Guaranteed Delay Margins for Adaptive Systems with State Variables Accessible

Megumi Matsutani¹, Anuradha Annaswamy², and Eugene Lavretsky³

Abstract—Robust adaptive control for plants whose state variables are accessible in the presence of time-delays is established in this paper. It is shown that a standard adaptive controller with a projection algorithm ensures global boundedness of the overall adaptive system for a range of non-zero delays. The upper bound of such delays, i.e. the delay margin, is explicitly computed.

I. INTRODUCTION

Adaptive control theory is a mature control discipline that has evolved over the past four decades and rigorously synthesized in [1]-[10]. Several attempts have been made to extend the robustness properties of adaptive systems to time-delays and unmodeled dynamics (see for example [3]-[6] and more recently [7]-[9]) by introducing modifications to the underlying adaptive law. Either these results are semi-global or global where the delay margin can be shown to exist but is not otherwise computable, or they are restricted to a small class of plants [7]. In contrast to these results, it was recently proven in [11] that global boundedness can be achieved for a first-order plant with a guaranteed delay margin using an adaptive law which includes a modification based on projections. This paper extends this result to higher dimensional plants with a scalar input, where states are accessible.

The adaptive law used in this paper was originally proposed in [12], rigorously analyzed in [5], [6], and revised and refined in [7], [13]. Unlike the standard practice of Lyapunov function based arguments which suffice when states are measurable, extensive arguments based on first principles are employed in order to prove the boundedness.

The problem is stated in Section II. The main result is stated in Section III and proved in Section IV. A flight control example is used to illustrate the order of magnitude of the analytically computable delay margin.

II. PROBLEM STATEMENT

An nth order plant with a scalar input and a parameter uncertainty is given by

\[ \dot{x}_p(t) = A_p x_p(t) + b_p u(t - \tau) \]  

where \( A_p \) is an unknown parameter and \( \tau \geq 0 \) is an unknown time delay.

A control law is chosen as

\[ u(t) = \theta^T(t)x_p(t) + r(t) \]  

where \( \theta(t) \) is time varying due to adaptation and \( r(t) \) is a reference input. The problem is to ensure bounded solutions with the control law as in (2) using a suitable adaptive law for adjusting \( \theta(t) \).

Equation (1) can be rewritten in the form

\[ \dot{x}_p(t) = A_p x_p(t) + b_p (u(t) + \eta(t)) \]

where

\[ \eta(t) = u(t - \tau) - u(t). \]

Therefore the system subject to the input time delay can be interpreted as a perturbed system by the unmodeled dynamics \( \eta(t) \).

A reference model is chosen as

\[ \dot{x}_m(t) = A_m x_m(t) + b_m r(t) \]

where \( A_m \) is Hurwitz. For any \( Q = Q^T > 0 \), there exists \( P = P^T > 0 \) which satisfies the Lyapunov equation

\[ A_m^T P + PA_m = -Q. \]

Therefore, with a bounded reference input, boundedness of \( x_m \) follows and we define \( \bar{x}_m \equiv \max_{t \geq t_0} \{x_m(t)\} \). For the sake of simplicity, we assume \( b_p \) is known and let \( b_m \equiv b_p \).

It is expected that the result in this paper can be extended directly for \( b_p \equiv \lambda b_m \), where \( \lambda \) is an unknown parameter. Assuming that \( \theta^* \) exists such that

\[ A_p + b_p \theta^* T = A_m \]

and defining the parameter and state errors

\[ \phi(t) = \theta(t) - \theta^* \]

\[ e(t) = x_p(t) - x_m(t) \]

the closed-loop adaptive system is given by

\[ \dot{x}_p(t) = A_m x_p(t) + b_p (\phi^T(t)x_p(t) + r(t) + \eta(t)) \]

and the error dynamics

\[ \dot{e}(t) = A_m e(t) + b_m (\phi^T(t)x_p(t) + \eta(t)) \]

The question is if a different adaptive law than a standard adaptive law [3]

\[ \dot{\theta}(t) = -\Gamma x_p(t)b_m^T P e(t) \]

where \( \Gamma \) is a symmetric positive definite matrix.

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¹M. Matsutani is with Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, MA, USA megunim at mit.edu

²A. M. Annaswamy is with Department of Mechanical Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, MA, USA anna at mit.edu

³E. Lavretsky is with Boeing company, Huntington Beach, CA, USA eugene.lavretsky at boeing.com
can ensure a delay margin. This is the problem addressed in Section III. In particular, the problem that will be addressed is the determination of a \( \tau^* \) such that a modified adaptive law together with the plant in (1) and the control law in (2) is guaranteed to have globally bounded solutions for all \( \tau \in [0, \tau^*] \).

### III. Boundedness in the Presence of Time Delay

#### A. Nonsingular Transformation

In this section, we will derive transformed state error \( E(t) \) and parameter error \( \theta(t) \) using transfer matrices \( C \) and \( M \) so that

\[
E(t) \equiv C e(t), \\
\theta(t) \equiv M \Theta(t).
\]

We refer to the \( i \)th component of the error states by \( E_i(t) \) and \( \theta_i(t) \) respectively, for \( i = 0, 1, \ldots, n - 1 \). The introduction of \( C \) and \( M \) are needed in order to identify crucial scalars that capture the dominant effect of the perturbation \( \eta \). We now describe the construction of \( C \) and \( M \).

