Box

B.4 Proof of Proposition 5.2

Proof: Let the coherent front-line $(P, O)$ be given and $\Sigma$ an $A$-live defence strategy for it, i.e., $P = P_\Sigma$ and $O = O_\Sigma$. From Prop. 4.3 we know that $O \subseteq [\tau]O \wedge [i]P$. It remains to verify $P_\Sigma \subseteq \langle\tau\rangle P_\Sigma \vee \langle i\rangle O_\Sigma$. So, let $x \in P_\Sigma$ be arbitrary and $\pi \in \Sigma$ a play with $\pi(n) = (x, A)$ for some $n \in \text{dom}(\pi)$. Since $\Sigma$ is $A$-live we must have $n + 1 \in \text{dom}(\pi)$. Then, by definition of plays, $\pi(n + 1) = (y, A^\gamma)$ for some corridor $x \xrightarrow{\gamma} y$. If $\gamma = \tau$ then $A^\gamma = A$ which means $y \in P_\Sigma$, and thus $x \in \langle\tau\rangle P_\Sigma$. If $\gamma = i$ then $A^\gamma = B$, from which it follows that $y \in O_\Sigma$ and further $x \in \langle i\rangle O_\Sigma$. This shows $P_\Sigma \subseteq \langle\tau\rangle P_\Sigma \vee \langle i\rangle O_\Sigma$ as claimed.
For the other direction consider a front line \((P, O)\) such that the conditions \(O \subseteq [\tau]O \land [\iota]P\) and \(P \subseteq \langle \tau \rangle P \lor \langle \iota \rangle O\) hold. Consider the relation

\[
T := \{(x, \tau, y) \mid x \xrightarrow{\tau} y \land x, y \in P\} \cup \{(x, \iota, y) \mid x \xrightarrow{\iota} y \land x \in P, y \in O\}.
\]

Because of the assumption \(P \subseteq \langle \tau \rangle P \lor \langle \iota \rangle O\) the relation \(T \subseteq P \times \{\tau, \iota\} \times S\) is left total. To defend \((P, O)\) we can pick any transition strategy \(\alpha : P \to \{\tau, \iota\} \times S\) contained in \(T\). We now show that \(\Sigma := \Pi^\alpha(P, O)\) is an \(A\)-live defence strategy for \((P, O)\).

\[\Sigma \text{ is consistent, } A\text{-live and that it defends } (P, O). \]

\[\square\]

**B.5 Proof of Proposition 5.3**

*Proof:* In view of Prop. 5.2 it suffices to show for the first part of the Proposition that the maximal consistent post-fixed points of \(\text{cfl-}M\) coincide with its maximal consistent fixed points. But this follows from Lem. A.6 in appendix A considering that \(\text{cfl-}M\) preserves front lines and that \(FL_M\) is chain complete.

For the second part of the Proposition consider that each coherent \((P, \neg P)\) is trivially maximal and thus \((P, \neg P)\) a fixed point of \(\text{cfl-}M\). From this \(P = \langle \tau \rangle P \lor \langle \iota \rangle \neg P\) follows directly. Vice versa, assume the equation \(P = \langle \tau \rangle P \lor \langle \iota \rangle \neg P\) holds. Then, by contraposition, \(\neg P = [\tau] \neg P \land [\iota] P\), which means \((P, \neg P)\) is a fixed point of \(\text{cfl-}M\) and thus coherent.

\[\square\]

**B.6 Proof of Proposition 5.4**

*Proof:* We assume throughout that \((P, O)\) is a front line, i.e., \(P \cap O = \emptyset\). We show that the following are equivalent:

1. \((P, O)\) is lazy
2. \((P, O)\) is a left inseparable and right stable element of \(\text{cfl-}M : FL_M \to FL_M\)
3. \(P \subseteq \mu X. (\langle \tau \rangle (P \land X) \lor \langle \iota \rangle O)\) and \(O \subseteq [\tau]O \land [\iota]P\).

Equivalence between (2) and (3) follows from the general result stated in Lem. A.9 based on Def. A.7.

(1) \(\Rightarrow\) (2) Assume that \((P, O)\) is lazy and defended by strategy \(\Sigma\). Then, \(P = P_\Sigma, O = O_\Sigma\) and \(\Sigma = \Pi^\alpha(P, O)\) for some transition strategy \(\alpha\). By Prop. 5.2 and Lem. A.9 \((P, O)\) is right stable for \(\text{cfl-}M\). We show that for all \(X \subseteq P\), if
Suppose we have already defined the sequence up to \( \alpha \), \( \beta \). \( P \) \& \( (\tau) (P \setminus X) \lor (\nu) O \) \( \beta \), then \( X \) \( \beta \). \( P \), too. This will prove that \( (P, O) \) is left inseparable for cfl-\( M \) by Lem. A.9 and condition (iv) of Lem. A.2.

Let \( X \) be some subregion of \( P \). Assume that for all \( m \in X \) if \( m \xrightarrow{\tau} y \) then \( y \notin P \setminus X \) and if \( m \xrightarrow{\nu} y \), then \( y \notin O \). By definition all rooms \( m \in P \) occur in some play \( \pi \in \Pi^a(P, O) \) with player \( A \) having the turn in \( m \). Since all plays are \( A \)-live, \( \alpha(m) = (\alpha_1(m), \alpha_2(m)) \in \{\nu, \tau\} \times S \) must be defined for all \( m \in P \) and \( m \xrightarrow{\alpha(m)} \alpha_2(m) \) must be a corridor in \( M \). Now, consider any \( m \in X \subseteq P \). We claim that \( \alpha_1(m) = \tau \) and \( \alpha_2(m) \in X \). First, suppose \( \alpha_1(m) = \nu \) instead. Then, from the assumption we get \( \alpha_2(m) \notin O \). But this is impossible since then \( A \) would give the turn to \( B \) in \( \alpha_2(m) \) and thus \( \alpha_2(m) \in O \) since \( O \) by definition are all positions occupied by \( B \) in any play of \( \Pi^a(P, O) \). Thus, we must have \( \alpha_1(m) = \tau \). The above assumption then implies \( \alpha_2(m) \notin P \setminus X \). But since necessarily \( \alpha_2(m) \in P \) by the same argument as before this implies \( \alpha_2(m) \in X \). Hence, we have shown

\[
\forall m \in X. \; \alpha_1(m) = \tau \; \text{and} \; \alpha_2(m) \in X.
\]

