Optimal Control Strategies for Maximizing the Performance of Variable Stiffness Joints with Nonlinear Springs

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Abstract—A very young robotics research topic is the use of elastic elements in joints to increase their performance. The idea of attaching the robot link to the motor via elastic elements is especially motivated by the possibility to store potential energy in the springs, which enables movement types that could not be implemented with rigid joints. After our recent results on the relation between the spring’s potential energy and the optimal control strategy for realising explosive motions using adjustable linear springs with a velocity-controlled motor, we extend in this paper these results to elastic joints with adjustable nonlinear springs. In particular, we build up on the method we first introduced for analysing Variable Stiffness Actuators with fixed motor positions and consider now additionally the control from the motor side. The application of the method is illustrated by investigating serial elastic joints as well as variable stiffness joints with nonlinear springs and it is shown how optimal control strategies for linear systems such as periodic bang-bang excitations with the system’s resonant frequency, change in this more general setting to energy-dependent excitations.

I. INTRODUCTION

The variety of joint designs with intrinsic elasticity is increasing vastly [16]. In order to compute control strategies, which exploit this elasticity in the best possible way, optimal control (OC) theory is the tool of choice. Many numerical results justifying the performance gain of complex multi-DoF robots with elastic joints have been published [3], [8], [10]. Nevertheless, complex models mostly prevent us from gaining analytical insights or fully understanding the underlying OC strategy. Our approach to the problem is to unveil the underlying control patterns for 1-DoF joints by finding analytical solutions to the corresponding OC problems in order to thoroughly understand the inherent capabilities of elastic robots.

Solving OC problems analytically is mostly a challenging task, especially when nonlinearity are present in the system dynamics. Consequently, most analytical results in literature regarding elastic joints apply to systems with linear (adjustable) springs (see for ex. [5], [6], [7], [9]). Nevertheless, in existing elastic joint designs the relation between the angular deflection and the elastic joint torque is mostly described by nonlinear (adjustable) functions ([4], [15], [17], [18]). Consequently, these results give only a limited insight to the control of elastic joints in real applications. Recently, we introduced a method to analyse the maximum performance of Variable Stiffness Actuators (VSA) with nonlinear springs assuming the motor position is fixed [11]. With this assumption we mainly focused on the optimal way to adjust the elastic properties of such actuators. The main aim of this paper is to extend this method by considering the control on the motor side and to apply the extended method to joints with various stiffness profiles in order to see how optimal motor control strategies, along which the system’s performance is maximized, are influenced by the nonlinearity in the springs.

The paper is organized as follows. Section II describes the used joint model and summarizes the OC problems which we deal with in order to analyse the joint’s maximum performance. In Section III, we discuss necessary conditions for the OC using Pontryagin’s Minimum Principle and show the relation between the OC and the system’s states and costates. Section IV summarizes our previous results regarding the description of the optimal costates in terms of the angular deflection, which turn out to be still valid for bang-bang optimal controls. After revealing the optimal switching structure and providing all the equations needed to construct the OC Trajectory, we illustrate the application of the method in Section V by computing OC strategies, which maximize the link velocity of serial elastic joints (SEA) and variable stiffness joints (VSI).

II. PROBLEM FORMULATION

The elastic joint model that we will use throughout the paper and which is illustrated in Figure 1 consists of one link and one motor, which are attached to each other via an adjustable elastic spring. The velocity of the motor is assumed to be bounded and directly controlled. Similarly, we assume that the joint torque $\tau_J$ in the spring is bounded by two continuously differentiable functions of the angular deflection $\tau_{J,1}(\phi), \tau_{J,2}(\phi)$ and directly controlled:

$$\tau_J(\phi, \sigma) = \frac{1}{2} (\tau_{J,2}(\phi) + \tau_{J,1}(\phi)) + \frac{\sigma}{2} (\tau_{J,2}(\phi) - \tau_{J,1}(\phi)),$$  \hspace{1cm} (1)

where $\tau_{J,1} < \tau_{J,2}$ holds for all $\phi > 0$ and the bounded stiffness controller $\sigma \in [-1,1]$ determines the torque profile of the spring along the system’s trajectory. In addition, we assume the two functions $\tau_{J,1}$ and $\tau_{J,2}$ to be strictly increasing and symmetrical with respect to the origin. The following two differential equations can then be used to fully describe the dynamics of the joint:

$$M\ddot{\theta} = \tau_J(\phi, u_2)$$ \hspace{1cm} (2)

$$\dot{\theta} = u_1 \in [-u_{1\text{max}}, u_{1\text{max}}],$$ \hspace{1cm} (3)

Fig. 1: Joint Model with Variable Stiffness Actuator
where $M$ stands for the link’s mass of inertia, $q$ for the link position, $\theta$ for the motor position, $\phi = \theta - q$ for the angular deflection and finally $u_1$ for the controlled motor velocity and $u_2 = \sigma$ for the stiffness controller.

