PATHWISE STOCHASTIC OPTIMAL CONTROL

L. C. G. Rogers

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Stochastic optimal control as we know it.
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Controlled Markov process with values in $\mathcal{X}$, with finite horizon, and objective

$$E \left[ \sum_{j=0}^{T-1} f_j(X_j, a_j) + F(X_T) \right]$$

to be maximized over adapted $a \in A$. 

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to be maximized over adapted \( a \in \mathcal{A} \). The value function

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V_n(x) \equiv \sup_{a \in \mathcal{A}} E\left[ \sum_{j=n}^{T-1} f_j(X_j, a_j) + F(X_T) \middle| X_n = x \right]
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satisfies the Bellman equations:
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$$V_n(x) = \sup_a \left[ f_n(x, a) + PV_{n+1}(x, a) \right], \quad V_T(x) = F(x).$$
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If no closed form solution exists, then numerics hit problems if the dimension of $\mathcal{X}$ is large (think of $\mathcal{X} = \mathbb{R}^{50}$):
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If no closed form solution exists, then numerics hit problems if the dimension of $\mathcal{X}$ is large (think of $\mathcal{X} = \mathbb{R}^{50}$):

- How do we store $V_n$?
- How do we compute integrals over $\mathcal{X}$?
- Use Monte Carlo for high-dimensional integration .... but what controls would we use along a simulated path?
Simplest example: optimal stopping.
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If we stop at time $\tau$, we get reward $Z_\tau$. The American option pricing problem (=optimal stopping problem) with horizon $T$ is to find stopping time $\tau^*$ such that

$$Y_0^* \equiv \sup_{\tau \in T} E[Z_\tau].$$
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It can be shown (R., Haugh & Kogan) that

$$Y_0^* = \inf_{M \in \mathcal{M}_0} \mathbb{E}\left[ \sup_{0 \leq s \leq T} (Z_s - M_s) \right].$$
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COULD A SIMILAR APPROACH WORK MORE GENERALLY??
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**COULD A SIMILAR APPROACH WORK MORE GENERALLY??**

- optimising over a *much* larger class;
- many possible laws to consider;
Problem formulation.

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$$E \left[ \sum_{j=0}^{T-1} f_j(X_j, a_j) + F(X_T) \right] \equiv E \left[ \sum_{j=0}^{T} f_j(X_j, a_j) \right],$$

to be max’ed over adapted $a \in \mathcal{A}$. 
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$$\Lambda_t(a) \equiv \prod_{r=0}^{t-1} \varphi(X_r, X_{r+1}; a_r) \equiv \prod_{r=0}^{t-1} \varphi_{r+1}(a_r) \in L^1(\mathcal{F}_t);$$
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so the problem is

$$V_0(X_0) = \sup_{a \in \mathcal{A}} v_0(X_0; a) \equiv \sup_{a \in \mathcal{A}} E^* \left[ \sum_{j=0}^{T} \Lambda_j(a) f_j(X_j, a_j) \right].$$
Main result, 1.

Fixing \( a \in A \), for any martingale \( M \),

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v_0(X_0; a) = E^* \left[ \sum_{j=0}^{T} \Lambda_j(a) f_j(X_j, a_j) \right]
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where \( \Delta M_{j+1} = P h_{j+1}(X_j, a_j) - h_{j+1}(X_{j+1}) \varphi_{j+1}(a_j) \), \( h_{T+1} \equiv 0 \).
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where $\Delta M_{j+1} = Ph_{j+1}(X_j, a_j) - h_{j+1}(X_{j+1}) \varphi_{j+1}(a_j)$, $h_{T+1} \equiv 0$. Hence

$$V_0(X_0) = \sup_{a \in A} v_0(X_0; a)$$

$$= \sup_{a \in A} E^* \left[ \sum_{j=0}^{T} \Lambda_j(a) \{ f_j(X_j, a_j) + Ph_{j+1}(X_j, a_j) - h_{j+1}(X_{j+1}) \varphi_{j+1}(a_j) \} \right]$$

$$= \sup_{a \in A} E^* \left[ h_0(X_0) + \sum_{j=0}^{T} \Lambda_j(a) \{ f_j(X_j, a_j) + Ph_{j+1}(X_j, a_j) - h_j(X_j) \} \right]$$

$$\leq E^* \left[ \sup_a \{ h_0(X_0) + \sum_{j=0}^{T} \Lambda_j(a) \{ f_j(X_j, a_j) + Ph_{j+1}(X_j, a_j) - h_j(X_j) \} \} \right]$$
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So

\[ V_0(X_0) \leq h_0(X_0) + E^* \left[ \sum_{j=0}^{T} \sup_a \Lambda_j(a) \{ f_j(X_j, a_j) + Ph_{j+1}(X_j, a_j) - h_j(X_j) \} \right] \]

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In fact, there is equality, and the equality is attained !!
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To see this, use the Bellman equation:

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Main result: remarks.

\[ V_0(X_0) = \min_{(h_j)} E^* \left[ \sup_a \sum_{j=0}^{T} \Lambda_j(a) \{ f_j(X_j, a_j) + P h_{j+1}(X_j, a_j) - h_{j+1}(X_{j+1}) \phi_{j+1}(a_j) \} \right]. \]
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- Well suited to Monte Carlo; simulate \( X \), and then maximise *pathwise* over \( a \);
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- This approach is a new \textit{strategy} for optimal control.
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Remarks.
- Well suited to Monte Carlo; simulate \( X \), and then maximise pathwise over \( a \);
- The pathwise optimisation can be done recursively;
- There is an infinite-horizon version of the result.
- Rockafellar & Wets, Wets, Back & Pliska study maximisation of a concave path functional over adapted processes by absorbing adaptedness constraint into a Lagrangian term and then doing pathwise max ...
  ... but what we do here requires no concavity assumption;
- This approach is a new strategy for optimal control.
- In fact, we have an infinite-dimensional linear program, where the choice variable is the RCD for a given \( X \).
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with objective $\min E(|X_T|^2 + \gamma \int_0^T |a_s|^2 \, ds)$, the solution is a linear feedback; the solution is (nearly) an OU process, very different from the Brownian motion we might use to simulate paths ..
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- Shorten the horizon, but think ahead! If we simply reduce $T$ to $T_1 < T$, and use objective $\min E\gamma \int_0^{T_1} |a_s|^2 \, ds$, then we will get things very wrong! Include something to account (even approximately) for significant upcoming costs.
Implementation outline.
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Probably very difficult unless there is some rich enough family \( \{ \psi(\cdot; \theta) : \theta \in \Theta \} \) of functions 'nice' in the sense that \( P\psi(x, a; \theta) \) is simply expressed.
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V^{(n+1)}_t(X^i_t) \equiv \sup_a \sum_{j=t}^T \Lambda_j(a) \{ f_j(X^i_j, a_j) + Ph_{j+1}(X^i_j, a_j) - h_{j+1}(X^i_{j+1})\psi_{j+1}(a_j) \} / \Lambda_t(a),
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**Step 7:** Increase \( n \) by 1 and return to Step 2.
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