The matrix \( M \) in (21) is chosen as follows. First we take

\[
c_0 = \frac{P_{bbm}}{p_{bb}}
\]

where \( P \) is given in (6) and \( p_{bb} = \sqrt{b_{mm}^2 P_{bbm}} \). We note that

\[
c_i^T b_m = \frac{b_{mm}^T P}{p_{bb}} b_m = p_{bb}.
\]

Then we pick \( n - 1 \) vectors \( c_i \) for \( i = 1, 2, \ldots, n - 1 \) which satisfy

\[
c_i^T P^{-1} c_j = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}
\]

where \( j = 0, 1, \ldots, n - 1 \). We therefore note that

\[
c_i^T b_m = c_i^T P^{-1} c_0 p_{bb} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n - 1.
\]

We obtain an invertible matrix \( C \) by defining

\[
C = \begin{bmatrix}
c_0^T \\
c_1^T \\
\vdots \\
c_{n-1}^T
\end{bmatrix}.
\]

From (14), (16) and (18), it can be seen that

\[
CP^{-1} C^T = I.
\]

Using \( P \) and \( C \), we choose \( M \) as

\[
M = p_{bb} C P^{-1}.
\]

Several approaches have been used in robust adaptive control to establish boundedness, one of which is the modification of the standard MRAC adaptive law. One such example is to modify the adaptive law with a projection algorithm [5], [6], [13]. The specific projection algorithm we will use in this paper is of the form

\[
\dot{\theta}(t) = M^{-1} w
\]

where \( w = [w_0 \ w_1 \ldots \ w_{n-1}]^T \) and

\[
w_i = \text{Proj}\left(\{M \theta(t)\}_i, -\{M \Gamma x_p(t) b_m^T P e(t)\}_i\right).
\]

The projection operator, \( \text{Proj} \), in (22) produces a scalar output with scalar arguments and is chosen to be

\[
\text{Proj}(\Theta, y) = \begin{cases} \frac{\theta_{\text{max}} - \Theta}{\theta_{\text{max}}^2 - \theta_{\text{max}}^2/\gamma} & \text{if } \Theta \in \Omega_1 \cap \Omega_0 \land y \Theta > 0 \\ y & \text{otherwise} \end{cases}
\]

where \( \theta_{\text{max}} \) and \( \varepsilon \) are positive constants, \( \theta_{\text{max}} = \theta_{\text{max}}^* + \varepsilon \), and

\[
\Omega_0 = \{ \Theta \in \mathbb{R}^I | -\theta_{\text{max}} \leq \Theta \leq \theta_{\text{max}} \}
\]

\[
\Omega_1 = \{ \Theta \in \mathbb{R}^I | -\theta_{\text{max}} \leq \Theta \leq \theta_{\text{max}} \}\]
D. Choice of Projection Algorithm Parameters

Throughout the paper, we use the following notations. Let
\[ s_A = \min \{ |\Re(\lambda_i(A))| \} \]
\[ s_A = \max \{ |\Re(\lambda_i(A))| \} \]  
where \( \lambda_i \) is the \( i \)th eigenvalue of a square matrix \( A \) and \( \Re(\lambda_i) \) denotes its real part. Also we let
\[ n_A = \| A \|_2, \]  
(31)
which is the 2-norm of a square matrix \( A \).

The projection algorithm (22) requires \( \theta_{i,\max} \) and \( \epsilon_i \) to be specified, whose selections are discussed below.

We assume that upperbounds \( \theta_{i,\max}^* \in \mathbb{R} \) on the uncertain parameter \( \theta^* \) are known and are defined as
\[ \theta_{i,\max}^* = \max_{\theta} |\vartheta_i^*| \quad i = 0, 1, \ldots, n - 1. \]  
(32)
where \( \vartheta^* = M \theta^* \), \( \vartheta_i^* \) refers to its \( i \)th component, and \( \theta^* \) given in (7). We choose the control parameters \( \theta_{i,\max} \) and \( \epsilon_i \) for \( i = 1, 2, \ldots, n - 1 \) such that
\[ \theta_{i,\max} - \epsilon_i \geq \theta_{i,\max}^*. \]  
(33)
We then choose \( \theta_{0,\max} \) and \( \epsilon_0 \) such that
\[ -\theta_{0,\max} - \epsilon_0 + \theta_{0,\max}^* \leq -\alpha_00 + \frac{(\| P' a_0 \| + (\| a_1 \| + \phi_{\max}' P_e)^2)}{2 p_e Q'} \]  
(34)
where \( \alpha_00, a_0, \) and \( a_1 \) are defined in (27), \( P' \) is the solution of
\[ A_m' P' + P' A_m = -Q' \]  
(35)
with \( Q' = Q'^T > 0, \)
\[ \phi_{\max}' = \sqrt{\sum_{i=1}^{n-1} (\theta_{i,\max} + \theta_{i,\max}^*)^2}, \]  
and \( p_e \) is an arbitrary positive number. It should be noted that choosing control parameters which satisfy (33) is always possible by taking \( \theta_{0,\max} \) large enough. We also note that (32) implies that the condition \( \vartheta^* \in \Omega_0 \) is satisfied. The choice of \( \theta_{0,\max} \) in (33) will become clear in section IV-F, where the boundedness of \( e \) and \( \theta \) are addressed.