This means, however, that strategy \( \alpha \) would keep \( A \) in sub-region \( X \), and hence in \( P \), indefinitely, which contradicts \( A \)-reactiveness as \( A \) never hands over to \( B \). So, \( X \) must be empty as claimed.

Now we tackle direction (2) \( \Rightarrow \) (1) and assume \( (P, O) \) is left inseparable, i.e., by Lem. A.9 and Lem. A.2 (iv) we have

\[
\forall \emptyset \neq X \subseteq P. \; X \wedge ((\tau)(P \setminus X) \lor (\nu) O) \neq \emptyset. \tag{2}
\]

Note that \( (P, O) \) is defensible according to Prop. 4.3 since by right stability \( O \subseteq [\tau]O \wedge [\nu]P \).

Now observe that if \( P = \emptyset \) any defence strategy \( \alpha \) will do. We know that \( (P, Q) = (\emptyset, O) \) is defensible. Since \( A \) never needs to make a move every play is trivially \( A \)-reactive. In fact, for front lines \( (\emptyset, O) \) all three notions of strategies \( A \)-live, \( A \)-reactive, \( A \)-terminating coincide. So, we may assume \( P \neq \emptyset \) in the following.

We are going to construct an \( A \)-reactive defence strategy \( \alpha \) for \( P \) incrementally as \( \alpha^0 \subseteq \alpha^1 \subseteq \cdots \subseteq \alpha \) from a sequence \( P_0 \subseteq P_1 \subseteq \cdots \subseteq P \) of positions such that \( \alpha^i \) at each stage \( i \) is defined for \( P_i \) as its domain of definition. The start of the approximation sequence is given as follows:

\[
P_0 := P \wedge (\nu) O = \{ m \in P \mid \exists m' \in O. \; m \xrightarrow{\nu} m' \}.
\]

Choosing \( X \) as \( P \) in our assumption (2) shows that \( P_0 \neq \emptyset \). Define \( \alpha^0 \) on all \( m \in P_0 \) by \( \alpha^0(m) = (\nu, y) \) where \( y \in O \) is some choice of position such that \( m \xrightarrow{\nu} y \). Now suppose we have already defined the sequence up to \( P_n \) and \( \alpha_n \). If \( P^n = P \) we are done. If \( P_n \subset P \) we continue with

\[
P_{n+1} := P_n \lor (P \wedge (\tau) P_n) = P_n \cup \{ m \in P \mid \exists m' \in P_n. \; m \xrightarrow{\tau} m' \}.
\]

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We claim that $P_n \subsetneq P_{n+1}$. First, since $P_n \subsetneq P$ we may take $P \setminus P_n \neq \emptyset$ for $X$ in our assumption (2), which gives $P \setminus P_n \land (\langle \tau \rangle P_n \lor \langle \iota \rangle O) \neq \emptyset$. Since $P \land \langle \iota \rangle O = P_0 \subseteq P_n$ by construction, we must have $P \setminus P_n \land (\tau) P_n \neq \emptyset$. But this is precisely the statement $P_{n+1} \setminus P_n \neq \emptyset$, hence $P_n \subsetneq P_{n+1}$. Now extend $\alpha_n$ to $\alpha_{n+1}$ by choosing, for every $m \in P_{n+1} \setminus P_n$ some $y$ such that $m \xrightarrow{\tau} y$ and $y \in P_n$. We continue this process until we have fully exhausted $P$, possible by iteration along arbitrary ordinals exploiting the Well-ordering Principle. For limit ordinals $\gamma$ we put $\alpha^\gamma := \bigcup_{\alpha < \gamma} \alpha^\iota$. In finite mazes we must eventually reach a stage where $k < \omega$ for which $P_k = P$ and $\alpha = \alpha_k$.

We claim that $\Pi^\alpha(P, O)$ is a lazy defence strategy for $P$. Because of the construction of $\alpha$ on $P$ it is clear that whenever $A$ finds himself in $P$ having the turn he will pass through a sequence of secret corridors through regions $P_i$ in descending order until he eventually (in finite time) reaches $P_0$, from which he exits region $P$ into $O$ through an observable corridor. On the other hand, if the game is in $O$ and it is $B$'s turn then assumption $O \subseteq [\tau] O \land [\iota] P$ tells us that all secret moves of $B$ will stay in $O$ and each visible move of $B$ will enter $P$. This means, the opponent outside of $P$ has no chance to conquer any of $A$'s territory $P$. So, $\Pi^\alpha(P, O)$ is consistent. Finally, all positions in which $A$ receives his turn, either in games starting with $A$ in $P$ or those starting with $B$ in $O$, must lie inside $P$. Since $A$ always eventually hands over the turn to $B$, the strategy is $A$-reactive. This completes the proof of Proposition 5.4. \hfill \Box

