Geometric Characterization of the Sporadic Groups
$Fi_{22}$, $Fi_{23}$, and $Fi_{24}$

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Communicated by A. A. Ivanov

Received October 22, 1992

Let $\Sigma_1, \ldots, \Sigma_4$ be the 3-transposition graphs for Fischer's sporadic groups $Fi_{21}, \ldots, Fi_{24}$. We classify connected locally $\Sigma_i$ graphs $\Theta = \Theta_{i+1}$ for $i = 1, 2, 3$. In the minimal case $i = 1$ we also assume that for every nondegenerate 4-circuit $abcd$ the subgraph $\Theta(a, b, c, d)$ is isomorphic to the disjoint union of three copies of $K_{3,3}$. If $i = 1, 2$ then $\Theta$ is isomorphic to $\Sigma_{i+1}$, whereas if $i = 3$ then $\Theta$ is isomorphic either to $\Sigma_4$ or to its 3-fold antipodal cover $3\Sigma_4$. © 1994 Academic Press, Inc.

1. Introduction

There have been extensive studies of the geometries of the classical groups as point–line systems with fixed local structure. This approach had led to many beautiful results. We refer the reader to a recent paper of Cohen and Shult [9] for a brief survey on that. This paper is an attempt to look at Fischer's sporadic groups from the aforementioned point of view.

It is worth mentioning broad activity in this area. Usually certain additional properties are assumed. First, some group action on the geometry (usually, it is a diagram geometry) is often assumed. This approach has led to study of some presentations for sporadic groups, and sometimes to proving the faithfulness of these presentations. For such studies of the groups under consideration, see Buekenhout and Hubaut [6], Meixner [18], Van Bon and Weiss [23, 24]. Another direction is via assuming some global property of the geometry, e.g., assuming that the related graph is strongly regular or distance regular (see the book by Brouwer et al. [2]). This book contains many results and many references to results of this kind.

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As well, the present paper may be viewed as an extension of Hall and Shult's work [15], where the same objects arising from classical groups generated by 3-transpositions have been characterized axiomatically.

Note that here we do not assume either any group action or global property. Only a few characterizations of sporadic groups under such weak assumptions are known to the author. Hughes [16, 17], has done this for the Higman–Sims group. Cuypers [11] has characterized Conway's second group. The author [20] has produced such a characterization for the Suzuki chain of groups.

It is worth mentioning the concept of Fischer space (see, e.g., Cuypers [12], Weiss [25, 26], Buekenhout [5]), which is in some sense dual to the concept of 3-transposition graphs. Lines of the Fischer space associated with a 3-transposition group \( G \) are triples of the 3-transposition graph, i.e., three points are on a line iff they, regarded as involutions of \( G \), generate \( S_3 \). Axiomatically, a Fischer space is a partial linear space with line size 3, whose planes are either affine or dual affine, and satisfying also certain irreducibility conditions. Finite Fischer spaces were classified (in other terms) in [14]. It should be stressed that the class of objects classified in our paper is somewhat larger than just a special family of duals of Fischer spaces. Indeed, \( 3\Sigma_4 \) is not the dual of any Fischer space at all. Finally, only in the final stage of the proof in the case of diameter 2 does it become clear that our objects admit 3-transposition groups (i.e., are associated with Fischer spaces).

Graphs with constant neighbourhood often arise as collinearity graphs of diagram geometries (see Buekenhout [4]). One may look at our result as a characterization of certain diagram geometries (see the discussion following the statement of Theorem 1.1).

Throughout the paper we consider undirected graphs without loops and multiple edges. Given a graph \( \Gamma \), let us denote the set of vertices by \( V = V\Gamma \), the set of edges by \( E = E\Gamma \). Let \( X \subseteq V\Gamma \). We denote by \( \langle X \rangle = \langle X \rangle\Gamma \) the subgraph \( \Xi \) of \( \Gamma \) induced by \( X \) (i.e., \( V\Xi = X, E\Xi = \{(u, v) \in E\Gamma | u, v \in X\} \)). Given two subgraphs \( \Gamma \) and \( \Delta \), the graph \( \Gamma \cup \Delta \) (resp. the graph \( \Gamma \cap \Delta \)) is the graph with vertex set \( V\Gamma \cup V\Delta \) (resp. \( V\Gamma \cap V\Delta \)) and edge set \( E\Gamma \cup E\Delta \) (resp. \( E\Gamma \cap E\Delta \)). Given \( v \in V\Gamma \), we denote \( \Gamma_v = \langle \{v\} \rangle \) and \( \Gamma_\Gamma = \langle \{\} \rangle \). Furthermore, \( \Gamma(X) = \bigcap_{x \in X} \Gamma(x) \). To simplify the notation we use \( \Gamma_{v_1, \ldots, v_k} \) instead of \( \Gamma(\{v_1, \ldots, v_k\}) \) and \( u \in \Gamma_{\ldots} \) instead of \( u \in \Gamma_{\ldots} \). As usual, \( v = v(\Gamma) = |V\Gamma| \), \( k = k(\Gamma) = v(\Gamma(x)) \), \( \lambda = \lambda(x, y) = \lambda(\Gamma) = v(\Gamma(x, y)) \), where \( x \in V\Gamma \), \( y \in \Gamma(x) \). Let \( y \in \Gamma_{\ldots}(x) \). We denote \( \mu = \mu(x, y) = \mu(\Gamma) = v(\Gamma(x, y)) \). Of course, we use \( k, \lambda, \mu \) if it makes sense, i.e., if those numbers are independent of the particular choice of the corresponding vertices. If \( \Delta \) is a (proper) subgraph of \( \Gamma \) we denote this fact by \( \Delta \subseteq \Gamma \) (resp. \( \Delta \subset \Gamma \)).
We denote the complete $n$-vertex graph by $K_n$, the complete bipartite graph with parts of size $n$ and $m$ by $K_{n,m}$, the circuit of length $n$ by $C_n$, and the empty graph by $\emptyset$.

