

ADAPTIVE λ -TRACKING OF NONLINEAR SYSTEMS WITH HIGHER RELATIVE DEGREE USING REDUCED-ORDER HIGH-GAIN CONTROL

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Abstract: In this paper an adaptive λ -tracking controller is presented that incorporates a reduced-order high-gain observer. It is shown that this controller can λ -stabilize a larger class of nonlinear systems than the controller described in Bullinger (2000). The controller is similar to the controllers for linear systems by Mareels (1984) and Mudgett and Morse (1989). In comparison to the controller proposed by Ye (1999), the proposed controller uses a simpler feedback law and can be applied to systems having a non-constant high-frequency gain.

Keywords: lambda-tracking, adaptive control, high-gain, reduced-order observer, robust stability, universal stabilization.

1. INTRODUCTION

In Bullinger (2000) an adaptive high-gain controller for λ -tracking of nonlinear single-input single-output systems with higher relative degree was outlined. This controller utilizes a high-gain observer of the same dimension as the relative degree and an adaptive observer state feedback. As is shown in Bullinger *et al.* (2000), Bullinger and Allgöwer (2000) and Bullinger (2000) this controller can universally stabilize a large class of systems. The full-order high-gain observer used in Bullinger (2000) can be seen as an estimator/filter of the output itself and its time derivatives, see (Bullinger and Allgöwer, 1997) and (Bullinger *et al.*, 1998).

Instead of using the filtered/estimated output in the observer state feedback, one might ask, if it is possible to directly use the measured output, thus leading to the use of a reduced-order observer. In this paper we present an adaptive λ -tracking controller that utilizes a reduced-order high-gain observer instead of a full-order one.

The advantage in using a reduced-order observer is that the dimension of the observer is reduced by one. The price to pay lays in the fact that the output enters directly the feedback part of the controller with a factor of $\hat{\kappa}^r$. This might cause problems, especially for a high relative degree r or a large adaptation parameter $\hat{\kappa}$.

The basic idea behind the reduced-order approach is to transform the higher-relative degree problem into one of relative degree one (see Ye, 1999). To achieve a stabilizing feedback, Ye proposed to choose the input u via backstepping. This normally results in a rather complicated controller. Another approach is to choose the input as a high-gain combination of the measured output y and of the observer states.

This approach has been taken for linear systems by Mareels (1984) and by Mudgett and Morse (1989). In this paper, we follow the line of Mudgett and Morse (1989) which itself is based on Luenberger (1964) for the observer design and the normal form the linear system is assumed to be in.

The use of a reduced-order observer for the stabilization of higher relative degree nonlinear systems has been already proposed in (Ye, 1999). There, the feedback is based on a backstepping approach, and does lead to a rather complicated control law. Our approach does lead to a simpler feedback structure.

This paper is organized as follows. In Section 2 we will present the considered system class and outline the proposed control scheme which consists of a reduced-order observer and an adaptive high-gain state and output feedback. In Section 3 we give the main stability result and shortly outline the proof.

2. SYSTEM CLASS AND CONTROLLER STRUCTURE

In this paper we consider the control of single-input single-output (SISO) systems that are time-invariant and affine in the input:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \quad (1a)$$

$$y = \mathbf{h}(\mathbf{x}). \quad (1b)$$

The input is denoted by u , the output by y . The state $\mathbf{x}(t)$ is in \mathbb{R}^n , $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ are continuous and locally Lipschitz functions mapping $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

The control objective of λ -tracking is to asymptotically track a reference signal $y_{ref}(\cdot)$ while tolerating a tracking error smaller than a user-defined λ . All states should remain bounded, i.e. $\mathbf{x} \in L_\infty$. The reference signal $y_{ref}(\cdot)$ is considered to be in $W^{1,\infty}$, the set of all bounded functions that are absolutely continuous on compact sub-intervals and whose first derivative is essentially bounded. This set includes almost all practically relevant signals.

To achieve this objective a reduced-order adaptive output-feedback controller is used. It consists of a reduced-order observer and an adaptive high-gain observer-state feedback, as described in Section 2.2, 2.3 and 2.4.

