Error Probability for M-ary Modulation using Hybrid Selection/Maximal-Ratio Combining in Rayleigh Fading

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Abstract— **In this paper, we derive *exact* expressions for the symbol error probability (SEP) of hybrid selection/maximal-ratio combining wireless systems in multipath fading environments. We consider coherent detection of several types of M-ary modulation for the case of independent Rayleigh fading on each diversity branch with equal signal-to-noise ratio averaged over the fading. We use a *virtual branch technique* which transforms the ordered physical branches, which are necessarily dependent, into independent and identically distributed virtual branches thereby permitting the derivation of *exact* SEP expressions.**

I. Introduction

Hybrid selection/maximal-ratio combining (H-S/MRC) is a diversity combining scheme where L out of N diversity branches are selected and combined using maximal-ratio combining (MRC). This technique provides improved performance over L branch MRC when additional diversity is available. Recently, H-S/MRC has been considered as an efficient means to combat multipath-fading [1], [2], [3]. The bit error rate (BER) performance of a H-S/MRC with $L = 2$ and $L = 3$ out of N branches was analyzed in [1], and it was pointed out that "the expressions become extremely unwieldy" for $L > 3$. The *average* signal-to-noise ratio (SNR) of H-S/MRC was derived in [2]. In [3], we introduced a "virtual branch" technique to succinctly derive the mean as well as the variance of the combiner output SNR of the H-S/MRC diversity system.

In this paper we extend [3] to derive analytical symbol error probability (SEP) expressions for several types of M-ary modulation with H-S/MRC for any L and N under the assumption of independent Rayleigh fading on each diversity branch with equal SNR averaged over the fading. A general form of the conditional SEP as a linear combination of terms involving a finite range integral is utilized. The proposed problem is made analytically tractable by transforming the physical diversity branches into the "virtual branch" domain which results in a simple derivation of the SEP for *arbitrary* L and N. Selection combining (SC) and MRC are shown to be special cases of our results and agree with previously published literature [4]. Numerical results are illustrated for quadrature phase shift keying (QPSK) and 16-ary quadrature amplitude modulation (16-QAM). Finally, remarks and conclusions are presented.

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II. Diversity Combining Analysis

A. Preliminaries

Let γ_i denote the instantaneous SNR of the $i^{\underline{\text{th}}}$ diversity branch defined by

$$\gamma_i \triangleq \alpha_i^2 \frac{E_s}{N_{0i}}, \tag{1}$$

where E_s is the average symbol energy, and α_i is the instantaneous fading amplitude and N_{0i} is the noise power spectral density of the $i^{\underline{\text{th}}}$ branch. We model the γ_i's as continuous random variables with probability density function (p.d.f.) $f_{\gamma_i}(x)$ and mean $\Gamma_i = \mathbb{E}\{\gamma_i\}$.

Let us consider a H-S/MRC diversity combining system with instantaneous output SNR of the form

$$\gamma_{\text{S/MRC}} = \sum_{i=1}^{L} \gamma_{(i)}, \tag{2}$$

where $\gamma_{(i)}$ is the ordered γ_i, i.e., $\gamma_{(1)} > \gamma_{(2)} > \ldots > \gamma_{(N)}$, N is the number of available diversity branches, and $1 \leq L \leq N$. Note that the possibility of at least two equal $\gamma_{(i)}$ is excluded, since $\gamma_{(i)} \neq \gamma_{(j)}$ *almost surely* for continuous random variables γ_i.[1]

For a Rayleigh fading channel, the p.d.f. of the α_i's is given by

$$f_{\alpha_i}(r) = \frac{2r}{\Omega_i} e^{-r^2/\Omega_i}, \qquad r \geq 0, \tag{3}$$

where $\Omega_i = \mathbb{E}\{\alpha_i^2\}$, and the p.d.f. of the instantaneous branch SNR is given by

$$f_{\gamma_i}(x) = \begin{cases} \frac{1}{\Gamma_i} e^{-\frac{x}{\Gamma_i}}, & 0 < x < \infty \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where the mean SNR of the $i^{\underline{\text{th}}}$ branch is $\Gamma_i = \mathbb{E}\{\gamma_i\} = \mathbb{E}\{\alpha_i^2\} \frac{E_s}{N_{0i}} = \Omega_i \frac{E_s}{N_{0i}}$. We assume that the γ_i's are