Lastly, we define
\[ \theta_{\max} = \sqrt{\sum_{i=0}^{n-1} \theta_{i,\max}^2}. \]  
(35)

E. Main result

**Theorem 1**: There exists a \( \tau^* \) such that the closed-loop adaptive system with the plant in (1), reference model in (5), control law in (2), and adaptive law in (21), (23), (24) together with the projection parameters as in (33) has globally bounded solutions for any initial conditions
\[ x_p(t) = \chi(t), \quad \theta(t) = \chi_0(t) \quad t \in [t_0 - \tau, t_0] \]  
(36)
where \( \chi(t) : \mathbb{R} \rightarrow \mathbb{R}^n, \chi_0(t) : \mathbb{R} \rightarrow \Omega_1, \) and \( \forall t \in [0, \tau^*]. \)

Theorem 1 implies that the overall adaptive system with the projection algorithm in the adaptive law has a nonzero time delay margin, \( \tau^* \).

F. Preliminaries

Prior to proving the main result, we include a few important definitions and specify a condition that the trajectory will be shown to satisfy.

1) Definitions:

**Definition 1**: We define regions \( A, B, \) and \( B' \) as follows (See Figure 1): Let \( z(t) = [e^T(t), \vartheta^T(t)]^T \).
\[ A = \{ z \in \mathbb{R}^{2n} | -\theta_{0,\max}^* \leq \vartheta_0 \leq \theta_{0,\max}^* \} \]
\[ B = \{ z \in \mathbb{R}^{2n} | -\theta_{0,\max} \leq \vartheta_0 < -\theta_{0,\max}^* \} \]
\[ B' = \{ z \in \mathbb{R}^{2n} | \theta_{0,\max} < \vartheta_0 \leq \theta_{0,\max}^* \} \]

**Definition 2**: We divide the boundary region \( B \) into two regions as follows (See Figure 1):
\[ B_L = \{ z \in \mathbb{R}^{2n} | -\theta_{0,\max} \leq \vartheta_0 \leq -{(\theta_{0,\max}^* + \epsilon_0)/2} \} \]
\[ B_U = \{ z \in \mathbb{R}^{2n} | -(\theta_{0,\max}^* + \epsilon_0/2) \leq \vartheta_0 < -\theta_{0,\max}^* \} \]

We note that \( B = B_L \cup B_U \), and that \( A, B_L, B_U, \) and \( B' \) are all regions in \( \mathbb{R}^{2n} \) that lie between two hyperplanes. All of these hyperplanes are specified using only one scalar state variable, \( \vartheta_0 \).

![Fig. 1. Definition of regions](image)

2) Constants: Let positive constants \( \delta \) and \( E_0 \) be defined by
\[ \delta \in (0 \ 1] \]  
(37)
and
\[ E_0 = \max_{t \in [t_0 - \tau, t_0]} \| C(\chi(t) - x_m(t)) \| + 2\delta, \]
\[ \frac{15}{\delta \gamma} (\theta_{\max}^2 + \gamma)(1 + \bar{x}_m), \beta \]  
(38)
where \( \beta > 0 \) is specified later in Lemma 4. From the definitions of \( E_0 \) and \( \delta \), it is immediate that \( E_0 - \delta > 0. \)
We also define a positive constant $E'$ by

$$E' = \max \left( \max_{t \in [t_0 - \tau, t_0]} \| C(\chi(t) - x_m(t)) \|, \sqrt{\frac{l^2 r_p}{1 - l^2 r_p}} E_0 \right)$$

(39)

where positive constants $l$ and $r_p$ are such that

$$r_p > 1,$$

(40)

$$\sqrt{\frac{l^2 r_p}{1 - l^2 r_p}} < 1.$$  

(41)

From the definition of $E'$, it follows that

$$E' < E_0.$$  

(42)

Using $r_p$, $E_0$ and $E'$, we further define

$$E = \sqrt{r_p} \sqrt{E_0^2 + E'^2}.$$  

(43)

Since $r_p > 1$, it is seen that

$$E > E_0.$$  

(44)

Also from the definitions of $E'$ and $E$, it can be proven that

$$1E \leq E'.$$  

(45)

3) Condition:

Condition 1: $\pi(t) \in \mathbb{R}^n$ is said to satisfy Condition 1 at time $t_0$ if

$$|\pi_0(t)| \leq E \forall t \in [t_0 - \tau, t_0],$$

(46)

$$|\pi_0(t_0)| = E_0 - \delta,$$  

(47)

$$\pi^T(t_0 - \tau) P \pi'(t_0 - \tau) \leq s_p E'^2$$

(48)

where $\pi_i$ is the $(i-1)$th component of $\pi$ and $\pi'(t) = [\pi_1 \pi_2 \ldots \pi_{n-1}]^T \in \mathbb{R}^{n-1}, t_0 \geq t_0, P'$ satisfies (34), and $E_0 \in \mathbb{R}, \delta \in \mathbb{R}$, and $E' \in \mathbb{R}$ are positive constants with $E_0 - \delta > 0$.

IV. PROOF OF THE MAIN RESULT

The closed-loop adaptive system has error dynamics equivalent to

$$\dot{e}(t) = A_m e(t) + b_m \left\{ (\theta^T(t) - \theta^* T) (e(t) + x_m(t)) + \eta(t) \right\}$$

(49)

with $\eta$ as in (4). The adaptive law in (21) and (22) can be rewritten as

$$\{ M \dot{\theta}(t) \}_i = \text{Proj} \left\{ \{ M \theta(t) \}_i, - \{ M \Gamma(e(t) + x_m(t)) b_m^T P e(t) \} \right\}.$$  

(50)

We first note that since $|\pi_0(t)| \leq \theta_{i, \text{max}}$, it follows from Lemma 2 that $|\phi_i(t)| \leq \theta_{i, \text{max}} \forall t \geq t_0$. Theorem 1 is therefore proved if the global boundedness of $e(t)$ is demonstrated.