2.1 Necessary assumptions and definitions

For the proposed controller the system needs to satisfy the following assumptions.

Assumption 1 (Normal form) *Their exists a change of coordinates, such that the system (1) can be transformed into the following normal form:*

$$\dot{\xi} = A(\xi, \eta)\xi + (\phi(\xi, \eta)^T \eta + u) \mathbf{b} + \mathbf{v}(\xi, \eta) \quad (2a)$$

$$\dot{\eta} = \chi(\xi, \eta)y + \theta(\eta) + \mathbf{w}(\xi, \eta), \quad (2b)$$

$$y = \xi_1, \quad (2c)$$

with

$$\begin{aligned} \xi(0) &\in \mathbb{R}^r, \quad \eta(0) \in \mathbb{R}^{n-r} \\ \mathbf{b} &= [0 \ \dots \ 0 \ 1]^T \in \mathbb{R}^r, \\ A(\xi, \eta) &= J_g(\xi, \eta) + E(\xi, \eta), \end{aligned}$$

and

$$\begin{aligned} E(\xi, \eta) &= \begin{bmatrix} \alpha_{r-1}(\xi, \eta) & \mathbf{0}^T \\ \alpha(\xi, \eta) & \hat{E} \end{bmatrix} \in \mathbb{R}^{r \times r}, \\ J_g(\xi, \eta) &= \begin{bmatrix} 0 & g(\xi, \eta) & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 0 \end{bmatrix} \in \mathbb{R}^{r \times r}. \end{aligned}$$

The matrix \hat{E} is assumed to be lower triangular.

This normal form is a generalization of the Byrnes-Isidori normal form (Byrnes and Isidori, 1984; Byrnes and Isidori, 1985; Isidori, 1995). If \mathbf{v} and A are equal to zero except their last row, then (2) is in Byrnes-Isidori normal form.

A similar normal form, but with constant $\alpha(\cdot)$ and $g(\cdot)$ is required in Ye (1999). As shown in Luenberger (1964) and Mudgett and Morse (1989), any linear system $\{\hat{A}, \hat{\mathbf{b}}, \hat{\mathbf{c}}^T, 0\}$ can be transformed into the form $\{A, \mathbf{b}, [1 \ \dots \ 0]^T, 0\}$ described above.

To achieve our results using a high-gain feedback, we have to additionally require boundedness of the nonlinearities.

Assumption 2 (bounded nonlinearities)

The nonlinearities, i.e. $E(\cdot)$, $\mathbf{v}(\cdot)$, $\mathbf{w}(\cdot)$, $\phi(\cdot)$ and $\chi(\cdot)$, are globally bounded.

In the following, we use a somehow unconventional assumption on the relative degree of the system, purely based on the unperturbed system.

Definition 1 (Unperturbed system) *The system (2) with $\mathbf{v}(\cdot) \equiv \mathbf{0}$ and $\mathbf{w}(\cdot) \equiv \mathbf{0}$, i.e.*

$$\dot{\xi} = A(\xi, \eta)\xi + (\phi(\xi, \eta)^T \eta + u) \mathbf{b} \quad (3a)$$

$$\dot{\eta} = \chi(\xi, \eta)y + \theta(\eta), \quad (3b)$$

$$y = \xi_1, \quad (3c)$$

will be called the unperturbed system.

Assumption 3 (known relative degree r)

The relative degree of the unperturbed system (3) is well defined and equal to r , the dimension of $\xi(t)$.

Remark 1 *The relative degree of the unperturbed system as used in Assumption 3 differs from the “classical” definition of a relative degree where a well-defined relative degree is equal to the relative degree of the linearized system. For example the system*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \tanh(x_r) \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= u \\
y &= x_1
\end{aligned} \tag{4}$$

has a classical relative degree of 2:

$$\ddot{y} = x_3 + \frac{1}{\cosh^2(x_r)} u.$$

However, as $\tanh(x_r)$ is bounded, we consider it as a “disturbance” term. Thus the unperturbed system of (4) has a relative degree of r . \diamond

In the proposed controller, non-constant high-frequency gains are allowed.