[1] In our context, the notion of "almost sure" or "almost everywhere" can be stated mathematically as: if $\mathcal{N} = \{\gamma_{(i)} = \gamma_{(j)}\}$, then $\Pr\{\mathcal{N}\} = 0$ [5], [6].

TABLE I

PARAMETERS FOR SPECIFIC MODULATION SCHEMES.

Modulation Scheme	K	$a_k(\theta)$	$\phi_k(\theta)$	Θ_k
MPSK	1	$\frac{1}{\pi}$	$\sin^2\left(\frac{\pi}{M}\right)\csc^2(\theta)$	$\pi\left(1-\frac{1}{M}\right)$
MQAM	2	$\frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)$	$\frac{3}{2(M-1)}\csc^2(\theta)$	$\frac{\pi}{2}$
		$-\frac{4}{\pi}\left(1-\frac{1}{\sqrt{M}}\right)^2$	$\frac{3}{2(M-1)}\csc^2(\theta)$	$\frac{\pi}{4}$
MPAM	1	$\frac{2}{\pi}\left(1-\frac{1}{M}\right)$	$\frac{3}{M^2-1}\csc^2(\theta)$	$\frac{\pi}{2}$
BFSK	1	$\frac{1}{\pi}$	$\frac{1}{2}\csc^2(\theta)$	$\frac{\pi}{2}$
BFSK_{min}	1	$\frac{1}{\pi}$	$\frac{1}{2}\left(1+\frac{2}{3\pi}\right)\csc^2(\theta)$	$\frac{\pi}{2}$
DE-BPSK	2	$\frac{2}{\pi}$	$\csc^2(\theta)$	$\frac{\pi}{2}$
		$-\frac{2}{\pi}$	$\csc^2(\theta)$	$\frac{\pi}{4}$
MSK	2	$\frac{2}{\pi}$	$\csc^2(\theta)$	$\frac{\pi}{2}$
		$-\frac{1}{\pi}$	$\csc^2(\theta)$	$\frac{\pi}{4}$

independent with equal average SNR, i.e., $\Gamma_i = \Gamma$ for $i = 1, \dots, N$, and thus $f_{\gamma_i}(x) = f(x)$ $\forall i$. Denoting $\gamma_{(N)} \triangleq (\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)})$, the joint p.d.f. of $\gamma_{(1)}, \gamma_{(2)}, \cdots, \gamma_{(N)}$ is [7]

$$f_{\gamma_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N}) = \begin{cases} \frac{N!}{\Gamma^N} e^{-\frac{1}{\Gamma}\sum_{m=1}^{N}\gamma_{(m)}}, & \gamma_{(1)} > \gamma_{(2)} > \cdots > \gamma_{(N)} > 0 \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

It is important to note that the $\gamma_{(i)}$'s are *no* longer independent, even though the underlying γ_i's are independent.

B. Virtual Branch Technique: The Key Idea

The analysis of H-S/MRC based on a chosen ordering of the branches at first appears to be complicated, since the SNR statistics of the ordered branches are *not* independent. Here, we alleviate this problem by transforming the ordered-branch variables into a new set of independent and identically distributed (i.i.d.) *virtual branches*, i.e., express the ordered-branch SNR variables as a linear function of i.i.d. virtual branch SNR variables. The key advantage of this formulation is that it permits the combiner output SNR to be expressed in terms of these i.i.d. virtual branch SNR variables. In this framework, the derivation of the SEP of H-S/MRC, involving the evaluation of nested N-fold integrals, reduces to the evaluation of a single integral.