### B.7 Proof of Proposition 5.5

**Proof:** Note that $\text{lfl-M}$ is monotone on front lines. Since front lines are chain complete we can apply Lem. A.6 to the function $\text{lfl-M}$ to prove the first part of the Proposition. As to the second part, assume $(P, \neg P)$ is lazy. Then it is trivially maximal and thus by the first part a fixed point of $\text{lfl-M}$. So, we know that $P = \mu X. (\langle \tau \rangle (P \land X) \lor \langle \iota \rangle \neg P)$. Reasoning the other direction, the assumption is $P = \mu X. \text{cfl-M}_1(P \land X, \neg P)$, where $\text{cfl-M}_1(X, Y) := \langle \tau \rangle X \lor \langle \iota \rangle Y$ is the first component of the maze function $\text{cfl-M}$. We obtain

\[
P = \mu X. \text{cfl-M}_1(P \land X, \neg P) = \text{cfl-M}_1(P \land \mu X. \text{cfl-M}_1(P \land X, \neg P), \neg P) = \text{cfl-M}_1(P \land P, \neg P) = \text{cfl-M}_1(P, \neg P) = \langle \tau \rangle P \lor \langle \iota \rangle \neg P,
\]

from which it follows by negation that $\neg P = [\tau] \neg P \land [\iota] P$, whence $(P, \neg P)$ is in fact a fixed point of $\text{lfl-M}$ and thus $(P, \neg P)$ is lazy (Prop. 5.4) as desired. \hfill \Box

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B.8 Proof of Proposition 5.7

Proof: Suppose \( (P, \neg P) \) is lazy. Take the constant two-world Kripke model \( K_P := (\{0,1\}, \leq, V) \) such that \( V(0) = V(1) = P \). Now \( (P, \neg P) \) in particular is a maximal defensible (cf. Prop. 4.3) front line, so valuation \( V(i) \) is a classical two-valued model of \( \phi_M \) as stated in Prop. 4.4. Trivially, then \( K_P \models \phi_M \). The crucial part is to show that \( K_P \) is a response model, i.e., that it cannot be properly weakened by any two-world model \( K' \leq K \), say \( K' = (\{0,1\}, \leq, V') \) and \( K' \models \phi_M \), where \( V'(1) = V(1) \) and \( V'(0) \subseteq V(0) \). We claim that \( V'(0) = P = V(1) \). The argument exploits the fixed point characterisation \( P = \mu X.f_P(X) \) where \( f_P(X) := \langle \tau \rangle(P \land X) \lor \langle i \rangle \neg P \) given in Prop. 5.5. If we can show that \( f_P(V'(0)) \subseteq V'(0) \) then \( V'(0) \) is a pre-fixed point of \( f_P \) and since \( P = \mu X.f_P(X) \) is the least pre-fixed point we can infer \( P \subseteq V'(0) \). This implies \( V'(0) = P \) as desired. Let us show that \( V'(0) \) is a pre-fixed point of \( f_P \), then. To this end pick any \( x \in f_P(V'(0)) = \langle \tau \rangle(P \land V'(0)) \lor \langle i \rangle \neg P = \langle \tau \rangle V'(0) \lor \langle i \rangle \neg P \). If \( x \in \langle i \rangle \neg P \) there is an implication (corridor) \( \neg y \supset x \) in \( \phi_M \) such that \( y \in \neg P = S \setminus P = S \setminus V'(1) \). Thus, \( K',0 \models \neg y \). Since \( K',0 \models \neg y \supset x \) we must have \( K',0 \models x \), whence \( x \in V'(0) \). If \( x \in \langle \tau \rangle V'(0) \) then \( \phi_M \) includes an implication \( y \supset x \) with \( y \in V'(0) \), i.e., \( K',0 \models y \). Again, the fact that \( K' \models \phi_M \) implies \( K',0 \models x \) which is precisely the condition \( x \in V'(0) \) that we are after. This completes the proof that \( f_P(V'(0)) \subseteq V'(0) \) and thus the proof that \( K \) is a response model.

For the other direction let us assume that \( P \subseteq S \) is a response model of \( \phi_M \). Then it is also a classical two-valued model of \( \phi_M \) and thus by Prop. 4.4 there exists a pre-fixed point \( \langle P,\neg P \rangle \) for every \( P \subseteq S \). This is equivalent to \( P \supseteq \langle \tau \rangle P \lor \langle i \rangle \neg P \), i.e., \( P \) is a pre-fixed point of \( f_P(X) = \langle \tau \rangle(P \land X) \lor \langle i \rangle \neg P \). We want to show it is the least one. So, let \( X \) be any subset of rooms such that \( X \supseteq \langle \tau \rangle(P \land X) \lor \langle i \rangle \neg P \). Consider the two-world linear Kripke model \( K = (\{0,1\}, \leq, V) \) where \( V(0) = P \land X \) and \( V(1) = P \). Since \( V(1) \) is a classical model of \( \phi_M \), \( K,1 \models \phi_M \) trivially. Further, it is not difficult to verify that \( K,0 \models \phi_M \) using the inequalities \( X \supseteq \langle \tau \rangle(P \land X) \lor \langle i \rangle \neg P \lor (\tau)P \lor (\iota)\neg P \supseteq \neg P \). Since \( P \) was assumed to be response model, \( V(0) \) cannot be a proper subset, we must have \( P \land X = V(0) = V(1) = P \). This means \( P \subseteq X \) and thus \( P \) is indeed the least pre-fixed point and in particular a fixed point of \( f_P(X) \). So, \( (P,\neg P) \) is a lazy front line by Prop. 5.5.

\[ \square \]

B.9 Proof of Lemma 5.8

Proof: We begin by introducing some useful auxiliary notation. For player \( U \in \mathbb{P} \) let

\[ U\text{-pos}(\pi) := \{ \text{pos}(\pi(n)) | n < \text{dom}(\pi) \land \text{turn}(\pi(n)) = U \} \]
be the set of positions occupied by \( U \) in the course of the play \( \pi \in \Pi \). So, if \( \Sigma \) is a strategy then \( P_\Sigma = \{ A\text{-pos}(\pi) \mid \pi \in \Sigma \} \) and \( O_\Sigma = \{ B\text{-pos}(\pi) \mid \pi \in \Sigma \} \). Further, given two transition strategies \( \alpha \) and \( \beta \) for players \( A \) and \( B \), respectively, a starting position \( m \) and starting player \( U \), there is a unique play \( play_{(\alpha,\beta)}(m, U) \) in which \( U \) starts off the game in position \( m \), player \( A \) plays strategy \( \alpha \) and player \( B \) strategy \( \beta \) as long as the strategies define a successor room for the corresponding player. Such a play is finite exactly if a player gets stuck in a dungeon.

