Statistical Methodology for Approximating $G/G/1$ Queues by the Strong Stability Technique

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Abstract: We consider a statistical methodology for the study of the strong stability of the $M/G/1$ queueing system after disrupting the arrival flow. More precisely, we use nonparametric density estimation with boundary correction techniques and the statistical Student test to approximate the $G/G/1$ system by the $M/G/1$ one, when the general arrivals law G in the $G/G/1$ system is unknown. By elaborating an appropriate algorithm, we effectuate simulation studies to provide the proximity error between the corresponding arrival distributions of the quoted systems, the approximation error on their stationary distributions and confidence intervals for the difference between their corresponding characteristics.

1 INTRODUCTION

In queueing theory performance evaluation may be challenging task, for example, in the $G/G/1$ queueing system, the Laplace transform and the generation function are not available in closed form (Kleinrock, 1975). For this reason there exists, when a practical study is performed in queueing theory, a common technique for substituting the real but complicated elements governing a queueing system by simpler ones in some sense close to the real elements. The queueing model so constructed represents an idealization of the real queueing one, and hence the stability problem arises. The stability problem in queueing theory is concerned with the domain within which the ideal queueing model may be taken as a good approximation of the real queueing system under consideration. In other words, we clarify the conditions for which the proximity in one way or another of the parameters of the system involves the proximity of the studied characteristics.

On the other hand, note that in practice all model parameters are imprecisely known because they are obtained by means of statistical methods. Such circumstances suggest to seek qualitative properties of the real system, i.e., the manner in which the system is affected by the changes in its parameters. These qualitative properties include invariance, monotonicity and stability. Its by means of qualitative properties that bounds can be obtained mathematically and approximations can be made rigorously (Stoyan, 1983).

Even if the concept of stability is the same in a general way, several approaches of the problem have been elaborated. This led to the diversity of the definitions and the methods of stability (Borovkov, 1984; Rachev, 1989). Moreover, there is a significant body of literature on perturbation bounds of Markov chains. One group of results uses the series expansion approach and the methods of matrix analysis (Heidergott and Hordijk, 2003; Heidergott et al., 2007). Another group employs the theory of operators and probabilistic methods (Aïssani and Kartashov, 1983; Kartashov, 1996; Rachev, 1989).

In this work we will place more emphasis on the strong stability method (Aïssani and Kartashov, 1983; Kartashov, 1996) which allows us to make both qualitative and quantitative analysis helpful in understanding complicated models by more simpler ones for which an evaluation can be made. This method, also called "method of operators" can be used to investigate the ergodicity and stability of the stationary and non-stationary characteristics of the imbedded Markov chains (Aïssani and Kartashov, 1984; Kartashov, 1996). In contrast to other methods, it supposes that the perturbations of the transition kernel are small with respect to some norms in the operators space. This stringent condition gives better stability estimates and enables us to find precise asymptotic expansions of the characteristics of the perturbed system.

Note that the first attempt to "measure" the performance of the strong stability method has been used

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in practice, and has been particularly applied to a simple system of queues (Bouallouche and Aïssani, 2006; Bouallouche and Aïssani,). The approach proposed is based on the classical approximation method where the authors perform the numerical proximity of the stationary distribution of an $Hyp/M/1$ (respectively $M/Cox2/1$) system by the one of an $M/M/1$ system when applying the strong stability method. For the first time, Bareche and Aïssani (Bareche and Aïssani, 2008) specify an approximation error on the stationary distributions of the $G/M/1$ (resp. $M/G/1$) and $M/M/1$ systems when the general law of arrivals (resp. service times) G is unknown and its density function is estimated by using the kernel density method. In (Berdjoudj et al., 2012), the authors use the discrete event simulation approach and the student test to measure the performance of the strong stability method through simple numerical examples for a concrete case of queueing systems (the $G/M/1$ queue after perturbation of the service law (Benaouicha and Aïssani, 2005), and the $M/G/1$ limit model for high retrial intensities (which is the classical $M/G/1$ system) after perturbation of the retrials parameter (Berdjoudj and Aïssani, 2003)). The same idea has been already investigated for an approximation analysis of the classical $G/G/1$ queue when the general law of service is unknown and must be estimated by different statistical methods, pointing out particularly the impact of those taking into account the correction of boundary effects (Bareche and Aïssani, 2011), see also the recent work of (Bareche and Aïssani, 2013).

Indeed, note that in practice all model parameters are imprecisely known because they are obtained by means of statistical methods. In this sense, our contribution concerns one aspect which is of some practical interest and has not sufficiently studied in the literature, for instance when a distribution governing a queueing system is unknown and we resort to nonparametric methods to estimate its density function. Besides, as the strong stability method assumes that the perturbation is small, then we suppose that the arrivals law of the $G/G/1$ system is close to the exponential one with parameter λ. This permits us to consider the problem of boundary bias correction (Bouezmarni and Scaillet, 2005; Chen, 2000; Schuster, 1985) when performing nonparametric estimation of the unknown density of the law G, since the exponential law is defined on the positive real line.