C. SEP for M-ary Modulation with H-S/MRC

The SEP for H-S/MRC in a multipath-fading environment is obtained by averaging the conditional SEP over the channel ensemble. This can be accomplished by averaging $\Pr\{e|\gamma_{\text{S/MRC}}\}$ over the p.d.f. of $\gamma_{\text{S/MRC}}$ as

$$\begin{aligned} P_{e,\text{S/MRC}} &= \mathbb{E}_{\gamma_{\text{S/MRC}}}\{\Pr\{e|\gamma_{\text{S/MRC}}\}\} \\ &= \int_0^\infty \Pr\{e|\gamma\} f_{\gamma_{\text{S/MRC}}}(\gamma) d\gamma, \end{aligned} \tag{6}$$

where $\Pr\{e|\gamma_{\text{S/MRC}}\}$ is the *conditional* SEP, conditioned on the random variable $\gamma_{\text{S/MRC}}$, and $f_{\gamma_{\text{S/MRC}}}(\cdot)$ is the p.d.f. of the combiner output SNR.

For a general class of modulation schemes, the conditional SEP can be expressed as a specific form of the following expression,

$$\Pr\{e|\gamma_{\text{S/MRC}}\} = \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta)\, e^{-\phi_k(\theta)\,\gamma_{\text{S/MRC}}}\, d\theta, \tag{7}$$

where $a_k(\theta)$, $\phi_k(\theta)$ and Θ_k are parameters particular to the specific modulation format and are independent of the instantaneous SNR, $\gamma_{S/MRC}$. Table I lists these parameters for some common coherent modulations: M-ary phase shift keying (MPSK), M-ary square quadrature amplitude modulation (MQAM) with $M = 2^l$ and l even, M-ary pulse amplitude modulation (MPAM), binary frequency shift keying (BFSK), BFSK with minimum correlation ($\text{BFSK}_{\min}$), coherent detection of differentially encoded BPSK (DE-BPSK), and precoded minimum shift keying (MSK) [8], [9], [10].

Since MRC requires channel phase estimates, it is generally used in conjunction with coherent modulation schemes. If channel phase estimates are not available, then one may resort to diversity combining techniques such as postdetection equal gain combining (EGC) with noncoherent or differentially coherent modulation. Nevertheless, in situations that dictate predetection combining, the mathematical form given by (7) is general and valid for noncoherent modulation techniques as well as coherent. For example, with noncoherent differentially detected BPSK, $a_1(\theta) = \frac{1}{2}\delta(\theta)$ and $\phi_1(\theta) = 1$, while noncoherent detection of BFSK has $a_1(\theta) = \frac{1}{2}\delta(\theta)$ and $\phi_1(\theta) = \frac{1}{2}$ as parameters.

Evaluating the SEP by substituting (7) into (6) involves a single integration to average over the channel ensemble. However it requires knowledge of the p.d.f. of $\gamma_{S/MRC}$. Alternatively, averaging over the channel ensemble can be accomplished using the technique of [11], [12], by substituting the expression for $\gamma_{S/MRC}$ directly in terms of the physical branch variables given by (2) into (6), and together with (7) to give

$$\begin{aligned} P_{e,S/MRC} &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta)\, \mathbb{E}_{\{\gamma_{(i)}\}}\left\{e^{-\phi_k(\theta)\sum_{i=1}^{L}\gamma_{(i)}}\right\} d\theta \\ &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \int_0^{\infty}\int_0^{\gamma_{(1)}}\!\!\cdots\int_0^{\gamma_{(N-1)}} e^{-\phi_k(\theta)\sum_{i=1}^{L}\gamma_{(i)}} \\ &\quad \times\ f_{\gamma_{(N)}}(\{\gamma_{(i)}\}_{i=1}^{N})\, d\gamma_{(N)}\cdots d\gamma_{(2)} d\gamma_{(1)}\, d\theta . \end{aligned} \tag{8}$$