We prove the statement of Lem. 5.8 for the special case of two front lines. The argument generalises to an arbitrary family without difficulties. Let \( \Sigma = \Pi^{\alpha}(P_1, O_1) \), for \( i = 1, 2 \), the two defence strategies in question and \( P = P_1 \cup P_2, O = O_1 \cup O_2 \). Our first step will be to prove that \( (P_1 \cup P_2) \cap (O_1 \cup O_2) = \emptyset \). By way of contradiction, assume that \( P \cap O \neq \emptyset \). Since \( (P_1, O_1) \) and \( (P_2, O_2) \) are front lines, \( P_1 \cap O_1 = \emptyset = P_2 \cap O_2 \). This means that one of \( P_1 \cap O_2 \) or \( P_2 \cap O_1 \) is non-empty.

So, suppose \( m \in P_1 \cap O_2 \). The play \( play_{(\alpha_1,\beta)}(m, A) \), for arbitrary strategy \( \beta \), is contained in \( \Sigma_1 = \Pi^{\alpha_1}(P_1, O_1) \), which is an eager defence strategy for \( (P_1, O_1) \). Hence all maximal plays in \( \Pi^{\alpha_1}(P_1, O_1) \), and thus all plays in this set are finite. We claim that the last player of the finite play \( play_{(\alpha_1,\beta)}(m, A) \) is \( B \). Otherwise, if \( A \) is the last player then either \( A \) is stuck in a dungeon or \( \alpha_1 \) is undefined in the last room reached. In either case \( play_{(\alpha_1,\beta)}(m, A) \) would be a maximal play in \( \Sigma_1 \). But this contradicts the assumption that \( \Sigma_1 \) defends \( (P_1, O_1) \) eagerly, i.e., in all maximal plays of \( \Sigma_1 \), opponent \( B \) is the last player. By a similar argument involving \( \Sigma_2 \) one shows that for all strategies \( \beta \), \( play_{(\alpha_2,\beta)}(m, B) \) is finite and has \( B \) as the last player. Thus,

\[
\forall \beta. \, play_{(\alpha_1,\beta)}(m, A) \in \text{Term}_A \quad \forall \beta. \, play_{(\alpha_2,\beta)}(m, B) \in \text{Term}_A,
\]

where

\[
\text{Term}_U := \{ \pi \mid \text{dom}(\pi) < \omega \& \pi_2(\text{dom}(\pi) - 1) = U \}.
\]

We conclude that \( play_{(\alpha_1,\alpha_2)}(m, A) \in \text{Term}_A \) and \( play_{(\alpha_2,\alpha_1)}(m, B) \in \text{Term}_A \). But the latter implies \( play_{(\alpha_1,\alpha_2)}(m, A) \in \text{Term}_B \) by swapping of players \( A \) and \( B \). Thus, \( \text{Term}_A \cap \text{Term}_B \neq \emptyset \) which is impossible since in every finite play only one player can have the final turn. This shows \( P_1 \cap O_2 = \emptyset \). A symmetric argument yields \( P_2 \cap O_1 = \emptyset \). This completes the proof that \( P \cap O = \emptyset \).

We now create an eager defence strategy for \( (P, O) \). Pick strategy \( \alpha \) such that for all \( m \in P_1 \) we have \( \alpha(m) := \alpha_1(m) \) and for all \( m \in P_2 \setminus P_1 \) take \( \alpha(m) := \alpha_2(m) \). For all other positions we may leave \( \alpha \) undefined. We claim that \( \Sigma := \Pi^{\alpha}(P, O) \) is an eager defence strategy for \( (P, O) \), i.e., that it is consistent and that all maximal plays in \( \Sigma \) are contained in \( \text{Term}_A \). Specifically, we will show for all \( \pi \in \Sigma \),

\[
A\text{-pos}(\pi) \subseteq P \quad \text{(3)}
\]

\[
B\text{-pos}(\pi) \subseteq O \quad \text{(4)}
\]

\[
\pi \text{ maximal in } \Sigma \Rightarrow \pi \in \text{Term}_A. \quad \text{(5)}
\]
The last part states that $\Sigma$ is $A$-terminating and the first two conditions imply consistency since then $P_2 \cap O_2 \subseteq P \cap O = \emptyset$. In other words, $\Sigma$ is an $A$-terminating defence strategy.

Observe that $\Pi^\alpha(P, O) = \Pi^\alpha(P_1, O_1) \cup \Pi^\alpha(P_2 \setminus P_1, O_2 \setminus O_1)$, so we can prove (3)-(5) in two parts. First, consider the plays in $\Pi^\alpha(P_1, O_1)$. Since by assumption $\Sigma_1 = \Pi^\alpha(P_1, O_1)$ is a defense strategy for $(P_1, O_1)$, we have $P_{\Sigma_1} = P_1$. But this means that $\Sigma_1 = \Pi^\sigma(P_1, O_1)$ for every strategy $\sigma$ that coincides with $\alpha_1$ on rooms in $P_1$. In particular, then, $\Pi^\alpha(P_1, O_1) = \Sigma_1$, and moreover $A\text{-pos}(\pi) \subseteq P_1 \subseteq P$ and $B\text{-pos}(\pi) \subseteq O_1 \subseteq O$ for all $\pi \in \Pi^\alpha(P_1, O_1)$. Finally, if such $\pi \in \Pi^\alpha(P_1, O_1)$ is maximal in $\Pi^\alpha(P, O)$ it must be maximal in $\Pi^\alpha(P_1, O_1)$ and thus $\pi \in \text{Term}_A$.