Let $\Gamma, \Delta$ be two graphs. We say that $\Gamma$ is locally $\Delta$ if $\Gamma(v) \cong \Delta$ for each $v \in V(\Gamma)$. More generally, if $\Delta$ is a class of graphs then $\Gamma$ is locally $\Delta$ if for each $v \in V\Gamma$ the subgraph $\Gamma(v)$ is isomorphic to a member of $\Delta$.

Let $\Gamma, \Gamma'$ be two graphs. We say that $\Gamma$ is a cover of $\Gamma'$ if there exists a mapping $\varphi$ from $V\Gamma$ to $V\Gamma'$ which maps edges to edges and for any $v \in V\Gamma$ the restriction of $\varphi$ to $\Gamma(v)$ is an isomorphism onto $\Gamma'(v)$. Note that, since the latter assumption in the definition of cover is usually omitted, our definition of cover is a bit nonobvious.

Suppose we have a chain of graphs $X_1, \ldots, X_n$, such that $X_i$ is locally $X_{i-1}$, $i = 2, \ldots, n$. Then for any complete $k$-vertex subgraph $T$ of $X_m$, $1 < k < m < n$ we have $X_m(VT) \cong X_{m-k}$.

We freely use basic facts concerning polar spaces (see, e.g., [7, 22]), generalized quadrangles (GQ, for short) (see, e.g., [2]) and their collinearity graphs. An $(i \times)$ extended polar space is a graph which is a locally $(i-1 \times)$ extended polar space (a $0 \times$ extended polar space is a polar space). Note that this definition slightly differs from the common one concerning extended polar spaces, given, say, in [8]. In the terminology of [8] we consider triangular extended polar spaces.

A 3-transposition group is a group containing a conjugacy class $C$ of involutions (called 3-transpositions) such that for any $a, b \in C$, one has either $(ab)^2 = 1$ or $(ab)^3 = 1$. The 3-transposition graph associated with a 3-transposition group is the graph with vertex set $C$, where two vertices are adjacent if the corresponding involutions commute. For a graph $\Gamma$, $\text{Aut}(\Gamma)$ denotes the automorphism group of $\Gamma$. Our group-theoretic notation is as in [10].

Let $\Sigma_1, \ldots, \Sigma_4$ be the 3-transposition graphs for Fischer's sporadic groups $Fi_{22}, \ldots, Fi_{24}$ [14]. Then $\Sigma_i$ is an $(i-1)$-fold extension of the polar space $\Gamma$, where $\Gamma$ is the polar space arising from a 6-dimensional GF(4)-vector space carrying a nondegenerate hermitian form. Also $\Sigma_4$ admits a unique (antipodal) 3-cover $3\Sigma_4$, which is a distance transitive graph in the sense of [2].

Theorem 1.1. Let $\Theta = \Theta_{i+1}$ be a connected locally $\Sigma_i$ graph ($i = 1, 2, 3$). In the minimal case $i = 1$ we also assume that, for any $C_3$-subgraph $\Xi$ of $\Theta$, one has $\Theta(\Xi) \cong K_{3,3} \cup K_{3,3} \cup K_{3,3}$. If $i = 1, 2$ then $\Theta \cong \Sigma_{i+1}$. If $i = 3$ then $\Theta \cong \Sigma_4$ or $3\Sigma_4$.

One may view the graphs $\Theta_{i+1}$ as the collinearity graphs of certain $c^i \cdot C_3$-geometries $\mathcal{G}(\Theta_{i+1})$, namely rank $i + 3$ geometries with diagram

```
\begin{verbatim}
  o --- o --- o --- o
  |  |  |  |
  v  v  v  v
  4  2
\end{verbatim}
```
Here $1 \leq i \leq 3$. Conversely, the elements of the geometry may be viewed as $k$-cliques of the graph with natural incidence, $k = 1, 2, \ldots, i + 1, i + 6, i + 21$.

Note that $(i + 21)$-cliques are of special interest, since it is well known that the geometries induced on them are $i$ times extended projective planes of order 4. Such extensions have been studied widely. In particular it has been shown that $i$ may not exceed 3. This implies that $\Sigma_4$ and $3\Sigma_4$ do not possess further extensions. It is well known that, for a given $i$, the $i$-extension of a projective plane of order 4 is unique (i.e., one obtains Steiner systems $S(3, 6, 22)$, $S(4, 7, 23)$, and $S(5, 8, 24)$, respectively).

Meixner [18] has proved the following result, which also follows from Van Bon and Weiss [23].

**RESULT 1.2.** Let $G$ be a residually connected flag-transitive $c^i \cdot C_3^i$-geometry, $1 \leq i \leq 3$. Then if the $C_3$-residues of $G$ are isomorphic to $G(\Sigma_1)$ then $G = G(\Sigma_{i + 1})$ (or $G(3\Sigma_4)$ if $i = 3$).

In his proof Meixner first shows that the collinearity graphs of the geometries under consideration are locally $\Sigma_i$, then classifies locally $\Sigma_i$ graphs having a certain automorphism group. We currently are almost able to repeat the second part of his work without any assumption on group action (almost, because if $i = 1$ we assume a bit more than local structure; cf. the statement of Theorem 1.1). Buekenhout and Hubaut [6] have proved a result similar to Result 1.2 for $i = 1$ under somewhat stronger assumptions (see also Del Fra, Ghinelli, Meixner, and Pasini [13]).