A. Transformed State Error Dynamics

In order to prove the boundedness of $e(t)$, we will utilize the transformed error, $\mathcal{E}(t)$, introduced in (12). The global boundedness of $e(t)$ is demonstrated if the global boundedness of $\mathcal{E}(t)$ is shown. In this section, we will derive the dynamics of $\mathcal{E}$. In what follows, $y \in \mathbb{R}^{n-1}$ is said to be a subvector of $x \in \mathbb{R}^n$ if its $j$th element satisfies

$$y_j = x_{j+1}, \ j = 1, \ldots, n - 1.$$  

We note that $c_i^T$ is the $i$th row vector of $C$. It follows from (12) that for $i = 0, \ldots, n - 1$

$$\dot{\mathcal{E}}_i(t) = c_i^T \dot{e}(t).$$  

(51)

Using the property in (17), we can rewrite $P$ in quadratic form such that

$$\sum_{j=0}^{n-1} c_j c_j^T = P.$$  

(52)

It can be shown from (49) and (52) that

$$\dot{\mathcal{E}}_i(t) = c_i^T A_m e(t)$$

$$= c_i^T A_m P^{-1} \left( \sum_{j=0}^{n-1} c_j c_j^T \right) e(t)$$

(53)

for $i = 1, \ldots, n - 1$. Noting the definition of $\alpha_{ij}$ in (25), (53) can be rewritten as

$$\dot{\mathcal{E}}_i(t) = \sum_{j=1}^{n-1} \alpha_{ij} \mathcal{E}_j(t) + \alpha_{i0} E_0(t).$$  

(54)

The definitions of $A'_m$ and $a_0$ in (26) and (27) imply that the subvector $\mathcal{E}'$ of $\mathcal{E}$ given by

$$\mathcal{E}'(t) = [\mathcal{E}_1 \mathcal{E}_2 \ldots \mathcal{E}_{n-1}]$$

(55)

satisfies the error dynamics

$$\dot{\mathcal{E}}'(t) = A'_m \mathcal{E}'(t) + a_0 E_0(t).$$

(56)

We now return to (51) and consider the special case when $i = 0$. Using the property in (15) and the definition of $\alpha_{ij}$ in (25), the dynamics of the critical state error $\mathcal{E}_0$ can be obtained from (49) as

$$\dot{\mathcal{E}}_0(t) = \sum_{j=0}^{n-1} \alpha_{0j} \mathcal{E}_j(t) + p_0 \eta(t)$$

$$+ p_0 \left( \theta^T(t) - \theta^* T \right) (e(t) + x_m(t)).$$

(57)

Defining

$$m_i(t) \equiv c_i^T x_m(t)$$

(58)

and from (13) and (52), the error equation (57) can be rewritten as

$$\dot{\mathcal{E}}_0(t) = \sum_{j=0}^{n-1} \alpha_{0j} \mathcal{E}_j(t) + \sum_{j=0}^{n-1} \left( \theta_j(t) - \theta^* \right) \left( \mathcal{E}_j(t) + m_j(t) \right) + p_0 \eta(t)$$

$$+ \left( a_1 + \theta^T(t) - \theta^* T \right) \mathcal{E}'(t) + \left( \theta(t) - \theta^* \right) m'(t).$$

(59)
where \( \vartheta^* \) is the subvector of \( \vartheta^* \).

Equations (56) and (59) represent the transformed tracking error dynamics. These equations show that the perturbation, \( \eta \), due to the time delay, \( \tau \), appears only in the dynamics of \( \mathcal{E}_0 \) and not in \( \mathcal{E}_{i}, i = 1, \ldots, n - 1 \).

In what follows, we will relate the boundedness of \( \mathcal{E}' \) to that of \( \mathcal{E}_0 \) using Lemma 1.

**Proposition 1:** Suppose

\[
|\mathcal{E}_0(t)| \leq W \quad t \in \mathcal{T}_s = [t_s,t_{ss}]
\]

where \( t_{ss} > t_s \geq t_0 \). Then

\[
V'(t_s + \Delta t) \leq \max \left( V'(t_s), \frac{1}{2} \mathcal{E}^T(t) P' \mathcal{E}(t), \right) \forall \Delta t > 0 \text{ s.t. } t_s + \Delta t \in \mathcal{T}_s,
\]

where \( V'(.) \) is defined as

\[
V'(t) = \frac{1}{2} \mathcal{E}^T(t) P' \mathcal{E}(t),
\]

\( P' > 0 \) as in (34), and a positive constant \( l \) defined by

\[
l = \frac{\overline{\mathcal{E}}}{\overline{P'}} \|a_0\| \leq l \mathcal{E}'(t_s + \Delta t) \leq max \left( \mathcal{E}^T(t_s) P' \mathcal{E}(t_s), \frac{1}{2} \mathcal{E}^T(t) P' \mathcal{E}(t), \right) \forall \Delta t > 0 \text{ s.t. } t_s + \Delta t \in \mathcal{T}_s.
\]

We refer the reader to [14], [15] for the detailed proofs of Proposition 1 and Corollary 1.