Regarding the play $\pi \in \Pi^\alpha(P_2 \setminus P_1, O_2 \setminus O_1)$ we distinguish two cases:

(i) If $\pi$ stays in $P_2 \setminus P_1$ for player $A$ all the time, i.e., if $A\text{-pos}(\pi) \subseteq P_2 \setminus P_1$, then both strategies $\alpha$ and $\alpha_2$ agree and thus $\pi \in \Pi^{\alpha_2}(P_2 \setminus P_1, O_2 \setminus O_1) \subseteq \Pi^{\alpha_2}(P_2, O_2)$. Again, $A\text{-pos}(\pi) \subseteq P_2 \subseteq P$ and $B\text{-pos}(\pi) \subseteq O_2 \subseteq O$. Furthermore, if $\pi$ is maximal in $\Pi^\alpha(P, O)$ then it is maximal in $\Pi^{\alpha_2}(P_2, O_2)$, too. Thus $\pi \in \text{Term}_A$ since $\Pi^{\alpha_2}(P_2, O_2)$ is an $A$-terminating defense strategy for $(P_2, O_2)$. So, the conditions (3)-(5) must hold for this choice of $\pi$.

(ii) Consider a play $\pi \in \Pi^\alpha(P_2 \setminus P_1, O_2 \setminus O_1)$ in which player $A$ at some point receives the turn outside of region $P_2 \setminus P_1$. Let $n < \text{dom}(\pi)$ be the first time when this happens. Up to this time $A$ always moved from $P_2 \setminus P_1$ according to transition strategy $\alpha_2$, which defends $(P_2, O_2)$ and thus makes sure that if the turn returns to $A$ it will be in $P_2$ and $B$ only moves from $O_2$. At stage $n$, then, $A$ is still inside $P_2$ but (for the first time) finds himself outside of $P_2 \setminus P_1$ which means he is in $P_1 \cap P_2$. Here, by definition $A$’s strategy $\alpha$ behaves like $\alpha_1$ which defends $(P_1, O_1)$. This means that from time $n$ onwards $A$ is always in region $P_1$ while $B$ is locked inside $O_1$. Overall, thus, $A\text{-pos}(\pi) \subseteq P_1 \cup P_2 = P$ and $B\text{-pos}(\pi) \subseteq O_1 \cup O_2 = O$, which proves (3) and (4). Finally, take the suffix of $\pi' := \pi(n) \cdot \pi(n+1) \cdots$ of $\pi$ starting at time $n$, which, as argued, satisfies $\pi' \in \Pi^{\alpha_1}(P_1, O_1)$. Since $\pi$ was maximal in $\Pi^\alpha(P, O)$, $\pi'$ must be maximal in $\Pi^{\alpha_1}(P_1, O_1)$ and thus $\pi' \in \text{Term}_A$. Since winning conditions are time invariant (i.e., independent of any finite initial part) this means $\pi \in \text{Term}_A$ which finally establishes (5) in case (ii).

This completes the proof that $\alpha$ is an eager defense strategy for $(P, O) = (P_1 \cup P_2, O_1 \cup O_2)$. \qed

**B.10 Proof of Proposition 5.10**

*Proof:* Prop. 5.11 states that eager front lines are the pfps of efl-$M$, which are the same as the inseparable elements (for inseparability see Appendix A) of cfl-$M$. 43
Thus, the maximal pfps are the maximal inseparable elements, which must also be fixed points of $cfl-M$. By Lem. A.4 the inseparable fixed point is the least fixed point. But we know that the least fixed point of $cfl-M$ is in fact $(\text{Win}, \text{Lose})$.

**B.11 Proof of Proposition 5.11**

Proof: By Lem. A.5 is suffices to show that eager front lines coincide with the inseparable elements of $cfl-M$. So, let us investigate the concrete conditions involved for a post-fixed point of $cfl-M$ to be inseparable. Applying Lem. A.2 (ii), a post-fixed point $(P, O)$ is inseparable for $cfl-M$ if for all $(X, Y) \subseteq (P, O)$ we have

$$((P \land \neg X, O \land \neg Y) \cap ([\tau]X \lor \langle \iota \rangle Y, [\tau]Y \land \langle \iota \rangle X) = ((P \land \neg X) \land ([\tau]X \lor \langle \iota \rangle Y), (O \land \neg Y) \land [\tau]Y \land \langle \iota \rangle X) \neq (\emptyset, \emptyset),$$

which is the same as

$$(a) \ (P \land \neg X) \land ([\tau]X \lor \langle \iota \rangle Y) \neq \emptyset \text{ or } (b) \ (O \land \neg Y) \land [\tau]Y \land \langle \iota \rangle X \neq \emptyset.$$  

($\Leftarrow$) Suppose front line $(P, O)$ satisfies (a) and (b). We show that $(P, O)$ is eagerly defensible. Consider the set of eager front lines $(P', O') \subseteq (P, O)$ included in $(P, O)$, which is non-empty, since $(\emptyset, \emptyset)$ is always eager, and let $(P^*, O^*)$ be the maximal such, which can be obtained as the set-theoretic union by Lem. 5.8. Let $\alpha$ be an eager transition strategy defending $(P^*, O^*)$, i.e., $\Pi^\alpha(P^*, O^*)$ is consistent and its maximal elements are contained in $\text{Term}_A$. We claim that $(P^*, O^*) = (P, O)$ which proves that $(P, O)$ is eager. The proof is by contradiction. Suppose $(P^*, O^*) \subsetneq (P, O)$. Then we know that

$$(a') \ (P \land \neg P^*) \land ([\tau]P^* \lor \langle \iota \rangle O^*) \neq \emptyset \text{ or } (b') \ (O \land \neg O^*) \land [\tau]O^* \land \langle \iota \rangle P^* \neq \emptyset.$$  