### 2. Proof of Theorem

Let $\Theta = \Theta_{i+1}$ be a connected locally $\Sigma_i$ graph $(i = 1, 2, 3)$. The crucial point in our proof is the recognition of the subgraphs $\Theta(u, v)$ of $\Theta$ induced on the common neighbourhood of two vertices $u, v$ at distance two. The following well-known general fact (see, e.g., [20]) has been used.

**Lemma 2.1.** Let $F$ be a locally $\Sigma_i$ graph, where $A$ satisfies (*). For any $u \in VF, v \in F^2(u)$ the graph $F(u, v)$ is locally $M^\ast$. independent of the particular choice of $u$ and $v$.

**Lemma 2.1.** Let $\Gamma$ be a locally $\Delta$ graph, where $\Delta$ satisfies (*). For any $u \in V\Gamma, v \in \Gamma^2(u)$ the graph $\Gamma(u, v)$ is locally $M^\ast$. independent of the particular choice of $u$ and $v$. 
Note that $\Delta = \Sigma_i$ satisfies (*). Indeed, the stabilizer of $u \in V\Delta$ in $\text{Aut}(\Delta)$ acts transitively on $\Delta_\Delta(u)$. This implies (*). Thus Lemma 2.1 holds for locally $\Delta$ graphs $\Theta$. We classify the locally $M_\Delta$ subgraphs of $\Delta$. It turns out that these subgraphs are (disjoint unions of) $i$ times extended $GQ(4, 2)$. For $i = 1$ [1] (resp. for $i > 1$ [19]) gives us the isomorphism type of these extended $GQ$.

Information on $\mu$-graphs gives us some control over the second neighbourhood of $\Theta$ as well. Indeed, adjacency in $\Theta_\Delta(u)$ may be interpreted in terms of intersections of two $\mu$-graphs, once we know the relationship between vertices of $\Sigma$ and $M_\Delta$-subgraphs.

The aforementioned outline for studying $\mu$-graphs of $\Theta$ is worked out in Subsection 2.1. In the remaining subsections the question of reconstruction of the second neighbourhood of $\Theta$ is considered. It turns out that for each $v \in \Theta_\Delta(u)$ there exists a unique $w \in \Theta_\Delta(u)$ such that $\Theta(u, v) = \Theta(u, w) = \Theta(v, w)$ (i.e., $\Theta$ is a triple graph). Except in a certain subcase of the case $i = 3$, we find that $\Theta$ has diameter two, and we complete the proof along the lines of [19], constructing certain automorphisms of $\Theta$ which leave $u$ and $\Theta(u)$ fixed and interchange $v$ and $w$ in triples. Finally, we consider the remaining subcase of case $i = 3$ and show that we get a (triple) cover of $\Sigma_\Delta$ as the only other possibility. There the author adopts many ideas from his paper [20].

It is worth noting, though we do not make use of this result, that the covers of $\Sigma_i$ were classified by Ronan [21].

2.1. Preliminaries

Here we present technical statements concerning certain subgroups of Fischer's groups and graphs associated with. Set $\Omega_i$ to be the graph of $(\pm)$-points of the $(i + 5)$-dimensional vector space over $\text{GF}(3)$ carrying a nondegenerate bilinear form with discriminant 1, where points are adjacent if and only if they are perpendicular with respect to this form, for $i = 1, 2, 3$. Equivalently, $\Omega_i$ is the 3-transposition graph for the group $O^{\varepsilon}_{i+5}(3)$, where the Witt index $\varepsilon$ equals $-$, empty, or $+$ according as $i = 1, 2, 3$.

The following statement is well known.

**Lemma 2.2.** Graphs $\Omega_1, \Omega_2, \Omega_3$ are strongly regular graphs with parameters $(v, k, \lambda, \mu)$ equal to $(126, 45, 12, 18)$, $(351, 126, 45, 45)$, $(1080, 351, 126, 108)$, respectively.

Consider the case $i = 1$. Let $\Gamma$ be the collinearity graph of $U_6(2)$-polar space, $H \cong O^{-}_6(3)$ be a subgroup of $G \cong F_{i+1} \cong U_6(2)$. There are three conjugacy classes of such subgroups in $U_6(2)$. Each such subgroup $H$ has a
unique orbit $\Omega_H$ of length 126 on $VT$ (note that the subgraph induced on this 126-orbit is isomorphic to $\Omega_1$). Let $O_k$ ($k = 1, 2, 3$ refers to the conjugacy classes of the $O_6^-(3)$-subgroups of $G$) be the orbits of the action of $G$ on the set of these 126-orbits, regarded as subsets of $VT$. Clearly, this action coincides with the action of $G$ on the union of conjugacy classes of $O_6^-(3)$-subgroups.

**Lemma 2.3.** Let $\Omega = \Omega_H \in O_1$, $H = G(\Omega)$ has orbits $O_{1j}$ ($j = 1, 2, 3$) on $O_1$ of lengths 1, 567, 840, respectively.

1. Let $\Omega' \in O_{1j}$. Then $|\Omega \cap \Omega'| = 30$ or 6 according as $j = 2$ or 3.

2. Let $k = 2$ or 3. Then $H$ has orbits $O_{kj}$ ($j = 1, 2$) on $O_k$ of lengths 112 and 1296, respectively. Let $\Omega' \in O_{kj}$. Then $|\Omega \cap \Omega'| = 45$ or 21 according as $j = 1$ or 2.

**Proof.** We exploit the following facts about the intersections of $O_6^-(3)$-subgroups of $G$. Here we prefer to work with the subgroups of index two of $H$ and its conjugates in $\text{Aut}(\Gamma)$. Such a subgroup in $G(\Omega)$ we denote by $F(\Omega)$.