It is why we use, in this paper, the tools of nonparametric density estimation to approximate the complex $G/G/1$ system by the simpler $M/G/1$ one, on the basis of the theoretical results addressed in (Aïssani and Kartashov, 1984) involving the strong stability of the $M/G/1$ system.

This article is organized as follows: In Section 2, we describe the considered queueing models and we present briefly the strong stability of the $M/G/1$ system. A review of boundary bias correction techniques in nonparametric density estimation is given in Section 3. The main new results of this paper are presented in Section 4, which shows the interest of combining these nonparametric methods with the strong stability principle for the study of the $M/G/1$ system. It also points out the importance of using the Student test to compare the characteristics of the two considered queueing systems, and presents a numerical case study based on simulation results.

2 STRONG STABILITY OF THE M/G/1 SYSTEM AFTER PERTURBATION OF THE ARRIVAL FLOW

2.1 Description of M/G/1 and G/G/1 Models

Consider a $G/G/1$ $(FIFO,\infty)$ queueing system with general service times distribution H and general inter-arrival times probability distribution G. The following notations are used: T_n (the arrival time of the n^{th} customer), θ_n (the departure time of the n^{th} customer), and γ_n (the time till the arrival of the following customer after θ_n). Let us designate by $\nu_n = \nu(\theta_n + 0)$ the number of customers in the system immediately after θ_n. ξ_n represents the service time of the n^{th} customer arriving at the system. It is proved that $X_n = (\nu_n, \gamma_n)$ forms a homogeneous Markov chain with state space $\mathbb{N} \times \mathbb{R}^+$ and transition operator $Q = (Q_{ij})_{i,j\geq 0}$, where $Q_{ij}(x,dy) = P(\nu_{n+1} = j, \gamma_{n+1} \in dy / \nu_n = i, \gamma_n = x)$, defined by (see (Aïssani and Kartashov, 1984)):

$$Q_{ij} = \begin{cases} q_j(dy), & \text{if } i = 0; \\ q_{j-i}(x,dy), & \text{if } i \geq 1, \ \ j \geq i; \\ p(x,dy), & \text{if } j = i-1, \ \ i \geq 1; \\ 0, & \text{otherwise}; \end{cases} \tag{1}$$

where

$$\begin{cases} q_j(dy) = \int P(T_j \leq u < T_{j+1}, T_{j+1} - u \in dy) dH(u); \\ q_j(x,dy) = \int_x^\infty P(T_j \leq u - x < T_{j+1}, T_{j+1} - (u-x) \in dy) dH(u); \\ p(x,dy) = \int_0^x P(x - u \in dy) dH(u). \end{cases}$$

Let us also consider an $M/G/1$ $(FIFO,\infty)$ system and denote by E_λ the inter-arrivals distribution (E_λ is an exponential distribution with parameter λ), and take the same distribution of service times than the $G/G/1$ one. We introduce the corresponding following notations: $\bar{T}_n$, $\bar{\theta}_n$, $\bar{\gamma}_n$, $\bar{\nu}_n = \bar{\nu}(\bar{\theta}_n - 0)$ and ξ_n defined as above. The transition operator $\bar{Q} = \left(\bar{Q}_{ij}\right)_{i,j>0}$

of the corresponding Markov chain $\bar{X}_n = (\bar{\nu}_n, \bar{\gamma}_n)$ in the $M/G/1$ system has the same form as in (1), where

$$\begin{cases} \overline{q}_j(dy) = p_j E_\lambda(dy), \ \ \overline{q}_j(x,dy) = p_j(x)E_\lambda(dy), \\ \overline{p}(x,dy) = p(x,dy); \\ p_j = \int exp(-\lambda u)\frac{(\lambda u)^j}{j!}dH(u); \\ p_j(x) = \int_x^\infty \exp(-\lambda(u-x))\frac{(\lambda(u-x))^j}{j!}dH(u). \end{cases}$$

Let us suppose that the arrival flow of the $G/G/1$ system is close to the Poisson one. This proximity is then characterized by the metric:

$$w^* = w^*(G, E_\lambda) = \int \varphi^*(t)|G - E_\lambda|(dt), \qquad (2)$$

where $|a|$ designates the variation of the measure a and φ^* is a weight function verifying the following conditions:

- φ^* is non decreasing;
- $\varphi^*(t+s) \leq \varphi^*(t)\cdot\varphi^*(s), \forall t,s \in \mathbb{R}^+$;
- $\varphi^*(0) = 1$.

We take $\varphi^*(t) = e^{\delta t}$, with $\delta > 0$. In addition, we use the following notations:

$$E^* = \int \varphi^*(t)E_\lambda(dt), \qquad (3)$$

$$G^* = \int \varphi^*(t)G(dt), \qquad (4)$$

$$w_0 = w_0(G, E_\lambda) = \int |G - E_\lambda|(dt). \qquad (5)$$

2.2 Approximation of the G/G/1 System by the M/G/1 One

In this section, we introduce some necessary notations and recall the basic theorem of the strong stability adapted to the studied case. For a general framework see (Aïssani and Kartashov, 1983; Kartashov, 1996; Benaouicha and Aïssani, 2005). In the sequel, when no domain of integration is indicated, an integral is extended over $\mathbb{R}^+$.