In case (a') there exists an element $x \in P \land \neg P^*$ such that $x \in [\tau]P^* \lor \langle \iota \rangle O^*$. If $x \in [\tau]P^*$ we select a $y \in P^*$ such that $x \xrightarrow{\tau} y$ and if $x \in \langle \iota \rangle O^*$ we pick a $y \in O^*$ such that $x \xrightarrow{\iota} y$. In first case we would have an extended strategy $\alpha' := \alpha \cup \{(x, \tau, y)\}$ and in the second $\alpha' := \alpha \cup \{(x, \iota, y)\}$. In either case, since $x \in P \land \neg P^*$ and $P^*$ is the domain of $\alpha$ this extension creates a partial function $\alpha'$ with domain $P' := P^* \cup \{x\} \supseteq P^*$. Also, since $(P, O)$ is a front line and $O^* \subseteq O$, we must have $x \notin O^*$, so $(P', O^*) \subseteq (P, O)$ again is a front line. One can show that $\alpha'$ is an A-terminating and consistent transition strategy for $(P', O^*)$. In case (b') there is a room $y \in O \land \neg O^*$ with $y \in [\tau]O^* \land \langle \iota \rangle P^*$. This means that $\alpha$ is in fact an eager strategy for front line $(P^*, O^* \cup \{y\}) \subseteq (P, O)$. In both cases we would have found properly larger eager front lines contained in $(P, O)$ which is impossible as $(P^*, O^*)$ was assumed to be maximal. Thus, $(P^*, O^*) = (P, O)$ as desired.

($\Rightarrow$) Vice versa, assume that $(P, O)$ is eager, with defense strategy $\Sigma = \Pi^\alpha(P, O)$ so that $P = P_\Sigma$ and $O = O_\Sigma$. Let any $(X, Y) \subseteq (P, O)$ be given. Suppose both (a)
and (b) are false, i.e., \((P \land \neg X) \land (\langle \tau \rangle X \lor \langle i \rangle Y) = \emptyset\) and \((O \land \neg Y) \land ([\tau] Y \land [i] X) = \emptyset\). We must show that \((X, Y) = (P, O)\). The assumption is equivalent to \(P \land \neg X \subseteq [\tau] \neg X \land [i] \neg Y\) and \(O \land \neg Y \subseteq \langle \tau \rangle \neg Y \lor \langle i \rangle \neg X\). Since \((P, O)\) is defensible we have \(O \subseteq [\tau] O \land [i] P\) by Prop. 4.3. Taken together we find

\[
P \land \neg X \subseteq [\tau] (\neg X) \land [i] (\neg Y) \tag{6}
\]

\[
O \land \neg Y \subseteq \langle \tau \rangle (O \land \neg Y) \lor \langle i \rangle (P \land \neg X). \tag{7}
\]

If there exists a room \(m \in O \land \neg Y \neq \emptyset\) we can consider the plays \(\emptyset \neq \Pi^\alpha(m, B) \subseteq \Pi^\alpha(P, O)\). We now give an inductive argument that there must be an infinite play \(\pi \in \Pi^\alpha(m, B)\) in which player \(B\) always remains inside \(O \land \neg Y\) and player \(A\) always inside \(P \land \neg X\). This is certainly true in the initial configuration \((m, B)\), viz. by assumption. Whenever \(B\) plays from \(O \land \neg Y\) then because of \((7)\) he has the option to stay in \(O \land \neg Y\) keeping the turn or to hand over to \(A\) in region \(P \land \neg X\). Next assume \(A\) plays from a room in \(P \land \neg X\). He uses strategy \(\alpha\) which is an eager defence strategy for \((P, O)\). This means that \(\alpha\) actually defines a move \((A\) does not stop!) and moreover this move is such that \(A\) keeps the turn in \(P\) or passes over to \(B\) in region \(O\). By \((6)\), however, these regions can be strengthened to \(P \land \neg X\) or \(O \land \neg Y\), respectively, whatever \(A\)'s move is. Thus, player \(B\) has a strategy to build a play \(\pi \in \Pi^\alpha(m, B)\) that is going on indefinitely in the restricted front line \((O \land \neg Y, P \land \neg X)\). But this contradicts the assumption that \(\Pi^\alpha(P, O)\) is eager, i.e., that all maximal plays terminate with \(B\) having the last turn. Hence, \(O \land \neg Y\) must be empty, which means \(Y = O\). But then \((6)\) implies that if player \(A\) ever gets to play from a room \(P \land \neg X\) he will never be able to give the turn to \(B\) (for this would necessarily have to be in \(O \land \neg Y\)), but instead just go round in an infinite loop in \(P \land \neg X\). This, too, would contradict the assumption that \(\alpha\) defends \((P, O)\) eagerly. Thus, \(P \land \neg X = \emptyset\), i.e., \(X = P\).

\[\Box\]

**B.12 Proof of Lemma A.6**

**Proof:** Let \((W, \sqsubseteq)\) be a chain complete partial ordering and \(f : W \rightarrow W\) a monotonic function on \(W\). Chain completeness means that every increasing \(\omega\)-sequence \(x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots\) has a least upper bound \(\bigsqcup_{i < \omega} x_i\).