(1) If $\Omega' \in O_1$ then $F(\Omega) \cap F(\Omega') \cong F(\Omega)$, $2^4$: $A_5$ or $3^{1+4} \cdot 2S_3$.

(2) If $\Omega' \in O_k$ ($k = 2, 3$) then $F(\Omega) \cap F(\Omega') \cong 3^4$: $A_6$ or $A_7$.

This follows from the information found in [10] concerning concrete types of subgroups normalized by outer automorphisms of $U_6(2)$, etc.

Now for any possible $X = F(\Omega) \cap F(\Omega')$ we look at the action of $X$ on the points of $\Gamma$ inside and outside $\Omega$. Using a computer, we calculate the orbits of this action and find that there is only one possibility for $|\Omega \cap \Omega'|$. (For instance, for $X \cong A_7$ we have found that $X$ has orbits of lengths 21 and 105 on $\Omega$ and has no orbits of length 21 outside $\Omega$.)

Consider the case $i = 2$. Let $\Gamma$ be the 3-transposition graph for $G \cong Fi_{22}$, $H \cong O_7(3)$ be a subgroup of $G$. There are two classes of such subgroups in $G$. Each such subgroup $H$ has a unique orbit $\Omega_H$ of length 351 on $VT$. Let $O_k$ ($k = 1, 2$ corresponds to the conjugacy classes of $O_7(3)$-subgroups of $G$) be the orbits of the action of $G$ on the set of these 351-orbits, regarded as subsets of $VT$.

**Lemma 2.4.** Let $\Omega = \Omega_H \in O_1$, $H$ has orbits $O_{1j}$ (resp. $O_{2j}$) ($j = 1, 2, 3$) on $O_1$ (resp. on $O_2$) of lengths 1,3159, 10,920 (resp. 1080, 364, 12,636), respectively.

1. Let $\Omega' \in O_{1j}$. Then $|\Omega \cap \Omega'| = 63$ or 7 according as $j = 2$ or 3.

2. Let $\Omega' \in O_{2j}$. Then $|\Omega \cap \Omega'| = 0$, 108, or 36 according as $j = 1, 2$, or 3.
Proof. Action of $H$ on $O_1$ (as well as the corresponding intersections) can be treated modulo the construction of $Fi_{23}$ as a transitive extension of $Fi_{22}$. Action on $O_2$ and corresponding intersections follow from information on the intersections given in [10] and by induction from Lemma 2.3. Empty intersection appears, in particular, because there are only two types of intersection in Lemma 2.3.

The following statement is well known.

**LEMMA 2.5.** Graphs $\Sigma_1, \Sigma_2, \Sigma_3$ are strongly regular graphs with parameters $(v, k, \lambda, \mu)$ equal to $(693, 180, 51, 45)$, $(3510, 693, 180, 126)$, $(31,671, 3510, 693, 351)$, respectively.

2.2. $\mu$-Graphs

Here $\Gamma = \Sigma_i$ ($i = 1, 2, 3$). Note that in the case $i = 1$ we are unable to classify all the locally $M_\Gamma$-subgraphs of $\Gamma$. However, our additional assumption on $\Theta$ says that for any $\mu$-graph $\Omega$ of $\Theta$ and for any pair $(x, y)$ of nonadjacent vertices of $\Omega$ the subgraph $\Omega(x, y)$ is isomorphic to a disjoint union of three copies of $K_{3,3}$. Thus we are able to restrict our attention to the locally $M_\Gamma$-subgraphs of $\Gamma$ satisfying the same condition that $\Omega$ satisfies.

The following statement gives us control over the isomorphism types of locally $M_\Gamma$-subgraphs $\Omega$ of $\Gamma$.

**PROPOSITION 2.6.** Let $\Omega$ be a locally $M_\Gamma$-subgraph of $\Gamma$ satisfying for $i = 1$ the additional condition that for any pair of $(x,y)$ of nonadjacent vertices $\Omega(x,y) \cong K_{3,3} \cup K_{3,3} \cup K_{3,3}$. Hence $\Omega$ is isomorphic to (a disjoint union of copies of $\Omega_i$, if $i = 3$) the graph(s) $\Omega_i$ defined in Subsection 2.1.

Proof. Since $M_\Gamma$ for $i = 1$ is isomorphic to $GQ(4,2)$, this case immediately follows from [1]. Thus we know the isomorphism type of $\Omega_1$. Since $\mu$-graphs of $\Sigma_i$ ($i < 4$) are connected, $\Omega_i$ is locally $\Omega_{i-1}$ for $1 < i < 4$. Thus in the remaining cases the classification of the graphs with this neighbourhood structure given in [19] is applicable. (Since [1] has not been published, it is worth noting that a proof for the case $i = 1$ can be easily deduced from [19], as well.)

Next, we establish the connection between certain subgroups of $Fi_{2i}$ and the subgraphs $\Omega$, whose isomorphism types $\Omega_i$ were determined in Proposition 2.6.

**PROPOSITION 2.7.** Let $\Omega \cong \Omega_i$ be a subgraph of $\Gamma$. Then there exists an $H \cong O_{i+5}(3)$-subgroup of Aut($\Gamma$) such that $V\Omega$ constitutes an orbit of $H$ in.
its action on $V\Gamma$, where the Witt index $e$ of $H$ equals $-$, empty, or $+$ according as $i = 1, 2, 3$.

Proof. The following proof was suggested by the referee. The original one was much longer.