Consider the σ-algebra $\mathcal{E}$, which represents the product $\mathcal{E}_1 \otimes \mathcal{E}_2$ ($\mathcal{E}_1$ is the σ-algebra generated by the countable partition of $\mathbb{N}$ and $\mathcal{E}_2$ is the Borel σ-algebra of $\mathbb{R}^+$).

We denote by $m\mathcal{E}$ the space of finite measures on $\mathcal{E}$, and we introduce the special family of norms of the form $\|m\|_v = \sum_{j\geq 0} \int v(j,y)|m_j|(dy)$, $\forall m \in m\mathcal{E}$, where v is a measurable function on $\mathbb{N}\times\mathbb{R}^+$, bounded below away from zero (not necessarily finite).

This norm induces a corresponding norm in the space $f\mathcal{E}$ of bounded measurable functions on $\mathbb{N}\times\mathbb{R}^+$, namely $\|f\|_v = \sup_{k\geq 0}\sup_{x\geq 0}[v(k,x)]^{-1}|f(k,x)|$, $\forall f \in f\mathcal{E}$, as well as a norm in the space of linear operators, namely $\|P\|_v = \sup_{k\geq 0}\sup_{x\geq 0}[v(k,x)]^{-1}\sum_{j\geq 0}\int v(j,y)|P_{kj}(x,dy)|$.

We associate to each transition kernel P the linear mapping $P : f\mathcal{E} \to f\mathcal{E}$ acting on $f \in f\mathcal{E}$ as follows, $(Pf)(k,x) = \sum_{j\geq 0}\int P_{kj}(x,dy)f(j,y)$. For $m \in m\mathcal{E}$ and $f \in f\mathcal{E}$ the symbol mf denotes the integral $mf = \sum_{j\geq 0}\int m_j(dx)f(j,x)$, and $f \circ m$ denotes the transition kernel having the form $(f\circ m)_{ij}(x,A) = f(i,x)m_j(A)$.

Remark 2.1. *Using the norm defined on the space of finite measures introduced in the current section, we can characterize the proximity of the two systems $G/G/1$ and $M/G/1$ by*

$$\|G - E_\lambda\|_v = \sum_{j\geq 0}\int_0^{+\infty} v(j,t)|G - E_\lambda|(dt). \qquad (6)$$

Taking in (6), $v(j,t) = \begin{cases} e^{\delta t}, & if \ \ j = 0, \\ 0, & if \ \ j \neq 0, \end{cases}$
where $\delta > 0$, one recovers the variational norm defined in (2).

Moreover, taking in (6), $v(j,t) = \begin{cases} 1, & if \ \ j = 0, \\ 0, & if \ \ j \neq 0, \end{cases}$
one recovers the variational norm defined in (5).

Defintion 2.1. *(see (Aïssani and Kartashov, 1983; Kartashov, 1996)) A Markov chain X with transition kernel P and invariant measure π is said to be strongly υ-stable with respect to the norm $\|.\|_\upsilon$, if $\|P\|_\upsilon < \infty$ and each stochastic kernel Q in some neighborhood $\{Q : \|Q - P\|_\upsilon < \varepsilon\}$ has a unique invariant measure $\mu = \mu(Q)$ and $\|\pi - \mu\|_\upsilon \to 0$ as $\|Q - P\|_\upsilon \to 0$.*

The following theorem determines the strong v-stability conditions of the $M/G/1$ system after a small perturbation of the arrivals law. It also gives the estimates of the deviations of both the transition kernels and the stationary distributions.

Theorem 2.1. *(Aïssani and Kartashov, 1984) Suppose that in the $M/G/1$ system, the following ergodicity condition holds:*
a) $\lambda\, \mathbf{E}(\xi) < 1$; **b)** $\exists a > 0$: $\mathbf{E}(e^{a\xi}) = \int e^{au}dH(u) < \infty$, *where ξ is a random variable representing the service times.*
Suppose also that $E^* < \infty$ and $\beta_0 = sup(\beta : H^*(\lambda - \lambda\beta) < \beta)$, where H^* is the Laplace transform of the probability density of the service times. Then, for all β such that $1 < \beta < \beta_0$, the Markov chain $\overline{X}_n$ is strongly $v-$stable for the function $v(n,t) = \beta^n[\exp(-\alpha t) + c^{-1}\varphi^*(t)]$, where:

$$\alpha > 0, \ \ c = \frac{\beta E^*}{1-\rho}, \ \ and \ \ \rho = \frac{H^*(\lambda - \lambda\beta) + \beta}{2\beta} < 1.$$