Let \(x\) be a maximal post-fixed point of \(f\). Then \(x \sqsubseteq f(x)\) and for all extensions \(x \subseteq x'\) in \(W\) that are post-fixed points of \(f\), we must have \(x' = x\) (this is what maximality means). Now, by monotonicity \(f(x) \sqsubseteq f(f(x))\). So, \(f(x)\) would be such a post-fixed point extending \(x\) and thus \(x = f(x)\). This proves that \(x\) is indeed a fixed point of \(f\). Moreover, \(x\) must be a maximal fixed point since any larger fixed point \(x \sqsubseteq x'\) is in particular a post-fixed point and thus \(x = x'\) as \(x\) was assumed to be maximal among post-fixed points. Vice versa, every maximal fixed point \(x\) is also a maximal post-fixed point. First, it is a post-fixed point, trivially. We show
that $x$ is also maximal as a post-fixed point, which is seen as follows. Let $x'$ be any other post-fixed point above $x$, i.e., $x \sqsubseteq x' \sqsubseteq f(x')$. Consider the nonempty set

$$PFP_{x'} := \{ y \in W \mid x' \sqsubseteq y \sqsubseteq f(y) \}$$

of all post-fixed points above $x'$. We claim that $PFP_{x'}$ is chain–complete. For if $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ is any increasing sequence of post-fixed points $x_i \in PFP_{x'}$, then by chain completeness of $W$ this sequence has a least upper bound $\bigcup_i x_i \in W$, which — as we now show — must be a post-fixed point, too: Observe that $x_i \sqsubseteq f(x_i) \sqsubseteq f(\bigcup_i x_i)$ by monotonicity and post-fixed points, for all $i < \omega$. Since $\bigcup_i x_i$ is the least upper bound, we have $\bigcup_i x_i \sqsubseteq f(\bigcup_i x_i)$, so $\bigcup_i x_i$ is a pfp of $f$. But this means that $\bigcup_i x_i \in PFP_{x'}$ as claimed and thus we have established that every increasing chain in $PFP_{x'}$ has an upper bound inside $PFP_{x'}$.

To cut things short we now appeal to Zorn’s Lemma to conclude that there must be a maximal element $x^* \in PFP_{x'}$.\footnote{Zorn’s Lemma (Every chain complete partially ordered set has a maximal element) works by “brute force” for general domain $W$, essentially constructing this maximal element $x^*$ by transfinite induction. In practical special cases one uses finiteness of the underlying domain $W$ or continuity of the function $f$ to show that $x^*$ is constructible by finite or $\omega$-iteration of $f$ already (c.f. [DP02]).}

From inseparability we thus conclude $\bigcup_i x_i = x^* = x$ as $x$ was assumed maximal among the fixed points. Thus $x$ is a maximal pre-fixed point as claimed.

\[ \square \]

B.13 Proof of Lemma A.2

Proof: We prove equivalence between the first two statements. The second and forth are simple variants of each other where the role of $x$ and $r \sqcap \neg x$ have been interchanged. The second and third translate into each other since $r \sqcap \neg x \sqcap f(x) = \bot$ is equivalent to $r \sqcap f(x) \sqsubseteq x$. Let us point out that $(i) \iff (iii)$ holds in any meet semi-lattice, for the others involving negation we invoke the laws of Boolean algebras.

$(i) \Rightarrow (ii)$ Let $r$ be inseparable and $x \sqsubseteq r$ such that $r \sqcap \neg x \sqcap f(x) = \bot$. We must show $x = r$ or $r \sqsubseteq x$, which is the same. From the assumption $r \sqcap \neg x \sqcap f(x) = \bot$ we get $r \sqcap f(x) \sqsubseteq r \sqcap x$. Then we compute $r \sqcap f(r \sqcap x) = r \sqcap f(r \sqcap x) \sqsubseteq r \sqcap f(x) \sqsubseteq r \sqcap x$. From inseparability we thus conclude $r \sqsubseteq r \sqcap x \sqsubseteq x$ as required.

$(ii) \Rightarrow (i)$ In the other direction assume that $r \sqcap \neg x \sqcap f(x) = \bot$ implies $x = r$ for all $x \sqsubseteq r$. Suppose we are given $y$ with $r \sqcap f(r \sqcap y) \sqsubseteq y$. This means $r \sqcap \neg(r \sqcap y) \sqcap f(r \sqcap y) = r \sqcap y \sqcap f(r \sqcap y) \sqsubseteq \neg y \sqcap y = \bot$. From the assumption where we choose $x$ to be $r \sqcap y$ we now obtain $r \sqcap y = r$, which is the same as $r \sqsubseteq y$. This proves that $r$ is inseparable.\[ \square \]
B.14 Proof of Lemma A.3

Proof: For stability the statement simply follows from the well-known fact that the join of post-fixed points of a monotone function is again a post-fixed point. Also, if \( r \) is a post-fixed point of \( f \) then \( f(r) \) is, too.

Now we turn to inseparability. Assume given a collection \( \{ r_i \mid i \in I \} \subseteq W \) of inseparable elements of \( f \). We claim that \( \bigvee_{i \in I} r_i \) is inseparable, too. To this end let \( x \subseteq \bigvee_i r_i \) together with \( \bigvee_i r_i \cap f(x) \subseteq x \). By distributivity this means that \( \bigvee_i (r_i \cap f(x)) \subseteq x \), which further implies \( r_i \cap f(x) \subseteq x \) for all \( i \in I \). Take any index \( i \in I \) and consider the element \( x \cap r_i \). Clearly, \( x \cap r_i \subseteq r_i \). Also, \( r_i \cap f(x \cap r_i) \subseteq r_i \cap f(x) = r_i \cap r_i \cap f(x) \subseteq r_i \cap x \) so that by inseparability of \( r_i \) (criterion (ii) of Lem. A.2) we can infer \( r_i \cap x = r_i \). But this implies \( r_i \subseteq x \) for all \( i \in I \), whence \( \bigvee_i r_i \subseteq x \). This shows that \( \bigvee_i r_i \) is indeed inseparable using criterion (iii) of Lem. A.2.