Assumption 4 (g positive and bounded from above) The high-frequency-gain $g(\xi, \eta)$ of the unperturbed system (3) is positive and bounded from above by a known constant \bar{g} , i.e.

$$\bar{g} \geq g(\xi, \eta) > 0.$$

To ensure stability of the proposed output feedback controller, we have to require that the zero dynamics is exponentially stable.

Assumption 5 (minimum phase) The zero-dynamics of the unperturbed system,

$$\dot{\eta} = \theta(\eta)$$

is globally exponentially stable.

Given that the outlined assumptions hold, we propose a controller consisting of a reduced-order observer and an adaptive observer-state and output feedback law to achieve λ -tracking.

2.2 Observer

The used reduced state observer is given by

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{b}u \tag{5}$$

with $\hat{x} \in \mathbb{R}^{r-1}$ and

$$\hat{b} = [0 \ \dots \ 0 \ 1]^T.$$

The observer matrix $\hat{A} \in \mathbb{R}^{(r-1) \times (r-1)}$ is identical to the linear part of $A(\cdot)$ of the system except that the first state $\xi_1 = y$ is not estimated:

$$\hat{A} = J + \hat{E},$$

where

$$J = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & 0 \end{bmatrix}.$$

In contrary to the case of the full-order observer, this observer is not directly adapted. The adaptation is indirectly caused by u which is a high-gain combination of y and of the observer states \hat{x} .

Remark 2 Systems of relative degree one have a reduced-order observer of dimension zero. The controller therefore reduces in this case to the controller of (Ilchmann and Ryan, 1994). \diamond

2.3 Observer-state and output feedback

The input is given by an observer state and output feedback. It depends on the adaptation parameter $\hat{\kappa}$:

$$u = -\mathbf{q}_\kappa^T \begin{bmatrix} y - y_{ref} \\ \hat{x} \end{bmatrix} \tag{6a}$$

where

$$\mathbf{q}_\kappa = \begin{bmatrix} \kappa^r & & & \\ & \kappa^{r-1} & 0 & \\ & & \ddots & \\ 0 & & & \kappa \end{bmatrix} \cdot \mathbf{q} \tag{6b}$$

with

$$\mathbf{q} = [q_0, \dots, q_{r-1}]^T. \tag{6c}$$

In comparison to the full-order case, not only the observer states, but also the output is used for feedback. The controller parameters q_i have to satisfy certain conditions, as described in the following, see also Section 3 and Theorem 1.

2.3.1. Constant high-frequency gain If the high-frequency gain g is constant, it is necessary that

$$J_g - \mathbf{b}\mathbf{q}^T \text{ is Hurwitz.} \tag{7}$$

It is easy to see, that the eigenvalues of $J_g - \mathbf{b}\mathbf{q}^T$ for $\mathbf{q} = [q_0, \dots, q_{r-1}]^T$ are the zeros of the polynomial $q_g(s) = s^r + \sum_{i=1}^{r-1} q_i s^i + q_0 g$. For $r \leq 2$, the Hurwitz condition is satisfied if q_0, q_1 are positive. For $r \geq 3$, this requires that $s^{r-1} + \sum_{i=1}^{r-1} q_i s^{i-1}$ is Hurwitz and that q_0 is sufficiently small, (see e.g. Bullinger, 2000).

2.3.2. Non-constant high-frequency gain The controllers by Mareels (1984), Mudgett and Morse (1989) and Ye (1999) require a constant high-frequency gain, whereas we allow a non-constant high-frequency gain. In case of a non-constant $g = g(\xi, \eta)$ it is required, that there exists a single Lyapunov matrix P for all $\tilde{g} \in (0, \bar{g}]$ for the matrix

$$J_{\tilde{g}} - \mathbf{b}\mathbf{q}^T,$$

i.e. that there exists a positive definite matrix P satisfying for all $\tilde{g} \in (0, \bar{g}]$

$$(J_{\tilde{g}} - \mathbf{b}\mathbf{q}^T)^T P + P (J_{\tilde{g}} - \mathbf{b}\mathbf{q}^T) < 0. \tag{8}$$

By Lemma 1 and 2, such a Lyapunov function exists for any $\bar{g} > 0$ if $s^r + \sum_{i=1}^{r-1} q_i s^i + \frac{\bar{g}}{2} q_0$ is Hurwitz and q_0 is sufficiently small.