In addition, if $G^* < \infty$, and $w_0 \leq \frac{(\beta_0 - \beta)}{\beta_0^2}$, then we have *the margin between the transition operators:*

$$\|Q-\overline{Q}\|_v \leq w^*(1+\beta)+w_0 G^*(1+\lambda\beta)\frac{\beta_0^4}{(\beta_0-\beta)^2},$$

where E^* and G^* are defined respectively in (3) and (4).
Moreover, if the general distribution of arrivals G is such that:

$$w^*(G,E_\lambda) \leq \frac{1-\rho}{2c_0(1+c)}(1+\beta+c_1)^{-1},$$

$$w_0(G,E_\lambda) \leq \frac{(\beta_0-\beta)}{\beta_0^2},$$

we obtain the deviation between the stationary distributions π and $\bar{\pi}$ associated, respectively, to the Markov chains X_n and $\bar{X}_n$, given by:

$$Er := \|\pi-\bar{\pi}\| \leq 2[(1+\beta)w^*+c_1 w_0]c_0 c_2(1+c), \quad (7)$$

where c_0, c_1, c_2 are defined as follows:

$$c_0^{'} \leq c_0,$$

where $c_0^{'} = 1+\frac{(1-\lambda m)(\beta-1)(2-\rho)E^*}{2(1-\rho)^2}$ *and* $m = \mathbf{E}(\xi)$,

$$c_1 = G^*(1+\lambda\beta)\frac{\beta_0^4}{(\beta_0-\beta)^2},$$

$$c_2 = \frac{(1-\lambda m)(\beta-1)(2-\rho)}{2(1-\rho)\beta}.$$

Remark 2.2. *The assumptions* **a)** *and* **b)** *of Theorem 2.1 imply the existence of a stationary distribution $\bar{\pi}$ for the imbedded Markov chain $\bar{X}_n$ in the $M/G/1$ system. This distribution has the following form:*

$\bar{\pi}(\{k\},A) = \bar{\pi}_k(A) = p_k E_\lambda(A),\ \forall\ \{k\} \subset \mathbb{N}$ *and* $A \subset \mathbb{R}^+$,
where $p_k = \lim_{n\to\infty} P(\bar{V}_n = k) = \lim_{t\to\infty} P(X(t)=k)$, *where, $X(t)$ represents the size of the $M/G/1$ system at time t.*
In general it is not possible to have stationary distribution $\lim_{t\to\infty} P(X(t)=k)$ of the $M/G/1$ system. However, one can compute its corresponding generating function $\Pi(z)$ given by (Kleinrock, 1975):

$$\Pi(z) = H^*(\lambda-\lambda z)\frac{(1-\rho)(1-z)}{H^*(\lambda-\lambda z)-z}, \quad (8)$$

where $\rho = \lambda\,\mathbf{E}(\xi)$, H^* represents the Laplace transform of the probability density of the service time ξ, and z is a complex number verifying $|z| \leq 1$. This formula is called Pollaczek-Khinchin formula. Its inversion allows us to find the stationary distribution Π.
Formula (8) permits us to compute the stationary distribution of the queue length in a $M/G/1$ system. Unfortunately, for the $G/G/1$ system, this exact formula is not known. So, if we suppose that the $G/G/1$ system is close to the $M/G/1$ system, then we can use formula (8) to approximate the $G/G/1$ system characteristics with prior estimation of the corresponding approximation error.

Given the bound in formula (7) in Theorem 2.1, it remains to compute w^* and w_0 and methods to do so will be discussed in the following.

3 BOUNDARY CORRECTION TECHNIQUES IN NONPARAMETRIC DENSITY ESTIMATION

Different standard types of nonparametric density estimate are performed. For a survey, see the monograph by Silverman (Silverman, 1986). The most known and used nonparametric estimation method is the kernel density estimate. If $X_1,\dots,X_n$ is a sample coming from a random variable X with probability density function f and distribution F, then the Parzen-Rosenblatt kernel estimator (Parzen, 1962; Rosenblatt, 1956) of the density $f(x)$ for each point $x \in \mathbb{R}$ is given by:

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K\left(\frac{x-X_j}{h_n}\right), \quad (9)$$

where K is a symmetric density function called the kernel and h_n is the bandwidth.

Even if the choice of the kernel K is not of a great importance, there would still remain the question of which window width h_n to choose. The problem of bandwidth selection has been extensively studied (for a survey, see (Jones et al., 1996)). The classical symmetric kernel estimate works well when estimating densities with unbounded support. However, when these latter are defined on the positive real line $[0,\infty[$, without correction, the kernel estimates suffer from boundary effects since they have a boundary bias (the expected value of the standard kernel estimate at $x=0$ converges to the half value of the underlying density when f is twice continuously differentiable on its support $[0,+\infty)$ (Bouezmarni and Scaillet, 2005; Schuster, 1985)). In fact, using a fixed symmetric kernel is not appropriate for fitting densities with bounded supports as a weight is given outside the support.

Several approaches have been introduced to get a better estimation on the border. Some of them proposed the use of particular kernels or bandwidths (Schuster, 1985), other techniques propose the use of estimators based on flexible kernels (asymmetric kernels (Bouezmarni and Scaillet, 2005; Chen, 2000) and smoothed histograms (Bouezmarni and Scaillet, 2005)). They are very simple in implementation, free of boundary bias, always nonnegative, their support matches the support of the probability density function to be estimated, and their rate of convergence for the mean integrated squared error is $O(n^{-4/5})$. Below, are briefly discussed the estimators which we will use in the context of this paper.