It is enough to check the claim that $\Omega$ is a triple subgraph of $\Gamma$. Indeed, in this case the subgroup generated by the 3-transpositions corresponding to $V\Omega$ is a subgroup of $\text{Aut}(\Gamma)$ and may be easily identified with $O^{e}_{i+3}(3)$, as required.

Now let us prove the claim for $i = 1$. Assume that $\{u, v, w\}$ is a triple of $S_2$, but not of $F$. Hence, by Lemma 2.2, $|\Omega(u, v, w)| = 18$. Clearly, $|\Gamma(u, v, w)| \geq |\Omega(u, v, w)|$. By a well-known property of polar spaces, the points of $\Gamma(u, v, w)$ constitute a proper geometric hyperplane of $GQ(4, 2)$ induced on $\Gamma(u, v)$, which cannot have more than 13 points. Hence $\{u, v, w\}$ is a triple of $\Gamma$. This completes the proof for $i = 1$.

Let $i > 2$. Assume that $\{x, y, z\}$ is a triple of $22$. Fix a vertex $u \in 22(x, y, z)$. By induction, $\{x, y, z\}$, being a triple of $\Omega(u)$, is a triple of $\Gamma(u)$. But triples of $\Gamma(u)$ are also triples of $\Gamma$, and we find that $\{x, y, z\}$ is a triple of $\Gamma$. \]

Let us turn to the relationship between the vertices of $\Gamma$ inside and outside a connected locally $M_{\Gamma}$-subgraph $\Omega$ of $\Gamma$.

**Lemma 2.8.** Let $v$ be a point of $\Gamma$ outside $\Omega$, $L = \Gamma(v) \cap \Omega$. Then, according to the value of $i$,

1. $|L| = 30$, and $L$ is isomorphic to the 2-clique extension $L_1$ of the point graph of $GQ(2, 2)$.
2. $|L| = 63$, and $L$ is isomorphic to the collinearity graph $L_2$ of $Sp_8(2)$-polar space.
3. If $L$ is not empty then $|L| = 120$, and $L$ is isomorphic to the collinearity graph $L_3$ of $O^{+}_6(2)$-affine polar space.

**Proof.** Here $H$ denotes an automorphism group of $\Omega$ provided by Proposition 2.7. Below we denote $\Lambda = \Gamma(v) \cap \Omega$.

Case $i = 1$. Since we know that $H$ has two orbits on the points of $\Gamma$, the first part of the lemma follows from a simple counting of the $\Gamma$-edges going from $\Omega$ outside.

Since each maximal clique of $\Omega$ is a hyperoval in a plane of $\Gamma$, each line through $v$ intersects $\Omega$ in zero or two points. Thus $L$ is the 2-clique extension of a subgraph $\Lambda$ of the point graph $\bar{X}$ of $GQ(4, 2)$. It is clear that each line of $\bar{X}$ intersects $\bar{X}$ in zero or three points. Thus $\bar{X}$ has lines of size 3. Moreover, if $l$ is such a line then $l \cup \bar{X}(l) = V\bar{X}$. It means that $\bar{X}$ is the point graph of a GQ, evidently of $GQ(2, 2)$. 
Case $i = 2$. Since we know that $H$ has two orbits on the points of $\Gamma$, the first part of the lemma follows from a simple counting of the $\Gamma$-edges going from $\Omega$ outside.

By case 1 of the current lemma each connected component of $L$ is isomorphic to a graph such that for each vertex the neighbourhood graph is the same as that for $\text{Sp}_6(2)$-polar space. According to J. Tits [22], such a graph is unique. It follows from the first part of the lemma that $L$ is connected.

Case $i = 3$. Here $H$ is not maximal in $G \cong F_{i23}$. According to [10], it is contained in a maximal subgroup isomorphic to $H$: $S_3$, which has exactly two orbits $W, W'$ on $V\Gamma$. It is straightforward to check that one of those orbits, say $W$, has length $3 \cdot |V\Gamma|$. Therefore $\Omega \subset W$, and $W$ contains two other copies, $\Omega'$ and $\Omega''$, of $\Omega$.

If $(x, y)$ is an edge between, say $\Omega$ and $\Omega'$, then, by case 2 of the current lemma, the subgraph $\Omega'(x)$ has valency 63. On the other hand, by [10], the stabilizer of $x$ in $H$ acts on $\Omega'$ transitively, so $\Omega'(x) = \Omega'$, a contradiction. Hence there are no edges between the copies of $\Omega$ in $W$.

Let $Y = \Gamma(v) \cap W$. By the same argument as in previous cases, we obtain $|Y| = 360$. Now, assuming that there exists an element $z$ of order 3 from $(H: S_3) \setminus H$ fixing $v$ and interchanging the copies of $\Omega$ inside $W$, we immediately get $|L| = 360/3 = 120$. A look at [10] shows us that each 3-element of $G$ cannot either fix $W$ pointwise or act fixed point freely on $V\Gamma$. Hence such a $z$ indeed exists, and the first part is proved.

By case 2 of the current lemma each connected component of $L$ is isomorphic to a graph such that for each $x \in VL$ the subgraph $L(x)$ is isomorphic to the collinearity graph of $\text{Sp}_6(2)$-polar space. By [15], or [9], there are exactly two such graphs, and only one of them has the required number of vertices. It follows from the first part of the lemma that $L$ is connected.

Now we are able to complete the classification of locally $M_{\Gamma}$-subgraphs of $\Gamma$. The following statement is an immediate consequence of Lemma 2.8.

**Lemma 2.9.** Let $\Omega$ be a locally $M_{\Gamma}$-subgraph of $\Gamma$. If $i = 1, 2$ then $\Omega$ is connected. If $i = 3$ there is a unique equivalence relation $\phi$ with class size three on the set of connected locally $M_{\Gamma}$-subgraphs, such that all the connected components of $\Omega$ lie in one class of $\phi$.