In order to make use of Pontryagin’s Minimum Principle [13], we need to first describe this system in terms of first-order differential equations. Choosing the state as $x = (\phi, \dot{q})^T$, we have:

$$f(u, x) := \dot{x} = \begin{pmatrix} \phi \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \frac{\tau_j(\phi, \phi_{\phi})}{M} \\ \frac{u_1 - x_2}{M} \end{pmatrix},$$

with $u = (u_1, u_2)^T$. For this given system dynamics, we will search piecewise continuous control trajectories which minimize a given cost functional. In order to concentrate on OC strategies which fully exploit the inherent capabilities of the discussed joints, the cost functionals we consider will only consist of a terminal cost in terms of the final states and/or the final time $t_f$:

$$J(u) = \partial(x(t_f), t_f).$$

Note that in Section V, we will be mainly concerned with optimal control strategies which maximize the link velocity and thus lead to explosive motions. Consequently, our particular choice of the states is well-suited for this purpose. Furthermore, as we discuss the construction of optimal control strategies, we will see that for this particular choice of the states the derived switching structure remains valid regardless of the form of the terminal cost. We now start applying OC theory to derive necessary conditions for the optimal controls $u^*$, which minimize $J(u)$ in (5).

III. NECESSARY CONDITIONS

In this section, we will make use of Pontryagin’s Minimum Principle to obtain necessary conditions for $u^*$. The Hamiltonian $\mathbb{H}$ for the system (4) and the cost functional (5) has the following form [12]:

$$\mathbb{H} = \lambda^T f = \lambda_1 u_1 - \lambda_1 x_2 + \lambda_2 \frac{\tau_j(x_1, u_2)}{M},$$

where $\lambda$ denotes the costates of the system. According to Minimum Principle, it is known that $\mathbb{H}$ is minimized by the optimal control $u^*$ along the optimal trajectory $x^*$ and $\lambda^*$:

$$\mathbb{H}(x^*, \lambda^*, u^*) \leq \mathbb{H}(x^*, \lambda^*, u),$$

for all $u \in [-u_{1\text{max}}, u_{1\text{max}}] \times [-1, 1]$. Taking a closer look at the relation for the controlled torque (1), together with our assumptions on $\tau_{j1}$ and $\tau_{j2}$, as well as the Hamiltonian (6) and the condition (7), we can find the dependence of $u^*$ on the optimal costates $\lambda^*$ and the angular deflection $x_1^*$ as follows:

$$u^*_1 = \begin{cases} -u_{1\text{max}} & \lambda_1^* > 0 \\ u_{1\text{max}} & \lambda_1^* < 0, \\ \text{singular} & \lambda_1^* = 0 \end{cases}; \quad u_2^* = \begin{cases} -1 & x_1^* \lambda_2^* > 0 \\ 1 & x_1^* \lambda_2^* < 0 \\ \text{singular} & x_1^* \lambda_2^* = 0 \end{cases}.$$  

According to (8), we can see that at the time the angular deflection $x_1^*$ becomes zero, a change in the stiffness profile is in general expected. This change makes intuitively sense, since at zero deflection the whole potential energy stored in the spring is converted to the link’s kinetic energy. Changing the torque profile at that instant by for instance decreasing it to its minimum value maximizes the angular deflection, one could obtain with the given energy [2]. On the other hand, establishing a physical reasoning between the costates and the optimal control is not straightforward. While constructing the optimal trajectory in Section IV, we will therefore describe the costates in terms of a normalized angular deflection, which will lead to a relation of the costates in terms of physical quantities. As we later discuss, this relation can be derived by making use of the dynamics of the costates, which is given by the partial derivative of the Hamiltonian with respect to $x$ [12]:

$$\lambda = -\frac{\partial \mathbb{H}}{\partial x} = \frac{(\frac{\partial \tau_j(x_1, u_2)}{\partial x_1}) \lambda_1}{\lambda_2}.$$  

According to (8), our derived necessary conditions do not say anything about the behaviour of the optimal control when there is a singularity. We conclude this section with a discussion about these singularities and show that the motor always rotates with the link velocity ($u_1^* = \dot{q}^*$) along these singular arcs.