3.1 Reflection Method

Schuster (Schuster, 1985) suggests creating the mirror image of the data in the other side of the boundary and then applying the estimator (9) for the set of the initial data and their reflection. $f(x)$ is then estimated, for $x \geq 0$, as follows:

$$\tilde{f}_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} [K(\frac{x - X_j}{h_n}) + K(\frac{x + X_j}{h_n})]. \qquad (10)$$

3.2 Asymmetric Gamma Kernel Estimator

A simple idea for avoiding boundary effects is using a flexible kernel, which never assigns a weight out of the support of the density function and which corrects the boundary effects automatically and implicitly. The first category of the flexible kernels consists of the asymmetric kernels (Bouezmarni and Scaillet, 2005; Chen, 2000) defined by the form

$$\hat{f}_b(x) = \frac{1}{n} \sum_{i=1}^{n} K(x,b)(X_i), \qquad (11)$$

where b is the bandwidth and the asymmetric kernel K can be taken as a Gamma density K_G with the parameters $(x/b+1, b)$ given by

$$K_G(\frac{x}{b}+1, b)(t) = \frac{t^{x/b} e^{-t/b}}{b^{x/b+1}\Gamma(x/b+1)}. \qquad (12)$$

3.3 Smoothed Histograms

The second category of flexible kernels consists of smoothed histograms (Bouezmarni and Scaillet, 2005) defined by the form

$$\hat{f}_k(x) = k \sum_{i=0}^{+\infty} \omega_{i,k} p_{ki}(x), \qquad (13)$$

where the random weights $\omega_{i,k}$ are given by

$$\omega_{i,k} = F_n(\frac{i+1}{k}) - F_n(\frac{i}{k}),$$

where F_n is the empiric distribution, k is the smoothing parameter and $p_{ki}(.)$ can be taken as a Poisson distribution with parameter kx,

$$p_{ki}(x) = e^{-kx} \frac{(kx)^i}{i!}, \qquad i = 0, 1, \ldots \qquad (14)$$

4 STATISTICAL TECHNIQUES FOR THE APPROXIMATION OF THE $G/G/1$ SYSTEM BY THE $M/G/1$ ONE

We want to apply nonparametric density estimation methods to determine the variation distances w_0 and w^* defined respectively in (2) and (5), the proximity error Er defined in (7) between the stationary distributions of the $G/G/1$ and $M/G/1$ systems and confidence intervals for the difference between the characteristics of the considered systems in the stationary state. We use the discrete event simulation approach (Banks et al., 1996) to simulate the according systems and we apply the Student test for the acceptance or rejection of the equality of the corresponding characteristics.

Note by Σ_1 and Σ_2 the $G/G/1$ and $M/G/1$ systems respectively. Let us repeat the simulation of the system $\Sigma_i (i = 1, 2)$ R_i times. Note by $\theta_i^{(j)}$ the theoretical value of the jth characteristic of Σ_i. At the rth repetition of the simulation of the system Σ_i, one obtains the estimate $Y_{ri}^{(j)}$ of the characteristic $\theta_i^{(j)}$. Suppose that the estimators $Y_{ri}^{(j)}$ are unbiased. This implies that $\theta_i^{(j)} = \mathbf{E}(Y_{ri}^{(j)}),\ r = 1, 2, \ldots, R_i,\ i = 1, 2.$

For the comparison of the two queueing systems Σ_1 and Σ_2, we take the number of simulations $R_1 = R_2 = R$ large enough. With a significance level α, the confidence interval for $\theta_1^{(j)} - \theta_2^{(j)}$ is given by the following form:

$$\begin{aligned} (\bar{y}_1^{(j)} - \bar{y}_2^{(j)}) - t_{(\alpha/2\,;\,\nu)}\sigma(\bar{y}_1^{(j)} - \bar{y}_2^{(j)}) &\leq \theta_1^{(j)} - \theta_2^{(j)} \\ &\leq (\bar{y}_1^{(j)} - \bar{y}_2^{(j)}) + t_{(\alpha/2\,;\,\nu)}\sigma(\bar{y}_1^{(j)} - \bar{y}_2^{(j)}), \end{aligned} \qquad (15)$$

where $\bar{y}_i^{(j)} = \frac{1}{R} \sum_{r=1}^{R} Y_{r_i}^{(j)},\ i = 1, 2,$ $\sigma(\bar{y}_1^{(j)} - \bar{y}_2^{(j)})$ is the standard error of the punctual specified estimator, $\nu = 2(R-1)$ is the number of degrees of freedom, and $t_{(\alpha/2\,;\,\nu)}$ is a value to be taken on the Student table.

To take a decision on how significant is the difference between the tow systems, we check if the confidence interval for $\theta_1^{(j)} - \theta_2^{(j)}$ contains the zero value or not. We have the following conclusions:

1. If the confidence interval for $\theta_1^{(j)} - \theta_2^{(j)}$ does not contain the zero (i.e. it is totally on the left-hand side or on the right-hand side of zero), then it is extremely probable that $\theta_1^{(j)} < \theta_2^{(j)}$ or $\theta_1^{(j)} > \theta_2^{(j)}$ respectively.