Remark 3 For systems with negative high frequency gain

$$g(\mathbf{x}) \leq \underline{g} < 0,$$

the control law has to be changed to:

$$u = +\mathbf{q}_{\hat{k}} \left[\frac{y - y_{ref}}{\hat{\mathbf{x}}} \right]. \quad \diamond$$

Remark 4 This observer state feedback controller is the limit case of a controller with a slower adaptation than the observer part. In contrary to the full-order case, the reduced-order controller does not have a constant Lyapunov matrix in coordinates scaled by \hat{k} . However, for any fixed \hat{k} , it is still possible to find such a Lyapunov matrix, see e.g. Khalil and Saberi (1987). \diamond

2.4 Adaptation of the feedback gain

The adaptation for the gain \hat{k} is chosen such that the gain is increased as long as the amplitude of the tracking error e is larger than the user-defined bound λ (the control objective).

The controller gain \hat{k} is defined as

$$\hat{k} = k^\alpha, \quad (9)$$

where $\alpha > 0$ is a user-defined parameter.

With $\lambda > 0$, $\gamma > 0$, $k(0) = k_0 > 0$ and $\epsilon > 0$ the adaptation parameter k is given by

$$\dot{k} = d_\lambda(e, k)^2; \quad (10a)$$

$$d_\lambda(e, k) = \frac{\gamma}{k^\gamma} \begin{cases} |e| - \lambda & \text{for } |e| > \lambda, \\ 0 & \text{for } |e| \leq \lambda, \end{cases} \quad (10b)$$

where $\tilde{\gamma}$ has to satisfy

$$\tilde{\gamma} \geq 2\alpha\epsilon - \frac{1}{2}. \quad (10c)$$

This adaptation law ensures a monotonous increase of the adaptation parameter k and thus of the controller parameter \hat{k} .

The parameter $\tilde{\gamma}$ slows the adaptation down. The more stable \hat{A} and $q_g(s)$ are the faster the adaptation can be chosen. This adaptation differs from the ones used in similar controllers. Mudgett and Morse (1989) and Ye (1999) do not need to slow down the adaptation for large k , i.e. $\tilde{\gamma} = 0$. Mareels (1984) requires that $\frac{d}{dt}k$ is bounded.

3. THEOREM AND PROOF

The following theorem states that combining the reduced observer (5) with the adaptive feedback (6) and using the adaptation law (9)-(10) to close the loop, achieves for any system satisfying Assumptions 1 to 5 that the tracking error asymptotically converges to the λ -strip.

Further the adaption converges, no finite escape time can occur and all states remain bounded.

The main theorem requires the following definitions of a subset of the Hurwitz polynomials and Hurwitz matrices.

Definition 2 A polynomial $p(s) = s^r + \sum_{i=1}^{r-1} s^i p_i + p_0$ is in the set $H(\epsilon, \mu)$ if there exists a symmetric, positive definite matrix P such that the companion matrix

$$A_c = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & -p_1 & \dots & -p_{r-1} \\ -p_0 & & & \end{bmatrix}$$

satisfies for $\Delta = \text{diag}\{0, 1, \dots, r-1\}$ the inequalities

$$A_c^T \cdot P + P \cdot A_c \leq -2\mu P \quad (11a)$$

$$(\Delta + \epsilon I) \cdot P + P \cdot (\Delta + \epsilon I) \geq 0. \quad (11b)$$