Since the statistics of the ordered-branches are *no* longer independent, the evaluation of (8) involves nested N-fold integrals, which are in general cumbersome and complicated to compute. This can be alleviated by transforming the instantaneous SNR of the ordered diversity branches, $\gamma_{(i)}$, into a new set of *virtual branch* instantaneous SNR's, V_n, using the following relationship:

$$\gamma_{(i)} = \sum_{n=i}^{N} \frac{\Gamma}{n} V_n . \tag{9}$$

It can be verified that the instantaneous SNR's of the virtual branches are i.i.d. normalized exponential random variables with p.d.f.'s given by

$$f_{V_n}(v) = \begin{cases} e^{-v}, & 0 < v < \infty \\ 0, & \text{otherwise}. \end{cases} \tag{10}$$

The characteristic function (c.f.) of V_n is given by the expression $\psi_{V_n}(j\nu) \triangleq \mathbb{E}\left\{e^{+j\nu V_n}\right\} = 1/(1 - j\nu)$. The instantaneous SNR of the combiner output can now be expressed in terms of the instantaneous SNR of the virtual branches as

$$\gamma_{S/MRC} = \sum_{n=1}^{N} b_n V_n , \tag{11}$$

where the coefficients b_n are given by

$$b_n = \begin{cases} \Gamma, & n \le L \\ \Gamma\frac{L}{n}, & \text{otherwise}. \end{cases} \tag{12}$$

Using the *independent* virtual branches, the N-fold nested integrals of (8) reduce to

$$\begin{aligned} P_{e,S/MRC} &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta)\, \mathbb{E}_{\{V_n\}}\left\{e^{-\phi_k(\theta)\sum_{n=1}^{N} b_n V_n}\right\} d\theta \\ &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \int_0^{\infty}\int_0^{\infty}\!\!\cdots\int_0^{\infty} e^{-\phi_k(\theta)\sum_{n=1}^{N} b_n V_n} \\ &\quad \times \prod_{n=1}^{N} f_{V_n}(V_n) dV_n\, d\theta . \end{aligned} \tag{13}$$

Exploiting the fact that the V_n's are independent, (13) becomes:

$$\begin{aligned} P_{e,S/MRC} &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \prod_{n=1}^{N} \mathbb{E}_{V_n}\left\{e^{-\phi_k(\theta) b_n V_n}\right\} d\theta \\ &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \prod_{n=1}^{N} \psi_{V_n}\left(-\phi_k(\theta) b_n\right) d\theta . \end{aligned} \tag{14}$$

Substituting the c.f. of V_n into (14) gives the final form of the SEP as

$$\begin{aligned} P_{e,S/MRC} &= \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \left[\frac{1}{1 + \phi_k(\theta)\Gamma}\right]^L \\ &\quad \times \prod_{n=L+1}^{N} \left[\frac{1}{1 + \phi_k(\theta)\Gamma\frac{L}{n}}\right] d\theta . \end{aligned} \tag{15}$$

Thus the derivation of the SEP for coherent detection of M-ary modulation using H-S/MRC, involving the N-fold nested integrals in (8), reduces to a single integral over θ with finite limits. The integrand is an N-fold product of a simple expression involving trigonometric functions. Note that the independence of the virtual branch variables plays a key role in simplifying the derivation.

III. Limiting Cases

A. Limiting Case 1: SC System

SC is the simplest form of diversity combining whereby the received signal from *one* of N diversity branches is selected [13]. The output SNR of SC is

$$\gamma_{SC} = \max_i \{\gamma_i\} = \gamma_{(1)} . \tag{16}$$

Note that SC is a limiting case of H-S/MRC with $L = 1$. Substituting $L = 1$ into (15), the SEP with SC becomes

$$P_{e,SC} = \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \prod_{n=1}^{N} \left[\frac{1}{1 + \phi_k(\theta)\Gamma\frac{1}{n}}\right] d\theta . \tag{17}$$

B. Limiting Case 2: MRC System

In MRC, the received signals from *all* diversity branches are weighted and combined to maximize the SNR at the combiner output [13]. The output SNR of MRC is