Finally, we show that \( f(r) \) is inseparable whenever \( r \) is. Again we assume \( x \subseteq f(r) \) together with \( f(r) \cap f(x) \subseteq x \). From this we get \( r \cap f(r \cap (x \cap r)) = r \cap f(x \cap r) = r \cap f(x \cap r) \cap f(x \cap r \cap f(x)) \subseteq r \cap f(x) \cap f(r) = x \cap r \), so by inseparability of \( r \), we must have \( x \subseteq r \cap r \), i.e., \( r \subseteq x \), from which it follows that \( f(r) \subseteq f(x) \). But then \( f(r) = f(r) \cap f(x) \subseteq x \), which shows that \( f(r) \) is inseparable, again by Lem. A.2 (iii).

\[ \square \]

B.15 Proof of Lemma A.4

Proof: Let \( r \) be an inseparable fixed point of \( f \) and \( s = \mu x. f(x) \) the least fixed point of \( f \), which must exist in a complete lattice. Clearly, \( s \subseteq r \) since \( r \) is a fixed point and \( s \) the least one. For the other direction, observe that \( r \cap f(s) = r \cap s \subseteq s \). Using characterisation (iii) of Lem. A.2 we conclude \( s = r \) as desired. Also, the least fixed point \( s \) is always inseparable. For suppose \( x \subseteq s \) is given arbitrarily such that \( s \cap f(x) \subseteq x \). By monotonicity and the fixed point property \( f(x) \subseteq f(s) = s \) and thus \( f(x) \subseteq s \cap f(x) \subseteq x \). Thus, \( x \) is a pre-fixed point of \( f \). But the least fixed point \( s \) is the meet of all pre-fixed points \( s = \bigwedge \{ x \mid f(x) \subseteq x \} \), so that \( s \subseteq x \), which proves that \( s \) is inseparable according to characterisation (iii) of Lem. A.2.

\[ \square \]

B.16 Proof of Lemma A.5

Proof: All we need to show is that the following conditions are equivalent:

\[ r \subseteq \mu x. f(r \cap x) \quad (8) \]
\[ \forall y. \ r \cap f(r \cap y) \subseteq y \Rightarrow r \subseteq y \quad (9) \]
We first show $(8) \Rightarrow (9)$, assuming that $r \sqsubseteq \mu x.f(r \cap x)$. Let any $y \in W$ be given such that $r \cap f(r \cap y) \subseteq y$. Consider the relative complement $r \rightarrow y$, which in a Boolean algebra is representable as $\neg r \sqcup y$. A Heyting algebra, where $r \rightarrow y$ is a weaker intuitionistic implication would do as well. Then, we have $f(r \cap (r \rightarrow y)) = f(r \cap y) \subseteq r \rightarrow y$ where the latter follows from the assumption $r \cap f(r \cap y) \subseteq y$. This means $r \rightarrow y$ is a pre-fixed point of $f_i(x) := f(r \cap x)$. But in any complete lattice the fixed-point of a monotone function is the least pre-fixed point (intersection of all pre-fixed points). Thus, we get $r \sqsubseteq r \cap \mu x.f(r \cap x) = r \cap \mu x.f(r \cap x) \subseteq \mu x.f(r \cap x)$. Hence, by $(9)$, we conclude $r \sqsubseteq \mu x. f(r \cap x)$ which is what we had to demonstrate.

\section*{B.17 Proof of Lemma A.8}

\textit{Proof:} Let $r = (r_1, r_2) \in W$ be inseparable for $f = (f_1, f_2)$. Consider any $(x, y) \subseteq r$ such that $r \cap ((Id \times \top) \circ f)(x, y) \subseteq (x, y)$. Observe that $((Id \times \top) \circ f)(x, y) = (f_1(x, y), \top)$, so our assumption comes down to $r_1 \cap f_1(x, y) \subsetneq x$ and $r_2 \cap \top = r_2 \subseteq y$. Since also $y \subseteq r_2$, we have in fact $r_2 = y$. But then $r \cap f(x, r_2) = (r_1 \cap f_1(x, r_2), r_2 \cap f_2(x, r_2)) = (r_1 \cap f_1(x, y), r_2 \cap f_2(x, r_2)) \subseteq (x, r_2)$ and thus by inseparability, Lem. A.2 (iii), noting that $(x, r_2) \subseteq r$, we get $(x, r_2) = r$ which implies $x = r_1$. Thus, $(x, y) = r$. This shows that $r$ is inseparable for $(Id \times \top) \circ f$ by the same criterion, hence $r$ is left inseparable for $f$. The same argument works for right inseparability.

\section*{B.18 Proof of Lemma A.9}

\textit{Proof:} From the proof of Lem. A.8 above it transpires that left/right inseparability are equivalent to

\begin{align*}
\forall x. (x \subseteq r_1 & \& r_1 \cap f_1(x, r_2) \subseteq x) \Rightarrow x = r_1 \\
\forall y. (y \subseteq r_2 & \& r_2 \cap f_2(r_1, y) \subseteq y) \Rightarrow y = r_2
\end{align*}

respectively. In other words, left inseparability of $(r_1, r_2)$ is the same as inseparability of $r_1$ with respect to the function $f_1(\cdot, r_2)$ where the second component has been frozen at $r_2$. So, we can apply Lem. A.5 to get equivalence with the condition $r_1 \subseteq \mu x.f_1(r_1 \cap x, r_2)$ as stated in Lem. A.9. A symmetric argument deals with right inseparability. The statement of Lem. A.9 concerning left/right stability follows directly from Def. A.1 considering that $((Id \times \top) \circ f)(x, y) = (f_1(x, y), \top)$ and $((\top \times Id) \circ f)(x, y) = (\top, f_2(x, y))$. 

\qed

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