Finally, we need the following technical statement.

**Lemma 2.10.** Let $\Omega$ be a locally $M_{\Gamma}$-subgraph of $\Gamma$, with the vertices $v$ and $w$ of $\Gamma$ lying outside $V\Omega$. Then $\Gamma(v) \cap \Omega = \Gamma(w) \cap \Omega$ implies $v = w$. The subgraph of $\Gamma$ induced on $V\Gamma \setminus V\Omega$ is connected.
Proof. Follows from the primitivity of the action of the stabilizer of $\Omega$ in $\text{Aut}(\Gamma')$ on the points outside $\Omega$. 

2.3. Final Part of the Proof

Let $\mathcal{X}$ be the graph defined on the set of connected locally $M_T$-subgraphs of $\Gamma$, with two vertices $\Omega, \Omega'$ adjacent if $\Omega \cap \Omega' \equiv L_i$ ($L_i$ are defined in Lemma 2.8). Let $\Theta$ be a connected locally $\Gamma$ graph, $u \in \mathcal{V}\Theta$, $v \in \Theta_2(u)$.

Cases $i = 1$ and $2$. We start with a technical statement.

**Lemma 2.11.** $\mathcal{X}$ has $4 - i$ connected components, which are the orbits of $Fi_{2i}$ on the set of locally $M_T$-subgraphs of $\Gamma$. $k(\mathcal{X}) = 567$ or $3159$ according as $i = 1$ or $2$.

**Proof.** Immediately follows from Lemmas 2.3, 2.4. 

Now are able to reach our goal. By Lemma 2.8, there are no edges coming from $v$ to $\Theta_3(u)$, that is, the diameter of $\Theta$ equals two. Lemma 2.10 provides a bijection between $\mathcal{O}(v) \cap \Theta_2(u)$ and $\mathcal{E}(\Theta(u, v))$. Therefore $\Theta$ is strongly regular with the same parameters as the $\Sigma_{i+1}$-graph. It follows from the fact that $\Gamma$ is a triple graph and the counting of $\mu$-graphs that $\Theta$ is a triple graph. Therefore $\Theta_2(u)$ is a double cover of a connected component of $\mathcal{X}$. Since this is a double cover, the covering map acts on $\Theta_2(u)$ fixed point freely. Thus for each $u \in V\Theta$ there exists an involution $g_u$ fixing $\Theta(u)$ pointwise and acting fixed point freely on $\Theta_2(u)$. The rest of the proof is straightforward along the lines of [19] and consists of identifying the group generated by $g_w$ ($w \in \Theta(u)$) with $\langle g_u \rangle \cdot Fi_{2i}$, and applying [14] to identify the group generated by all the $g_u$ with $Fi_{2i+1}$.

Case $i = 3$. Let us recognize the situation with intersections of locally $M_T$-subgraphs in this case.

Let $H \cong O_{10}^+(3)$: $S_3$ be a subgroup of $F = Fi_{23}$. There is a unique class of such subgroups in $F$. Each such subgroup has an orbit $\overline{\Omega}$ of length $3 \cdot 1080$ on $VT$, whose elements are the vertices of the subgraphs belonging to $\phi(\Omega)$ for a connected locally $M_T$-subgraph $\Omega$. Let $O$ be the orbit of the action of $F$ on the set of these $3 \cdot 1080$-orbits regarded as subsets of $VT$.

**Lemma 2.12.** Let $\Pi = \overline{\Omega} \in O_1$. $H$ has the orbits $O_j$ on $O$ of lengths 1, 28, 431, 109200, respectively ($j = 1, 2, 3$).

Let $\Pi' \in O_2$. Then $|\Pi \cap \Pi'| = 360$. The subgraph induced on $\Pi \cap \Pi'$ is isomorphic to the disjoint union of three copies of $L_3$. Moreover, for any
$\Omega \in \phi(\Omega)$ there exists a unique $\Omega' \in \phi(\Omega') = \Pi'$ such that $\Omega \cap \Omega' \cong L_3$. Let $\Pi' \in O_3$. Then $8 \leq |\Pi \cap \Pi'| < 360$.

**Proof.** Action of $H$ on $O$ (as well as the corresponding intersections) can be treated modulo the construction of $Fi_{24}$ as a transitive extension of $Fi_{23}$.  

Let $\overline{E}$ be the graph defined on $O$ such that two vertices $\phi(\Omega), \phi(\Omega')$ are adjacent if for some $\Omega, \in \phi(\Omega)$ there exists $\Omega'_1 \in \phi(\Omega')$ such that $\Omega \cap \Omega'_1 \cong L_3$. By Lemma 2.12, actually for any $\Omega, \in \phi(\Omega)$ such a $\Omega'_1$ exists. Clearly, $\overline{E}$ is connected. Thus $\overline{E}$ is the quotient of $\Xi$ defined by $\phi$, where $\Xi$ is defined at the beginning of the current subsection.

**Lemma 2.13.** $\Xi$ is connected, $k(\Xi) = 28,431$.

**Proof.** Clearly $k(\Xi) = k(\overline{E}) = 28,431$. If $\Xi$ were disconnected, it would happen only if it has three connected components, but it is impossible, since the simple group $Fi_{23}$ acts vertex transitively on it. 

We need one more technical statement.

**Lemma 2.14.** Let $F \equiv \Sigma$. Suppose that a subgraph $\Delta$ of $\Gamma$ satisfies the following property: for any $x \in VA$ one has $\Gamma(x) \setminus \Delta(x) = \Gamma(x, t^y)$ for some $t^y \in \Gamma_2(x)$. Then $\Delta = \Gamma_2(t)$ for some $t \in V\Gamma$.