**B. Transformed Parameter Error Dynamics**

Similar to Section IV-A, we now focus on the transformed parameter error, \( \vartheta(t) \), in (13). From (21) and (22), we obtain for \( i = 0, \ldots, n - 1 \)

\[
\dot{\vartheta}_i(t) = \text{Proj} \left( \vartheta_i - \gamma p_{bb,c}^T(e(t) + x_m(t)) b_m^T Pe(t) \right) = \gamma p_{bb} \text{Proj} \left( \vartheta_i - (\mathcal{E}_i(t) + m_i(t)) b_m^T Pe(t) \right).
\]

We also note that \( b_m^T Pe(t) = p_{bb,c}^T e(t) = p_{bb} \mathcal{E}_0(t) \) from (12) and (14). Therefore, we obtain

\[
\dot{\vartheta}_i(t) = \gamma' \text{Proj} \left( \vartheta_i - (\mathcal{E}_i(t) + m_i(t)) \mathcal{E}_0(t) \right), \quad i = 0, \ldots, n - 1
\]

where \( \gamma' = \gamma p_{bb}^2 \). It will be shown that \( \mathcal{E}_0 \) is the main component of interest. We therefore examine (65) for \( i = 0 \) in more detail. From (23), it follows that

\[
\dot{\vartheta}_0(t) = -\gamma' (\mathcal{E}_0(t) + m_0(t)) \mathcal{E}_0(t)
\]

if \( [z \in A] \lor [(z \in (B \cup B')) \land (\mathcal{E}_0(t) + m_0(t)) \mathcal{E}_0(t) \vartheta_0 \geq 0] \)

\[
(i) \quad |\mathcal{E}_0(t)| < E_0 \quad \forall t \in [t_a, t_a + \Delta t]
\]

\[
(ii) \quad \exists t_b \in [t_a, t_a + \Delta t] \text{ s.t. } z(t_b) \in B_L
\]

\[
\dot{\vartheta}_0(t) = -\left( \frac{\vartheta_0^2_{\text{max}} - \vartheta_0^2}{\vartheta_0^2_{\text{max}} - \vartheta_0^2} \right) \gamma' (\mathcal{E}_0(t) + m_0(t)) \mathcal{E}_0(t)
\]

\[
\text{if } \left[ (z \in (B \cup B')) \land (\mathcal{E}_0(t) + m_0(t)) \mathcal{E}_0(t) \vartheta_0 < 0 \right].
\]

It is observed that \( \dot{\vartheta}_0 < 0 \) when \( |\mathcal{E}_0| > \bar{m}_0 \) where \( \bar{m}_0 = \max_{t \geq t_0} |m_0(t)| \).

Equation (65) for \( i = 1, \ldots, n - 1, (66), \) and (67) constitute the complete adaptive law.

**C. Complete Transformed State Error Dynamics**

The two errors in the adaptive system are the state error, \( \mathcal{E} \), and the parameter error, \( \vartheta \). The former is given by (57) and (56), and the latter by (65) for \( i = 1, \ldots, n - 1, (66), \) and (67). Of the \( 2n \) states, two scalar variables, \( \mathcal{E}_0(t) \) and \( \vartheta_0(t) \), are shown to be crucial. We note that while \( \eta \) explicitly appears in the dynamics of \( \mathcal{E}_0 \), it does not appear in the dynamics of \( \mathcal{E}_i, i \geq 1 \). From the adaptive law in (66) and (67), it is obvious that \( \theta_0 \) has a nonlinear dependence on \( \mathcal{E}_0 \).

We also observe that for all \( i \geq 1 \), the transformed parameter dynamics are linearly dependent on \( \mathcal{E}_0 \), as shown in (65). The effects of such dependencies on the transformed state and parameter errors will become clear in the following section.

**D. Outline of the Proof**

The proof is completed using the following four phases.

**Phase I:** The transformed error \( \mathcal{E}(t) \) satisfies Condition 1 for some \( t = t_0 \); this implies that the state \( z \) has to enter \( B \) at \( t_0 \) \( t_a, t_a + \Delta T_{in,max} \), where \( \Delta T_{in,max} > 0 \) is a finite constant (see Figure 2(a)).

**Phase II:** When the trajectory enters \( B \), the parameter enters the boundary of the projection algorithm; \( \mathcal{E} \) is shown to be bounded by making use of the underlying linear time-varying system (see Figure 2(b)).

**Phase III:** There exists \( \Delta T_{out,min} \) such that the trajectory reenters \( A \) at \( t_c > t_b + \Delta T_{out,min} \) with \( |\mathcal{E}_0(t_c)| < \bar{m}_0 \) (see Figure 2(c)).

**Phase IV:** The trajectory has only two alternatives: (II-A): \( |\mathcal{E}_0(t)| < \varepsilon - \delta \quad \forall t > t_c \) which proves Theorem 1; (II-B): \( \mathcal{E}_0(t) \) satisfies Condition 1 for some \( t_d > t_c \). If the latter case holds, we replace \( t_a \) by \( t_d \) and repeat Phases I through IV.

We refer the reader to [14], [15] for the detailed proofs of Phases I - IV. In below we only state the key theorem, lemmas and propositions in proving these phases.

**E. Phase I: Entering the boundary**

The goal of this section is to prove the following proposition.