Remark 5 Equation (11a) implies, that $H(\epsilon, \mu)$ is a subset of the Hurwitz polynomials. It can easily be shown that if $p(s) \in H(\underline{\epsilon}, \bar{\mu})$ then

$$p(s) \in H(\epsilon, \mu) \text{ for all } \epsilon \geq \underline{\epsilon} \text{ and for all } \mu \leq \bar{\mu}. \quad \diamond$$

Remark 6 For polynomials of degree $r = 1$ (systems with relative degree one), $\Delta = 0$. In that case, $\epsilon = 0$ and $\tilde{\gamma} = 0$ are valid choices. \diamond

Definition 3 A matrix is in the set $H(\epsilon, \mu)$ if its characteristic polynomial is in $H(\epsilon, \mu)$.

Using these definitions we can state the following:

Theorem 1 Reduced-order λ -tracker

Assume that for all $\tilde{g} \in (0, \underline{g}]$, \hat{A} and $q_{\tilde{g}}(s)$ are of class $H(\epsilon, \mu)$ for some $\epsilon > 0$, $\mu > 0$ and that $q(s)$ satisfies (7) or (8). Then the application of the λ -tracker (5), (6), (9), (10) to any system satisfying Assumptions 1 to 5 with any reference signal $y_{ref}(\cdot) \in W^{1,\infty}$ results in a closed-loop system which independently of the initial values $\mathbf{x}(0) \in \mathbb{R}^n$, $\hat{\mathbf{x}}(0) \in \mathbb{R}^r$, $k(0) > 0$ has a unique solution existing on the whole half axis $t \in [0, \infty)$ and, moreover,

$$a) (\mathbf{x}(\cdot), \hat{\mathbf{x}}(\cdot), k(\cdot)) \in L_\infty(0, \infty),$$

$$b) \lim_{t \rightarrow \infty} |y(t) - y_{ref}(t)| \leq \lambda.$$

Proof (of Theorem 1)

Outline of the Proof of Theorem 1. The proof of Theorem 1 is similar to the one in (Bullinger, 2000). The main difference lie in the different closed loop equations. This makes the use of different coordinate transformations necessary.

Proof 1.a Boundedness of the adaption parameters. Due to lack of space, this part of the proof is only sketched here. The proof is done by contradiction. It is based on the fact that for sufficiently large adaptation parameter, it is possible to find a Lyapunov function. This implies that the closed loop is then exponentially stable. The consequence is that the adaptation parameter cannot be unbounded.

1.b) Boundedness of the observer states.

As $k(\cdot)$ is bounded, $d_\lambda(\cdot) \in \mathcal{L}_2([0, \omega])$. This, (10) and the Hölder inequality implies that

$$|e(\cdot)| \in \mathcal{L}_\infty([0, \omega]). \quad (12)$$

By boundedness of k there exist a k_ω such that

$$k(t) \leq k_\omega \text{ for all } t \in [0, \omega].$$

Defining $\hat{A}_\omega = \hat{A} - \hat{\mathbf{b}}\hat{\mathbf{q}}_{k_\omega}^T$, $\hat{A}_1 = \hat{A} - \hat{\mathbf{b}}\hat{\mathbf{q}}_k^T - \hat{A}_\omega$, (5) is equivalent to

$$\dot{\hat{\mathbf{x}}} = \hat{A}_\omega \hat{\mathbf{x}} + \hat{A}_1 \hat{\mathbf{x}} - q_{k_1} \hat{\mathbf{b}}e, \quad (13)$$

where \hat{A}_ω is Hurwitz, $\|\hat{A}_1\|$ decreases monotonically to zero and $\|q_{k_1} \hat{\mathbf{b}}\|$ is bounded. Therefore, it follows from Variation of Constants that $\hat{\mathbf{x}}$ is bounded, i.e.

$$\hat{\mathbf{x}}(\cdot) \in \mathcal{L}_\infty^r([0, \omega]).$$

As $u = -\mathbf{q}_k \hat{\mathbf{x}}$, it follows that also

$$u(\cdot) \in \mathcal{L}_\infty([0, \omega]). \quad (14)$$

1.c) Boundedness of the states of the plant.