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{N} \gamma_i = \sum_{i=1}^{N} \gamma_{(i)} \,. \tag{18}$$

Note that MRC is a limiting case of H-S/MRC with $L = N$. Substituting $L = N$ into (15), the SEP with MRC is

$$P_{e,\mathrm{MRC}} = \sum_{k=1}^{K} \int_0^{\Theta_k} a_k(\theta) \left[\frac{1}{1 + \phi_k(\theta)\Gamma} \right]^N d\theta \,. \tag{19}$$

IV. Numerical Results

In this section, we illustrate the SEP results derived for H-S/MRC. The notation H-L/N is used to denote H-S/MRC that selects and combines L out of N branches. Note that H-1/1 is a single branch receiver, and H-1/N and H-N/N are N-branch SC and N-branch MRC, respectively.

Figure 1 shows the SEP for coherent detection of MPSK with M=4 (QPSK) versus the average SNR per branch for various L with $N = 4$. Note that SC and MRC upper and lower bound, respectively, the SEP for H-S/MRC. It is seen that most of the gain of H-S/MRC is achieved for small L, e.g., the SEP for H-S/MRC is within 1 dB of MRC when $L = N/2$.

Figure 2 shows the SEP for coherent detection of QPSK versus the average SNR per branch for various N with $L = 2$. Note that, although the incremental gain with each additional combined branch becomes smaller as N increases, the gain is still significant even with $N = 8$. Furthermore, for $L = 2$ at a 10^{-3} SEP, H-S/MRC with $N = 8$ requires about 10 dB lower SNR than 2-branch MRC.

Similar results for coherent detection of MQAM with M=16 (16-QAM) are plotted in Figs. 3-4. These results show the same characteristics as QPSK illustrated in Figs. 1-2.

V. Conclusions

We derived exact SEP expressions for coherent detection of several types of M-ary modulation with H-S/MRC in multipath-fading wireless environments. A general expression was derived in terms of the parameters of the specific modulation schemes. With H-S/MRC, L out of N diversity branches are selected and combined using MRC. This technique provides improved performance over L branch MRC when additional diversity is available. We considered independent Rayleigh fading on each diversity branch with equal SNR's, averaged over the fading. We analyzed this system using a "virtual branch" technique which resulted in a simple derivation of the SEP for *arbitrary* L and N.

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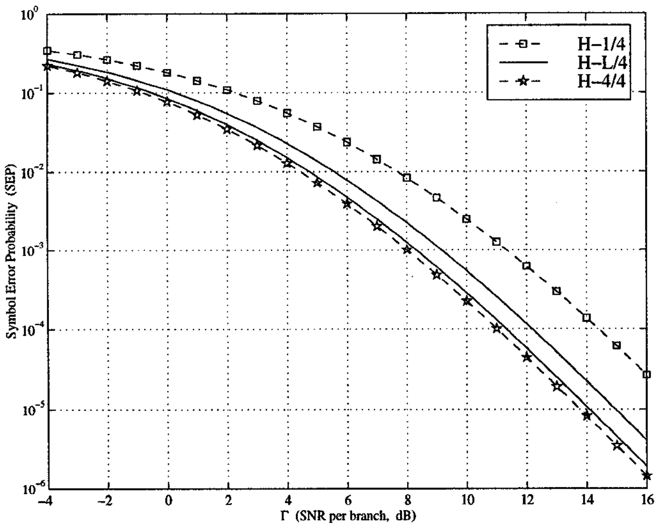

Fig. 1. The symbol error probability for coherent detection of QPSK with H-S/MRC as a function of the average SNR per branch in dB for various L with $N = 4$. The curves are parameterized by different L starting from the upper curve representing H-1/4, to the lowest curve representing H-4/4.