**Proposition 2:** Let \( \mathcal{E}(t) \) satisfy Condition 1 at \( t = t_0 \) with \( \delta, E_0, E' \) given in (37), (38), (39) respectively and \( z(t_a) \) \( A \) where \( z = [\mathcal{E}' \quad \vartheta^T]^T \). Then

\[
(i) \quad |\mathcal{E}_0(t)| < E_0 \quad \forall t \in [t_a, t_a + \Delta T]
\]

\[
(ii) \quad \exists t_b \in [t_a, t_a + \Delta T] \text{ s.t. } z(t_b) \in B_L
\]
where
\[
\Delta T = \frac{\delta}{b_0 E + b_1},
\]
\[
b_0 = B + B', \quad b_1 = (\Phi + 2\frac{c_e}{2\varepsilon} \theta_{\text{max}})\bar{m} + 2p_0 \bar{r},
\]
\[
B = |a_0| + |\theta_0^\ast| + (1 + 2\frac{c_e}{2\varepsilon})\theta_{\text{max}},
\]
\[
B' = \|a_1\| + \|\theta'\| + (1 + 2\frac{c_e}{2\varepsilon})\theta_{\text{max}}.
\]

The proof of Proposition 2 follows the equivalent scalar case proposition in [11] with similar arguments. In addition, Proposition 1 is satisfied.

F. Phase II: In the boundary region \(B\)

We return to the overall adaptive system, which can be written using (1), (2), and (7) as
\[
\dot{x}_p(t) = \left\{ A_m - b_m \theta^T \right\} x_p(t) + b_m \left\{ \theta^T (t - \tau) x_p(t - \tau) + r(t - \tau) \right\}
\]
which leads to the error dynamics
\[
\dot{e}(t) = A_m e(t) - b_m \theta^T x_p(t) + b_m \theta^T (t - \tau) x_p(t - \tau) + b_m (r(t - \tau) - r(t)).
\]  
(72)

Noting that \(E = Ce\), we then obtain
\[
\dot{E}(t) = M_0 E(t) + M_1 d(t - \tau) + R(t)
\]
where the matrices \(M_0, M_1\), and the vector \(R\) are defined as
\[
M_0 \equiv A_m - c_I \theta^T, \quad M_1 \equiv c_I \theta (t - \tau)
\]
\[
R(t) \equiv -p_{bb} c_I \theta^T x_m(t) + p_{bb} c_I \theta^T (t - \tau) x_m(t - \tau) + p_{bb} c_I (r(t - \tau) - r(t))
\]
where \(c_I = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T\).

Let the trajectory stay in \(B\) for \(t \in (t_b, t_c)\) for some \(t_c > t_b\). From the definition of \(B\), it follows that
\[
\vartheta_0(t) = -\vartheta_{0,\text{max}} - \epsilon_0(t) \quad \text{for} \quad t \in (t_b, t_c)
\]
where
\[
|\epsilon_0(t)| \leq \varepsilon_0.
\]

The main result in this section is the proof that \(E(t)\) is guaranteed to converge to a bounded set if the trajectory remains in \(B\).

Before we proceed to the main theorem, we study the properties of \(M_0 + M_1\) while in \(B\). Let us define the following set
\[
\Omega_B = \{(M_0, M_1)\mid \bar{z} \in B\}.
\]

Lemma 3: There exists a \(q > 0\) such that
\[
(M_0 + M_1)^T \mathcal{P} + \mathcal{P}(M_0 + M_1) < -q I
\]
is satisfied for all \((M_0 + M_1) \in \Omega_B\), where \(\mathcal{P}\) is a constant matrix defined as
\[
\mathcal{P} = \mathcal{I}^T \mathcal{P} \mathcal{I}
\]
with
\[
\mathcal{P} = \begin{bmatrix} P' & 0 \\ 0 & p_c \end{bmatrix}, \quad \mathcal{I} = \begin{bmatrix} I_{0 \times (n-1)} & 0 \\ 0_{(n-1) \times 1} \end{bmatrix}.
\]

The choice of the projection parameters satisfying (33) is used to prove the lemma. Lemma 3 is the key property of the time-varying system (73)-(75).

Lemma 4: Consider the uncertain time-varying system (73)-(75) and the selection of the projection parameters satisfying (33). Let the solutions of the system lie in \(B\) for \(t \in (t_b, t_c)\). Then there exists \(\beta > 0\) such that for any \(\tau \leq \bar{\tau}\),
\[
V(E(t)) \leq \max \left\{ V(E(t_b)), \bar{s} p \beta^2 \right\} \quad \forall t \in (t_b, t_c)
\]
where
\[
V(E) = E^T \mathcal{P} E
\]
and
\[
\bar{\tau} = \frac{1}{(4 + \delta) \theta_{\text{max}} \bar{s} \sqrt{\frac{\bar{s} p}{\bar{s} p} q}}, \quad \delta > 0.
\]

The proof of Lemma 4 follows from the equivalent scalar case propositions in [11] with similar arguments.
The proof is a vector version of Theorem 2 in [11] and utilizes Lemma 3, Proposition 6.7 in [16], model transformation, and Razumikhin Theorem. We conclude this section with the following proposition. Proposition 3: If \( \tau \leq \bar{\tau} \), then \( \| \mathcal{E}(t) \| < E \forall t \in [t_b, t_c] \).

We refer the reader to [14], [15] for the proof of Proposition 3.