**Proof.** The major part of the following argument was suggested by the referee.

Fix $v \in VA$ and $t = t^v$. Let $t'$ be the third vertex in the triple of $\Gamma'$ containing $v$ and $t$. The subgraphs $\Gamma_2(t)$ and $\Gamma_2(t')$ satisfy the conditions of the lemma. We show that $\Delta$ coincides with one of $\Gamma_2(t)$ and $\Gamma_2(t')$.

Let $x \in \Delta(v)$. There are exactly two triples $\{x, t_{11}, t_{12}\}$ and $\{x, t_{21}, t_{22}\}$ such that $\Gamma(x, v, t, t_{\alpha, \beta}) = \Gamma(x, v, t)$, where $\alpha, \beta = 1, 2$. Indeed, by Lemmas 2.4 and 2.8, $\Gamma(x, v, t)$ is a maximal-by-size possible proper intersection of $\Gamma(x, a)$ and $\Gamma(x, b)$ for $a, b \in \Gamma_2(x)$. Hence $\Gamma(x, v, t, t_{\alpha, \beta}) = \Gamma(x, v, t) = \Gamma(x, t, t_{\alpha, \beta})$, where $\alpha, \beta = 1, 2$. Therefore for $\alpha, \beta = 1, 2$ one has that $t_{\alpha, \beta}$ belongs to the Fischer subspace $\Pi$ generated by $x, v,$ and $t$, which is isomorphic to the dual affine plane of order 2. Thus, the triples under question are the two triples of $\Pi$ through $x$.

Moreover, these triples $\{x, t_{11}, t_{12}\}$, $\{x, t_{21}, t_{22}\}$ meet the triple $\{v, t, t'\}$ in $\Pi$. So, without loss of generality we have $t = t_{11}$, $t' = t_{21}$.

Thus, without loss of generality we can take $t^x$ to be $t$.

Let $y \in \Delta(v, x)$. Then, repeating the arguments of the first part of the proof with $y$ and $v$ or with $y$ and $x$, we see that the triple $\{y, t^y, (t^y)'\}$ must meet both the triples on $v$ and $t$ on $x$ and $t$, and thus contains $t$. Now, by connectedness of $\Delta(v)$ and $\Gamma_2(z)$ for any
z \in \mathcal{V}_\Gamma$, we find that all triples obtained in this way contain $t$ and $\Delta = \Gamma_2(t)$.

We start with identifying $\Gamma_2(u)$. As a by-product, we settle the case leading to $\Theta \equiv \Sigma_4$.

By Lemma 2.9, the number of connected components of $\Theta(u, v)$ equals $m = m_v = m_{u,v}$ for $1 \leq m \leq 3$. Let $w \in \Theta(v) \cap \Theta_2(u)$, $\Theta(u, v, w) \neq \emptyset$. Clearly $m_w = m$. By Lemma 2.8, $\Theta(u, v, w)$ is isomorphic to the disjoint union of $m$ copies of $L_3$. Set $A = A^w$ to be a connected (partial) subgraph of $\Theta_2(u)$ containing $v$, with two vertices $x, y$ joined if $\Theta(u, x, y) \equiv \Theta(u, v, w)$. Lemma 2.10 provides the natural bijection between $\Theta(v)$ and the set of locally $M_\Gamma$-subgraphs $\Omega$ of $\Gamma = \Theta(u)$ such that $\Omega \cap \Theta(u, v) \equiv \Theta(u, v, w)$.

Let $m \geq 2$. Then $A$ is a cover of $\mathcal{E}$ defined by $y \mapsto \Theta(u, y)$ for $y \in \mathcal{V}A$. (This is straightforward for $m = 3$. For $m = 2$ it follows from the fact that for any $\phi$-equivalence class of connected locally $M_\Gamma$-subgraphs of $\Gamma$ exactly two of the three members of the class appear as connected components of $\Theta(u, y)$, $y \in \mathcal{V}A$). Since

$$\mu(\mathcal{E}) = 5832 > 3 \cdot 1080 = \sup_{x \in \mathcal{V}\Theta, y \in \Theta(x)} |\Theta(x, y)|$$

(cf. [3] on the value of $\mu(\mathcal{E})$), $A$ is either a connected proper cover, or all the distance two vertices of $\mathcal{V}$ are joined by edges inside $\Theta_2(u)$. Since the latter implies $|\Theta_2(u) \cap \Theta(v)| > k(\Theta)$, it is nonsense. By consideration of the neighbourhood of a vertex from $\Theta(u, v)$, the foldness of this cover is two.

Let $m = 3$. In this case, by Lemmas 2.8 and 2.9, $\Theta(v) = A(v) \cup \Theta(u, v)$. By counting the edges between $\Theta(u)$ and $\Theta_2(u)$, we have $\Theta_2(u) = \Lambda$. The same arguments as in the already considered cases $i = 1$ or $2$ complete the proof of $\Theta \equiv \Sigma_4$.

Thus we may assume from now on that $m \leq 2$. Let $m = 2$. Counting edges between $\Theta(u)$ and $\mathcal{V}A$, we see that

$$|\Theta_2(u)| < 2|\mathcal{V}A| < 2|\Theta_2(u)|.$$  \hspace{1cm} (1)

Let $v' \in \Theta_2(u) \setminus \mathcal{V}A$. Since $|\mathcal{V}A'| < |\mathcal{V}A|$, we have $m_{v'} = 1$. But $\Lambda^v$ is a (possibly, improper) cover of $\mathcal{E}$ defined by $y \mapsto \Theta(u, y)$ for $y \in \mathcal{V}A'$. So $|\mathcal{V}A'| \geq |\mathcal{V}\mathcal{E}| > |\mathcal{V}A|$, a contradiction to (1).