By (12) e is bounded almost everywhere, as is y_{ref} by assumption. Thus, also y is bounded almost everywhere.

In a first step, we analyze the internal dynamics of (2),

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\theta}(\boldsymbol{\eta}) + \boldsymbol{\chi}(\boldsymbol{\xi}, \boldsymbol{\eta})y + \mathbf{w}(\boldsymbol{\xi}, \boldsymbol{\eta}).$$

It follows from Variation of Constants that

$$\boldsymbol{\eta}(\cdot) \in \mathcal{L}_\infty^{n-r}([0, \omega]).$$

In a second step we look at the remaining states, i.e. at $\boldsymbol{\xi}$, which satisfy

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{J}\boldsymbol{\xi} + \mathbf{E}(\cdot)\boldsymbol{\xi} + \mathbf{b} \left(\boldsymbol{\phi}^T(\cdot)\boldsymbol{\eta} + u(\cdot) \right) + \mathbf{v}(\cdot) \\ &= (\mathbf{J} + \mathbf{E}(\cdot))\boldsymbol{\xi} + \tilde{\mathbf{v}}, \end{aligned} \quad (15)$$

where $\tilde{\mathbf{v}} \in \mathcal{L}_\infty^r([0, \omega])$, As \mathbf{E} is lower triangular, it follows from the observability of (15) that $\tilde{\boldsymbol{\xi}} \in \mathcal{L}_\infty^r([0, \omega])$, see (Bullinger, 2000).

1.d) Global existence of a unique solution.

As k , \mathbf{x} and $\hat{\mathbf{x}}$ are bounded on $[0, \omega]$, it follows by maximality of ω that $\omega = \infty$.

1.e) Convergence of the tracking error. It remains to show b). For this we prove that $\lim_{t \rightarrow \infty} d_\lambda(e(t), k(t)) = 0$. Since $e(\cdot)$ and $k(\cdot)$ are bounded it follows that $\dot{k}(\cdot) = d_\lambda(\cdot)^2 \in \mathcal{L}_\infty([0, \omega])$. From $\dot{e} = g(\alpha_{r-1}\xi_1 + \xi_2) - \dot{y}_{ref}$ and using the boundedness of $\boldsymbol{\xi}(\cdot)$ we conclude that $\dot{e}(\cdot) \in \mathcal{L}_\infty([0, \infty))$. As

$$\frac{d}{dt}d_\lambda^2 = 2d_\lambda \left(\frac{\gamma}{k^\gamma} \frac{e\dot{e}(\cdot)}{|e|} - \tilde{\gamma} \frac{\dot{k}}{k} d_\lambda \right) \in \mathcal{L}_\infty([0, \omega]),$$

$d_\lambda(\cdot)^2$ is uniformly continuous. This, together with $d_\lambda(\cdot)^2 \in \mathcal{L}_1([0, \infty))$ yields, by Barbălat's Lemma (Barbălat, 1959) that $\lim_{t \rightarrow \infty} d_\lambda(t)^2 = 0$.

This completes the proof. \blacksquare

CONCLUSIONS

This paper presents an adaptive λ -tracking controller which utilizes a reduced-order high-gain observer. By slightly modifying the adaptation law compared to Mareels (1984) and Mudgett and Morse (1989), the same properties can be achieved for nonlinear systems as has been shown for linear systems. The controller achieves λ -tracking, namely it is guaranteed that all states and the adaptation parameter remain bounded and that the tracking error $y - y_{ref}$ asymptotically converges to the λ -strip. The width of this strip is a parameter which chosen by the user and does usually depend on the specifications, on model uncertainties and on the quality of the measurement. Stability and convergence of the adaptation is proven for tracking a large class of smooth reference trajectories. The design of the controller is very simple and intuitive, only few parameters have to be tuned.

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APPENDIX

5. SPECIAL HIGH-GAIN NORMAL FORM

The following lemma is needed in the proof of Theorem 1. It allows to consider non-constant high-frequency gains.