2. If the confidence interval for $\theta_1^{(j)} - \theta_2^{(j)}$ contains the zero, then there is no statistical obviousness affirming that the jth characteristic of one of the systems is better than that of the other.

Table 1: Performance measures with different estimators.

	$g(x)$	$g_n(x)$	$\tilde{g}_n(x)$	$\hat{g}_b(x)$	$\hat{g}_k(x)$
Mean arrival rate λ	1.6874	1.5392	1.6503	1.6851	1.6840
Traffic intensity of the system $\frac{\lambda}{\mu}$	0.1562	0.1578	0.1570	0.1564	0.1567
Variation distance w_0	0.0096	0.1287	0.0114	0.0102	0.0105
Variation distance w^*	0.0183	0.2536	0.0311	0.0206	0.0224
Error on stationary distributions Er	0.0356		0.0452	0.0378	0.0377

In the context of this paper, we consider the following characteristics:

- $\bar{n}_i, i = 1,2$, mean number of customers in the system i.
- $\bar{\omega}_i, i = 1,2$, output rate in the system i.
- $\bar{t}_i, i = 1,2$, sojourn mean time of a customer in the system i (response time of the system).
- $\bar{\rho}_i, i = 1,2$, occupation rate of the system i.

To realize this work, we elaborated an algorithm which follows the following steps:

1) Generation of a sample of size n of general arrivals distribution G with theoretical density $g(x)$.

2) Use of a nonparametric estimation method to estimate the theoretical density function $g(x)$ by a function denoted in general $g_n^*(x)$.

3) Calculation of the mean arrival rate given by: $\lambda = 1/\int x dG(x) = 1/\int x g(x)dx = 1/\int x g_n^*(x)dx$.

4) Verification, in this case, of the strong stability conditions given in the subsection 2.2. For calculation considerations, the variation distances w_0 and w^* are given respectively by: $w_0 = \int |G - E_\lambda|(dx) \simeq \int |g_n^* - e_\lambda|(x)dx$ and $w^* = \int e^{\delta x}|G - E_\lambda|(dx) \simeq \int e^{\delta x}|g_n^* - e_\lambda|(x)dx$, where $\delta > 0$.

5) Computation of the minimal error on the stationary distributions of the considered systems according to (7).

6) Application of the Student test to determine the difference between the corresponding characteristics of the considered systems according to (15).

Simulation studies were performed under the Matlab 7.1 environment. The Epanechnikov kernel (Silverman, 1986) is used throughout for estimators involving symmetric kernels. The bandwidth h_n is chosen to minimize the criterion of the "least squares cross-validation" (Jones et al., 1996). The smoothing parameters b and k are chosen according to a bandwidth selection method which leads to an asymptotically optimal window in the sense of minimizing L_1 distance (Bouezmarni and Scaillet, 2005).

4.1 Simulation Study

We consider a $G/G/1$ system such that the general inter-arrivals distribution G is assumed to be a Gamma distribution with parameters $\alpha = 0.7, \beta = 2$, denoted $\Gamma(0.7,2)$, with a theoretical density $g(x)$ and the service times distribution is Cox2 with parameters: $\mu_1 = 3$, $\mu_2 = 10$, $a = 0.005$.

By generating a sample coming from the $\Gamma(0.7,2)$ distribution, we use the different nonparametric estimators given respectively in (9)-(14) to estimate the theoretical density $g(x)$

For these estimators, we take the sample size $n = 200$ and the number of simulations $R = 100$. For the construction of confidence intervals using the Student test, we take the significance level $\alpha = 0.05$. Hence the number of degrees of freedom $\nu = 198$ and $t_{(\alpha/2\ ;\ \nu)} = t_{(0.025\ ;\ 198)} = 1.96$.

Curves of the theoretical and estimated densities are illustrated in Figure 1. Different performance measures are listed in Table 1.

The confidence intervals are shown respectively in Tables 2, 3, 4, 5 and 6, by giving their lower and upper bounds.

Table 2: Confidence intervals with theoretical density $g(x)$.

Characteristics difference	Lower bound	Upper bound
$\bar{n}_1 - \bar{n}_2$	-0.0204	0.0098
$\bar{\omega}_1 - \bar{\omega}_2$	-0.0120	0.0016
$\bar{t}_1 - \bar{t}_2$	-0.0523	0.0273
$\bar{\rho}_1 - \bar{\rho}_2$	-0.0087	0.0025

Table 3: Confidence intervals with Parzen-Rosenblatt estimator $g_n(x)$.