Hence $m = 2$ is nonsense. In what follows we assume $m = m_{x,y} = m(\Theta) = 1$ does not depend on the particular choice of $x \in \mathcal{V}\Theta$ and
$y \in \Theta_2(x)$. As we have already observed, $\Lambda$ is a cover of $\Xi$. Since

$$\max_{a \in V\Theta, b \in \Theta_2(a)} |\Theta(a, b)| \geq \mu(\Xi)/3 > \mu(\Xi),$$

$\Lambda$ is either a connected proper cover, or all the distance two vertices of $\Lambda$ are joined by the edges inside $\Theta_2(u)$. By the same argument as in the case $m \geq 2$, the latter is impossible. By consideration of the neighbourhood of a vertex from $\Theta(u, v)$, the foldness of the cover is two.

Now we claim that $\Theta$ is a distance-regular graph with the same intersection array as $3\Sigma_4$.

First, let us show that $\Lambda = \Theta_2(u)$. Suppose false, i.e., there exists an edge $(x, y)$ in $\Theta_2(u)$, which does not belong to $\Lambda$. Clearly, $\Theta(u, x) \not= \Theta(u, y)$. Hence $v(\Theta(u, x, y)) = 0$. Consider $\Theta(x)$. The vertex $y$ belongs to a subgraph $\Delta$ from $\phi_3(\Theta(x, u))$. Hence there are at most $k(\Delta) = 351$ vertices of $\Theta(x, y)$ belonging to $\Theta_2(u)$. Moreover, for any vertex $w$ from $\Theta(x, y)$ not in this 351-set $\Delta(y)$, we have $\Theta(u, x, w) \not= \Theta(u, y, w) \not= L_3$, i.e., $(x, w)$ and $(y, w)$ belong to $E\Lambda$. Since $\Theta(x, y) \not= \Sigma_2$, we have that there are at least 2808 such vertices $w$. Recall that $\Lambda$ is a cover of $\Xi$. Hence there exists a pair of distance two vertices of $\Xi$, which have at least 2808 common neighbours. Therefore $\mu(\Xi) = 5832 \geq 3 \cdot 2808$, a contradiction.

Let $v \in \Theta_2(u)$, $w \in \Theta_3(u) \cap \Theta(v)$. Denote $\Gamma = \Theta(w)$. We claim that $\Delta = \Theta_2(u) \cap \Gamma$ satisfies the conditions of Lemma 2.14. From the above consideration of $\Theta_2(u)$, we know that for any $x \in \Theta_3(u)$ the subgraph $\Theta_2(u) \cap \Theta(x)$ is the disjoint union of the two subgraphs from $\phi_3(\Theta(x, u))$ distinct from $\Theta(x, u)$. Hence for any $x \in V\Delta$ the subgraph $\Gamma(x) \cap \Theta_2(u)$ is isomorphic to $\Gamma(x, t^*)$, where $t^* \in \Gamma_2(x)$. Thus Lemma 2.14 is applicable, and $\Delta = \Gamma_2(t)$ for some $t \in \Gamma V\Gamma$.

Let $w' \in \Gamma(t)$. By the same arguments, $\Theta(w') \cap \Theta_2(u) = \Theta(w') \cap \Theta_2(t')$ for some $t' \in \Theta(w')$. Suppose $t \neq t'$. Note that $t' \in \Gamma V\Gamma$. Clearly $t' \not= \Gamma_2(t')$. Since $\lambda(\Gamma(t)) = 693 < v(\Gamma(t'), t')) - 1$, we have $\Gamma(w', t') \cap \Gamma_2(t) \not= \emptyset$, a contradiction to the definition of $t'$.

Thus $\Theta(t) \subseteq \Theta_2(u)$. Hence $t \in \Theta_4(u)$. Now a standard counting of edges between $\Theta_3(u)$ and $\Theta_2(u)$ shows that $\Theta$ has the same intersection array as $3\Sigma_4$. It remains to show that $\Theta$ is a cover of $\Sigma_4$. This is enough to complete the proof. Indeed, it implies that the universal cover of $\Sigma_4$, which has the same neighbourhood as $\Sigma_4$, has the same intersection array as $3\Sigma_4$. So this universal cover is isomorphic to $\Theta$, and $\Theta \cong 3\Sigma_4$.

Since $\Theta_3(u) \cong \Sigma_3 \cup \Sigma_3$, to be at distance 4 in $\Theta$ is an equivalence relation on $V\Theta$. Then it is straightforward to check that for any $x \in \Theta(u)$ there exists a unique $y \in \Theta(t) \cap \Theta_4(x)$. Thus our equivalence relation is well-defined on $E\Theta$, and we may define the quotient graph, whose vertices
(resp. edges) are equivalence classes of vertices (resp. edges) of $\Theta$. Clearly, the restriction of our quotient map to the neighbourhood of any $z \in V\Theta$ is an isomorphism. Therefore our quotient graph is locally $\Sigma_3$ and of diameter 2, hence isomorphic to $\Sigma_4$.

The proof of the theorem is completed.

ACKNOWLEDGMENT

The author thanks the referee for numerous comments and improvements, particularly concerning proofs of Proposition 2.7 and Lemma 2.14.

Note added in proof. In the subsequent paper "Extending polar spaces of rank at least 3," submitted for publication, the author removes the extra assumption on $Cu$-subgraphs in the case $i = 1$ (cf. Theorem 1.1).

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