Lemma 1 *If $q_{\tilde{g}}(s) \in H(\epsilon, 2\mu)$ with*

$$q_{\tilde{g}}(s) = s^r + \sum_{i=1}^{r-1} s^i q_i + q_0 \tilde{g}, \quad (16)$$

then

$$q_{\tilde{g}}(s) \in H(\epsilon, \mu) \text{ for all } |g - \tilde{g}| \leq \mu. \quad (17)$$

Proof (of Lemma 1) As $q_{\tilde{g}}(s) \in H(\epsilon, 2\mu)$ it is easy to show, that there exist a symmetric, positive definite matrix $P_{\tilde{g}}$ satisfying

$$A_{\tilde{g}}^T P_{\tilde{g}} + P_{\tilde{g}} A_{\tilde{g}} \leq -4\mu P_{\tilde{g}} \quad (18)$$

with

$$A_{\tilde{g}} = \begin{bmatrix} 0 & \tilde{g} & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & & & \ddots & & \\ 0 & & & & 1 & \\ -q_0 & \dots & \dots & \dots & \dots & -q_{r-1} \end{bmatrix}.$$

With

$$\mathbf{b} = [0 \ \dots \ 0 \ 1]^T, \quad \mathbf{q} = [q_0 \ \dots \ q_{r-1}]^T,$$

$$J_{\tilde{g}} = \begin{bmatrix} 0 & \tilde{g} & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & & & \ddots & & \\ 0 & & & & 1 & \\ 0 & & & & & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 0 & 0 & & \\ & & & \ddots & & \\ 0 & & & & & 0 \end{bmatrix}$$

we can write

$$A_{\tilde{g}} = J_{\tilde{g}} - \mathbf{b}\mathbf{q}.$$

Then,

$$J_{\tilde{g}} = J_{\tilde{g}} + (g - \tilde{g})X.$$

Defining $Q_{\tilde{g}} = P_{\tilde{g}}^{-1}$ and restrict P such that $\mathbf{q} = P\mathbf{b}$, it follows from (18) that

$$J_{\tilde{g}}Q_{\tilde{g}} + Q_{\tilde{g}}J_{\tilde{g}}^T - 2\mathbf{b}\mathbf{b}^T \leq -4\mu Q_{\tilde{g}}.$$

Therefore,

$$J_{\tilde{g}}Q_{\tilde{g}} + Q_{\tilde{g}}J_{\tilde{g}}^T + (g - \tilde{g})(XQ_{\tilde{g}} + Q_{\tilde{g}}X) - 2\mathbf{b}\mathbf{b}^T \leq -4\mu Q_{\tilde{g}}.$$

As $\|X\| = 1$,

$$\mu \geq |g - \tilde{g}| \geq |g - \tilde{g}| \|X\|$$

which implies that

$$2\mu Q_{\tilde{g}} \geq |g - \tilde{g}| (XQ_{\tilde{g}} + Q_{\tilde{g}}X^T).$$

Therefore,

$$J_{\tilde{g}}Q_{\tilde{g}} + Q_{\tilde{g}}J_{\tilde{g}}^T - 2\mathbf{b}\mathbf{b}^T \leq -2\mu Q_{\tilde{g}},$$

or, equivalently,

$$A_g^T P_g + P_g A_g \leq -2\mu P_g,$$

which concludes the proof. \blacksquare

A straightforward result of Lemma 1 is the following lemma.

Lemma 2 *Let $q_g(s) = s^r + \sum_{i=1}^{r-1} q_i s^i + gq_0$. If*

$$q_g(s) \in H(\epsilon, \mu) \text{ for all } |g - \tilde{g}| \leq \mu, \quad (19)$$

then for any $\hat{g} > 0$

$$\hat{q}_g(s) \in H(\epsilon, \mu) \text{ for all } |g - \tilde{g}| \leq \hat{g}\mu, \quad (20)$$

where $\hat{q}_g(s) = s^r + \sum_{i=1}^{r-1} q_i s^i + \frac{g}{\hat{g}}q_0$.