G. Phase III: Exiting from the boundary

**Proposition 4:** Let \( z(t_b) \in B_L \). Then either

(i) \( z(t) \in B \forall t \geq t_b \), or

(ii) there exists \( t_c \) such that \( z(t_c) \in A \) and \( z(t) \in B \forall t \in [t_b, t_c) \).

In addition, in case (ii),

\[ t_c - t_b \geq \Delta T_{exit,\text{min}} \]

where

\[ \Delta T_{exit,\text{min}} = \frac{2\bar{\varepsilon}_0}{\gamma m_0^2} \]

and

\[ [\mathcal{E}_0(t_c)] < \bar{m}_0 \].

The proof for Proposition 4 is almost identical to the equivalent scalar case proposition in [11].

H. Phase IV: Return to Condition 1

So far, we have shown the following:

**Phase I.** At \( t = t_a \), \( \mathcal{E}(t) \) satisfies Condition 1. Then \( z(t_b) \in B_L \) for \( t_b < t_a + \Delta T_{in,\text{max}} \), with \( [\mathcal{E}_0(t)] < E_0 \forall t \in [t_a, t_a + \Delta T] \).

**Phase II.** Defining \( t_c \) such that \( z(t) \in B \forall t \in [t_b, t_c) \), if \( \tau \leq \bar{\tau} \), then \( \| \mathcal{E}(t) \| < E \forall t \in [t_b, t_c) \).

**Phase III.** Either (a) \( t_c = \infty \), or (b) \( t_c \geq t_b + \Delta T_{exit,\text{min}} \) where \( z(t_c) \in A \) and \( [\mathcal{E}_0(t_c)] < \bar{m}_0 \).

The following proposition contains the main result of this section:

**Proposition 5:** Either \( \mathcal{E}(t) \) returns to Condition 1 for some \( t = t_q \) or the boundedness of \( \mathcal{E}(t) \) is immediate.

We refer the reader to [14], [15] for the proof of Proposition 5.

I. Final Part of the Proof

The above phases imply that starting \( t = t_a \), there are only one of three possibilities: (i) The trajectory stays in Phase II for all \( t \geq t_1 \) for some finite \( t_1 \geq t_b \); (ii) The trajectory stays in Phase IV-A for all \( t \geq t_2 \) for some \( t_2 \geq t_c \); (iii) The trajectory visits all four phases infinitely often. The discussions in sections IV-E through IV-H imply that in all three cases (i)-(iii), \( \mathcal{E}(t) \) always remains bounded, which proves Theorem 1. In particular, it follows from (68), Lemma 4, and (84) that in all cases (i)-(iii), if \( \tau \leq \tau^* \) defined as

\[ \tau^* = \min \left\{ \Delta T_{exit,\text{min}}, \bar{\tau} \right\} \]

then,

\[ [\mathcal{E}_0(t)] \leq E, \quad \| \mathcal{E}'(t) \| \leq \sqrt{\frac{E}{2p^*}} E' \), \quad \forall t \geq t_0 \]

and hence

\[ |z(t)| \leq M \quad \forall t \geq t_0, \quad (87) \]

where \( M \equiv \sqrt{E^2 + \frac{\bar{\varepsilon}^2}{2p^*} E'^2 + \theta^2_{\text{max}}} \), proving global boundedness.

J. Delay Margin of the Adaptive System

From (81), (83), and (85), we obtain that the solutions of the overall adaptive system are bounded for all \( \tau \leq \tau^* \). Hence, the delay margin is given by \( \tau^* \), with

\[ \tau^* = \min \left\{ \frac{2\bar{\varepsilon}_0}{\gamma m_0^2}, \frac{1}{(4 + \delta)\theta_{\text{max}}\bar{m}_0 \sqrt{\frac{s_p}{s_L}} q} \right\}. \]

We note that every quantity on the right hand side in (88) depends on \( A_m, b_m, P, \gamma, \theta_{\text{max}}, \) and \( r \). Therefore \( \tau^* \) is computable a priori. However, determining the largest such \( \tau^* \) requires the solution of a nonlinear constrained optimization problem.

V. Numerical Example

In this section we demonstrate using a simple example as to how the main result in this chapter can be used to obtain the delay margin of adaptive system. We consider the short period dynamics of a conventional aircraft that are approximated by a second-order plant with a scalar input.

From [17], short period dynamics of a fixed-wing aircraft with zero bank angle can be expressed as

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\gamma}
\end{bmatrix} = \begin{bmatrix}
-L_{\alpha} & -L_q \\
\lambda_{\alpha} M_{\alpha} & \lambda_{\gamma} M_{\gamma}
\end{bmatrix} \begin{bmatrix}
\alpha \\
q
\end{bmatrix} + \lambda_\delta \begin{bmatrix}
0 \\
M_{\delta}
\end{bmatrix} \begin{bmatrix}
\delta \\
d_{\text{trim}}
\end{bmatrix}
\]

(89)

where \( \alpha \) is the aircraft angle of attack (radians), \( q \) is the body pitch rate along the stability axis (radians/sec), and \( \delta \) is the total differential elevator deflection (radians). The scalars \( \lambda_\delta > 0, \lambda_{\alpha}, \) and \( \lambda_{\gamma} \) represent uncertainties in the parameter values, and \( d_{\text{trim}} \) denotes an unknown trim input component. The nominal values are given as \( \lambda_{\alpha} = \lambda_{\gamma} = \lambda_\delta = 1 \) and \( d_{\text{trim}} = 0 \). In the following numerical example, we assume that there are no uncertainties in the control effectiveness and the trim input, i.e. \( \lambda_\delta = 1 \) and \( d_{\text{trim}} = 0 \). We also assume that the size of uncertainties are known with \( \lambda_{\alpha} \in [0.6, 1] \) and \( \lambda_{\gamma} \in [0.7, 1] \). The remaining parameters represent the so-called aircraft stability and control derivatives. The values of the stability and control derivatives used in this example are \( L_{\alpha} = 0.6582, L_q = -0.9705, M_{\alpha} = -3.3105, M_{\gamma} = -1.4741, \) and \( M_{\delta} = -3.6764 \). These values can be found by numerical linearization of a nonlinear aircraft model.