A. Singularity of the Optimal Control

We start looking at the case when the optimal stiffness controller $u_2^*$ is singular. This case occurs if $x_1^* \lambda_2^*$ or $\lambda_2^*$ is zero in a finite time-interval along the optimal trajectory. Assume first that $\lambda_2^* = 0$ holds. This means that $\lambda_2^* = \lambda_1^*$ is zero, as well. Since the origin is an isolated point of the system (9), the optimal costates $\lambda^*$ will then remain zero along the whole trajectory. Consequently, our derived necessary conditions (8) will not give any information regarding the optimal control strategy. The OC problems, which we will deal with in this paper, will be restricted to those for which $\lambda^*$ is not identicaly zero. Note that this restriction does in general not prevent us from deriving the optimal control strategies which maximize the joint’s performance. Indeed, according to Minimum Principle $\lambda^*$ can not be identically zero for time-optimal control problems, since our system dynamics does not explicitly depend on time [14]. Furthermore, Minimum Principle also provides in general non-zero boundary conditions for the optimal costates. For instance, for the OC problem of maximization the link velocity at a fixed final time $t_f$, we have [12]:

$$J = -x_2(t_f) \Rightarrow \lambda^*(t_f) = \frac{\delta J}{\delta x_{t_f}} \neq 0. \quad (10)$$

When illustrating the application of our method to construct the OC strategy, we will solve this particular OC problem for different torque profiles.

For the OC problems we consider in this paper, a singularity of $u_2^*$ can thus only occur if $x_1^*$ stays at zero. In this case no elastic joint torque will be present and the time-derivative of the angular deflection will also be equal to zero so that we have (see (4)):

$$u_1^* = x_2^* = \text{const.},$$

along the singular arc. In addition, since $|u_1^*|$ is constrained by the maximum motor velocity $u_{1\text{max}}$, this particular singularity can only occur if the link moves with a velocity less than or equal to the maximum motor velocity, $|x_2^*| \leq u_{1\text{max}}$. Furthermore, if the inequality is satisfied strictly, not only $u_2^*$ but also $u_1^*$ becomes singular. Now we are going to look when a singularity in the motor control $u_1^*$ can occur.
In order for $u_1^*$ to be singular, $\lambda_1^* = \dot{\lambda}_2^* = 0$ must hold. Consequently, $\lambda_2^*$ must remain constant on the singular trajectory. In addition, since $\lambda_1^*$ remains at zero, its time derivative $\dot{\lambda}_1^* = \frac{\partial \tau_j(x^*, u^*)}{\partial x_1} \lambda_2^*$ must also be equal to zero. Consequently, two possibilities exist: $\lambda_2^* = 0$ or $\frac{\partial \tau_j(x^*, u^*)}{\partial x_1} = 0$. Since we assume that $\lambda^*$ is not identically zero, we only need to investigate the case when $\dot{\lambda}_1^* \lambda_2^* = 0$ equals to zero.

Using (1), we can find the following expression for the partial derivative $\frac{\partial \tau_j(x_1, u_2^*)}{\partial x_1}$:

$$\frac{\partial \tau_j(x_1, u_2^*)}{\partial x_1} = (1 + u_2) \frac{d\tau_{j,2}}{dx_1} + \frac{1 - u_2}{2} \frac{d\tau_{j,1}}{dx_1}, \quad (12)$$

where we know that $u_2 \in [-1, 1]$ holds for the stiffness controller and that both derivatives $\frac{d\tau_{j,1}}{dx_1}$ and $\frac{d\tau_{j,2}}{dx_1}$ are non-negative, since $\tau_{j,1}$ and $\tau_{j,2}$ are strictly increasing functions of $x_1$. Furthermore, these derivatives cannot remain at zero if $x_1$ changes. We can then conclude that the partial derivative (12) can only be zero in a finite time interval $t \in [t_s, t_s + \varepsilon]$ with $\varepsilon > 0$, if $x_1^*(t_s)$ remains at the same value $x_1^*(t_s)$ and the derivatives $\frac{d\tau_{j,1}}{dx_1}$ and/or $\frac{d\tau_{j,2}}{dx_1}$ are equal to zero at this particular deflection. Now, if $u_2^*$ is not on the singular arc we have for a constant stiffness controller $u_2^*$:

$$u_1^*(t) = x_2^*(t) = x_2^*(t_s) + \frac{\tau_j(x_2^*(t_s), u_2^*)}{M}(t - t_s), \quad (13)$$

since the torque in the spring does not change when the deflection and $u_2^*$ remains constant. Note that since $u_1^*$ is bounded, the duration of this particular singular trajectory is also limited by the inequality $|x_2^*(t)| \leq u_{1,\text{max}}$. Finally, if $u_2^*$ is singular as well, we know from our previous discussions that $x_1^*(t_s)$ equals to zero and (11) holds.