Characteristics difference	Lower bound	Upper bound
$\bar{n}_1 - \bar{n}_2$	-0.0394	-0.0120
$\bar{\omega}_1 - \bar{\omega}_2$	-0.0305	-0.0106
$\bar{t}_1 - \bar{t}_2$	-0.0923	-0.0416
$\bar{\rho}_1 - \bar{\rho}_2$	-0.0637	-0.0398

Discussion: Figure 1 shows that the use of nonparametric density estimation methods taking into account the correction of boundary effects improves the quality of the estimation (compared to the curve of of the Parzen-Rosenblatt estimator, those of mirror image, asymmetric Gamma kernel and smoothed histogram

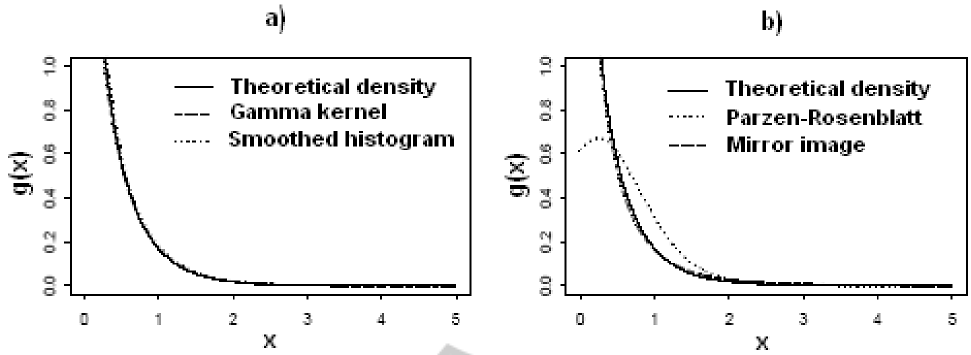

Figure 1: Theoretical density $g(x) = \Gamma(0.7, 2)(x)$, and estimated densities.

Table 4: Confidence intervals with mirror image estimator $\tilde{g}_n(x)$.

Characteristics difference	Lower bound	Upper bound
$\bar{n}_1 - \bar{n}_2$	-0.0389	0.0089
$\bar{\omega}_1 - \bar{\omega}_2$	-0.0314	0.0122
$\bar{t}_1 - \bar{t}_2$	-0.0565	0.0697
$\bar{\rho}_1 - \bar{\rho}_2$	-0.0128	0.0130

Table 5: Confidence intervals with asymmetric Gamma kernel estimator $\hat{g}_b(x)$.

Characteristics difference	Lower bound	Upper bound
$\bar{n}_1 - \bar{n}_2$	-0.0213	0.0018
$\bar{\omega}_1 - \bar{\omega}_2$	-0.0175	0.0019
$\bar{t}_1 - \bar{t}_2$	-0.0379	0.0365
$\bar{\rho}_1 - \bar{\rho}_2$	-0.0085	0.0027

Table 6: Confidence intervals with smoothed histogram $\hat{g}_k(x)$.

Characteristics difference	Lower bound	Upper bound
$\bar{n}_1 - \bar{n}_2$	-0.0254	0.0012
$\bar{\omega}_1 - \bar{\omega}_2$	-0.0138	0.0022
$\bar{t}_1 - \bar{t}_2$	-0.0386	0.0296
$\bar{\rho}_1 - \bar{\rho}_2$	-0.0092	0.0038

estimators are closer to the curve of the theoretical density). We note in Table 1 that the approximation error on the stationary distributions of the $G/G/1$ and $M/G/1$ systems was given when applying nonparametric density estimation methods by considering the correction of boundary effects such in the cases of using the mirror image estimator ($Er = 0.0452$), asymmetric Gamma kernel estimator ($Er = 0.0378$) and smoothed histogram ($Er = 0.0377$). In addition, these two last errors are close to the one given when using the theoretical density $g(x)$ ($Er = 0.0356$). But, when applying the Parzen-Rosenblatt estimator which does not take into account the correction of boundary effects, the approximation error Er on the stationary distributions of the quoted systems could not be given. This shows the importance of the smallness of the proximity error of the two corresponding arrival distributions of the considered systems, characterized by the variation distances w_0 and w^*.

We notice also that in the cases of using the theoretical density $g(x)$, the mirror image estimator $\tilde{g}_n(x)$, the Gamma kernel estimator $\hat{g}_b(x)$ and the smoothed histogram $\hat{g}_k(x)$, with a significance level $\alpha = 0.05$, we do not reject any hypotheses since all the intervals contain zero (see Tables 2, 4, 5 and 6). This means that, with this level $\alpha = 0.05$, the corresponding characteristics of the two considered systems are not significantly different. In addition, we remark that the confidence intervals are very close. That gives an idea about the accuracy of the approximation. But in the case of using the Parzen-Rosenblatt estimator $g_n(x)$, with the same level $\alpha = 0.05$, all the hypotheses are rejected since all the intervals do not contain zero (see Table 3). This means that the risk of wrongly rejecting these hypotheses is of order 5%. So, we prefer to say that the corresponding characteristics of the two systems considered are significantly different.
The theoretical results (Aïssani and Kartashov, 1984) are then illustrated by numerical results. Indeed, it is noted that in practice, for a low margin between the service laws of the $G/G/1$ and $M/G/1$ queueing systems, it is possible to approximate the $G/G/1$ systems characteristics with the corresponding ones of the $M/G/1$ system when the general distribution G of arrivals is unknown and must be estimated. In addition, this approximation is as much accurate than w^* and w_0 are small. Note also that contrary to non parametric methods taking into account the boundary bias correction, the classical kernel density estimate is inappropriate for determining this type of approximation.