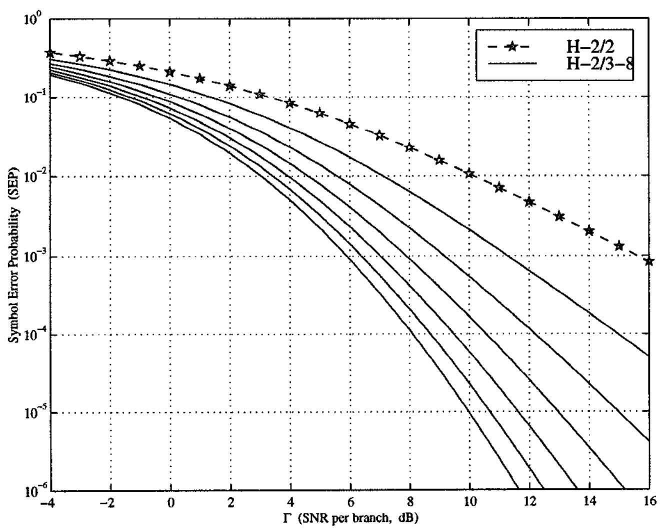

Fig. 2. The symbol error probability for coherent detection of QPSK with H-S/MRC as a function of the average SNR per branch in dB for various N with $L = 2$. The curves are parameterized by different N starting from the upper curve representing H-2/2, to the lowest curve representing H-2/8.

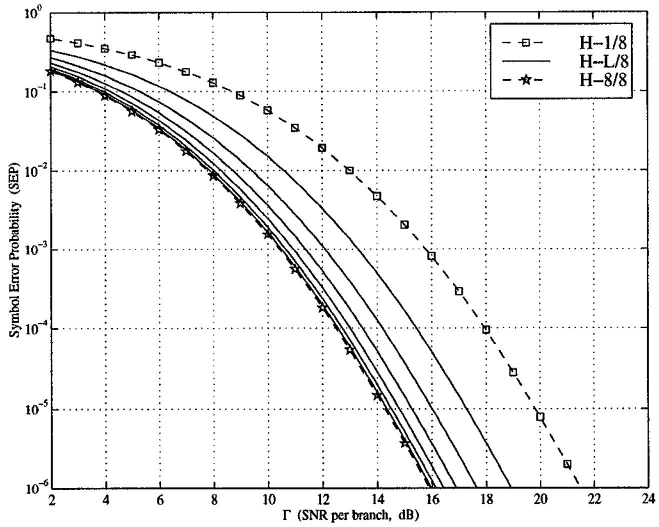

Fig. 3. The symbol error probability for coherent detection of 16-QAM with H-S/MRC as a function of the average SNR per branch in dB for various L with $N = 8$. The curves are parameterized by different L starting from the upper curve representing H-1/8, to the lowest curve representing H-8/8.

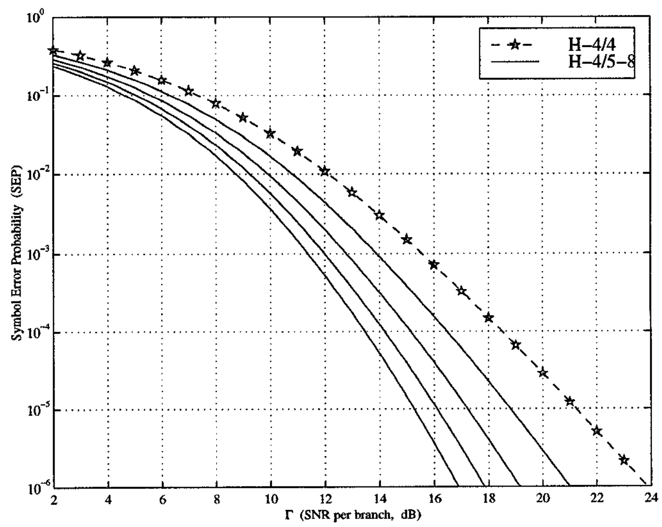

Fig. 4. The symbol error probability for coherent detection of 16-QAM with H-S/MRC as a function of the average SNR per branch in dB for various N with $L = 4$. The curves are parameterized by different N starting from the upper curve representing H-4/4, to the lowest curve representing H-4/8.