A state-feedback controller

\[ \delta = \theta_{\text{cmd}}^T(t) x_p + k_{\delta} \delta_{\text{cmd}} \]

is used where \( \theta_x = [\theta_{\alpha} \theta_{\gamma}] \) and \( \delta_{\text{cmd}} \) is the commanded elevator deflection from the pilot. We derive the nominal dynamics from (89) by assuming no parametric uncertainty (i.e. \( \lambda_{\alpha} = \lambda_{\gamma} = 1 \)) as

\[ \dot{x}_p = A_{p,nom} x_p + b_{p} \delta, \]

(90)
The Linear Quadratic (LQ) optimal control design technique [18] is applied to the dynamics in (90) to obtain a nominal controller. In this example, the values minimizing the cost function $J = \frac{1}{2}\int_{0}^{T} \dot{x}_p^T (Qf + k_x^T R_f k_x) x_p dt$ with $Q_f = \text{diag}(2I)$ and $R_f = I$ are calculated as $k_x = [-0.2816 \quad -0.7434]$. This nominal gain is used as an initial condition for $\dot{\theta}_s(t)$, i.e., $\theta_s(t_0) = k_x$. The feed forward gain $k_b$ is designed to produce the angle of attack such that $k_b = 1/g_\alpha$, where $[g_x \quad g_q]^T = -(A_{p,nom} + b_pk_x)^{-1}b_p$ is the steady state gain of (90). Therefore the closed-loop dynamics with the nominal controller can be written as

$$\dot{x}_m = A_m x_m + b_m \delta_{cmd}$$

(91)

where $A_m = A_{p,nom} + b_pk_x$ and $b_m = b_p$. Equation (91) will serve as a reference model. An adaptation can be then introduced into $\theta_s(t)$ as

$$\dot{\theta}_s = \gamma \text{Proj}(\dot{\theta}_s - (E_i + m_i)E_0) \quad i = 0, 1$$

where $E$, $\theta$, and $m$ are the transformed state error, parameter, and reference state, as introduced in (12), (13), and (58), respectively. The transfer matrix $C$ is constructed from $A_m$ and $b_m$ as discussed in Section III, and given as $C = \begin{bmatrix} 0.1247 & 0.4572 \\ -0.4572 & 0.1247 \end{bmatrix}$. Similarly, $M$ and $A_m$ are constructed from (20) and (26). We then choose the projection parameters from (32) and (33), and the size of uncertainties in $\lambda_\theta$, $\lambda_b$ as $\theta_{i,\max} = 6.0$, $\theta_{i,\max} = 1.5$, and $\varepsilon = 0.01$. We also set the adaptation gain to $\gamma = 10.83$ based on ad-hoc tuning and assume that $\delta_{cmd}$ is such that $|\alpha_m(t)| \leq 0.1745(\text{radians})$, $|\gamma_m(t)| \leq 0.6109(\text{radians/sec}) \quad \forall t \geq t_0$ which leads to $\bar{m}_\theta = 0.3010$ from (58). $\bar{g}$ is set to 1 and $\bar{g}_p$, $\bar{g}$ are calculated from $A_m$, $\theta_{i,\max}$, $\varepsilon_i$, and $\varepsilon*$. We can therefore calculate the delay margin using (88) as $\tau^* = 6.8(\text{msec})$. According to numerical simulation studies, it was observed that the actual delay margin of the adaptive system is around 0.070(\text{sec}). It can be therefore argued that the analytically computable delay margin established in this section is not overly conservative.

VI. CONCLUDING REMARKS

In this paper, the result in [11] is extended and robust adaptive control of general nth order plants with a scalar input in the presence of time-delays is established. As opposed to traditional robust adaptive control proofs most of which require Lyapunov based arguments, the dynamics of the state variables are closely investigated using first principles, thus resulting in a trajectory analysis denoted by four phases. This analysis ensured the global boundedness of the tracking error. As can be seen in the proof of Theorem 1, there are two crucial pieces of the proof that both involve the projection algorithm, a modification made to the standard adaptive law. The first criticality is the guarantee that the trajectory will enter the boundary region in a finite time (phase I). The projection algorithm then ensures that all signals remain bounded while $z$ is inside the boundary, which is the second crucial piece of the proof. These two crucial points helped to establish global boundedness of the challenging control problem discussed in this paper.

Needless to say, due to the introduction of a higher dimensional plant, additional complexities needed to be addressed. The proposed adaptive control law applies the projection algorithm on the transformed parameter state component-wise. The transformation involves a matrix $M$ which is constructed utilizing the direction of the input vector $b_p$.

Together with [11], these results clearly demonstrate that adaptive systems with plants whose state variables are accessible have a guaranteed delay margin, solving a non-trivial problem in this field.

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