5 CONCLUSIONS AND FURTHER RESEARCH

Simulation studies presented in this paper show the importance of some aspects in the application of the strong stability method to queueing systems. First of all, the smallness of the perturbation done has a significant impact on the determination of the proximity of the considered systems and hence on the approximation error on their stationary distributions. On the other hand, when statistical methods are used to estimate an unknown density function in a considered system, we cannot ignore the problem of boundary effects.

To summarize, we show the interest of some statistical techniques (nonparametric estimation methods with boundary bias techniques and Student test) to measure the performance of the strong stability method in a $M/G/1$ queueing system after perturbation of the arrival flow. Indeed, we note that practically, for a low margin between the arrival laws of the $G/G/1$ and $M/G/1$ systems, and by taking into account the boundary effects when using nonparametric density estimation to estimate the unknown arrivals law G in the $G/G/1$ system, it is possible to approximate the $G/G/1$ system's characteristics by the corresponding ones of the $M/G/1$ system.

A closely field of practical interest can be described as follows: when modeling insurance claims, one could be interested in the loss distribution which describes the probability distribution of payment to the insured. It is a positive variable, hence the presence of the boundary bias problem. The asymmetric Beta kernel estimates are suitable for estimating this type of heavy-tailed distributions.

REFERENCES

Aïssani, D. and Kartashov, N. (1983). Ergodicity and stability of Markov chains with respect to operator topology in the space of transition kernels. *Compte Rendu Academy of Sciences U.S.S.R, ser.*, A 11:3–5.

Aïssani, D. and Kartashov, N. (1984). Strong stability of the imbedded Markov chain in an $M/G/1$ system. *Theory of Probability and Mathematical Statistics, American Mathematical Society*, 29:1–5.

Banks, J., Carson, J., and Nelson, B. (1996). *Discrete-Event System Simulation*. Prentice Hall, New Jersey.

Bareche, A. and Aïssani, D. (2008). Kernel density in the study of the strong stability of the $M/M/1$ queueing system. *Operations Research Letters*, 36 (5):535–538.

Bareche, A. and Aïssani, D. (2011). Statistical techniques for a numerical evaluation of the proximity of $G/G/1$ and $G/M/1$ queueing systems. *Computers and Mathematics with Applications*, 61 (5):1296–1304.

Bareche, A. and Aïssani, D. (2013). Combinaison des méthodes de stabilité forte et d'estimation non paramétrique pour l'approximation de la file d'attente $G/G/1$. *Journal européen des systèmes automatisés*, 47 (1-2-3):155–164.

Benaouicha, M. and Aïssani, D. (2005). Strong stability in a $G/M/1$ queueing system. *Theor. Probability and Math. Statist.*, 71:25–36.

Berdjoudj, L. and Aïssani, D. (2003). Strong stability in retrial queues. *Theor. Probab. Math. Stat.*, 68:11–17.

Berdjoudj, L., Benaouicha, M., and Aïssani, D. (2012). Measure of performances of the strong stability method. *Mathematical and Computer Modelling*, 56:241–246.

Borovkov, A. (1984). *Asymptotic Methods in Queueing Theory*. Wiley, New York.

Bouallouche, L. and Aïssani, D. Performance Analysis Approximation in a Queueing System of Type $M/G/1$.

Bouallouche, L. and Aïssani, D. (2006). Measurement and Performance of the Strong Stability Method. *Theory of Probability and Mathematical Statistics, American Mathematical Society*, 72:1–9.

Bouezmarni, T. and Scaillet, O. (2005). Consistency of Asymmetric Kernel Density Estimators and Smoothed Histograms with Application to Income Data. *Econometric Theory*, 21:390–412.

Chen, S. X. (2000). Probability Density Function Estimation Using Gamma Kernels. *Ann. Inst. Statis. Math.*, 52:471–480.

Heidergott, B. and Hordijk, A. (2003). Taylor expansions for stationary markov chains. *Adv. Appl. Probab.*, 35:1046–1070.

Heidergott, B., Hordijk, A., and Uitert, M. V. (2007). Series expansions for finite-state markov chains. *Adv. Appl. Probab.*, 21:381–400.

Jones, M. C., Marron, J. S., and Sheather, S. J. (1996). A brief survey of bandwidth selection for density estimation. *J.A.S.A.*, 91:401–407.

Kartashov, N. V. (1996). *Strong Stable Markov Chains*. TbiMC Scientific Publishers, VSPV, Utrecht.

Kleinrock, L. (1975). *Queueing Systems, vol. 1*. John Wiley and Sons.

Parzen, E. (1962). On estimation of a probability density function and mode. *Ann. Math. Stat.*, 33:1065–1076.

Rachev, S. (1989). The problem of stability in queueing theory. *Queueing Systems*, 4:287318.

Rosenblatt, M. (1956). Remarks on some non-parametric estimates of a density function. *Ann. Math. Statist.*, 27:832–837.

Schuster, E. F. (1985). Incorporating support constraints into nonparametric estimation of densities. *Commun. Statist. Theory Meth.*, 14:1123–1136.

Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. Chapman and Hall, London.

Stoyan, D. (1983). *Comparison Methods for Queues and Other Stochastic Models (English translation)*. In: Daley, D.J. (Ed.), Wiley, New York.