We have thus shown that both optimal controls will in general be bang-bang unless the motor and link velocity are equal to each other ($\ddot{x}_1^* = 0$) and either the angular deflection or the instantaneous stiffness remains at zero.

IV. CONSTRUCTING THE OPTIMAL TRAJECTORY

In the previous section we showed that both controls in $u^*$ switch between their minimum and maximum values along the optimal trajectory, unless a singular trajectory exists. In this section, we will focus on the trajectory of the states and costates for bang-bang controls and show how to systematically construct the optimal trajectory when we first ignore any singular arcs.

For a constant motor velocity $u_1$, it is possible to describe the dynamics of the angular deflection in terms of a second order differential equation:

$$M \ddot{x}_1 + \tau_{j,i}(x_1) = 0, \quad (14)$$

where $i \in \{1, 2\}$ and $\tau_{j,i}$ is obtained whenever $u_2 = -1$ and $\tau_{j,2}$ whenever $u_2 = 1$. This second order differential equation corresponds now exactly to the dynamics of a VSJ with a fixed motor position $\theta$ and fixed stiffness controller $\sigma = u_2$. Consequently, it describes a conservative system with the constant energy $E = \frac{1}{2} M \dot{x}_1^2 + \int_{0}^{x_1^*} \tau_{j,i}(\xi) d\xi$ so that we have the following relation between the angular velocity and deflection:

$$\dot{x}_1(x_1) = \pm \sqrt{\frac{2(E - \int_{0}^{x_1^*} \tau_{j,i}(\xi) d\xi)}{M}}. \quad (15)$$

Note that since we ignore any singular arcs, we know that $x_1$ and $\dot{x}_1$ cannot be equal to zero simultaneously and we always have $E > 0$ along non-singular trajectories. Consequently, for constant $u^*$ the angular deflection $x_1$ and its time derivative $\dot{x}_1$ will oscillate periodically such that the maximum velocity $\phi_{max}$ is obtained whenever $x_1 = 0$ and the maximum deflection $\phi_{max}$, whenever $\dot{x}_1 = 0$. Furthermore, since the resulting system (14) is equivalent to the VSJ with only one stiffness controller as analysed in [11], we can also express the trajectory of the costates in terms of the angular deflection. Indeed, most of our previous results still apply for the construction of the OC strategy for the more general system (4) with motor control. Nevertheless, we need to take care about some additional properties of the OC trajectory such as the discontinuity of the angular velocity $\ddot{x}_1 = u_1^* - x_2^*$, whenever $u_1^*$ switches.

We will next follow two main steps to introduce our method to compute OC strategies for variable stiffness joints with velocity-controlled motors: We first summarize our results from [11] regarding the description of the costates along the state trajectory, where we additionally analyse different torque profiles which can be used to describe the different torque characteristics of some existing designs such as the DLR FSJ [17]. Secondly, based on the solution of the costates we discuss the switching structure of both controls $u^*$ and clarify the behaviour of the optimal trajectory at the switching positions.

A. The Costates along the Angular Deflection ($u = \text{const.}$)

In this subsection, we want to shortly describe the solution of the costates in terms of the normalized angular deflection $\bar{x}_1$ defined as:

$$\bar{x}_1 := \frac{x_1}{\phi_{\text{max}}} \in [-1, 1]. \quad (16)$$

It is possible to write the costates as a function of this normalized deflection for constant $u$ and thus constant $E$. Indeed, substituting the derivatives $\lambda_2 = \frac{d\lambda_2}{dx_1} \phi_{\text{max}}$ and $\lambda_2 = \frac{d\lambda_2}{dx_1} \phi_{\text{max}}$ into the differential equation (9) yields [11]:

$$\lambda_2(\bar{x}_1) = \left(\frac{\phi_{\text{max}}}{\bar{x}_1}\right) \lambda_2(\bar{x}_1), \quad \lambda_2(\bar{x}_1) = \frac{\phi_{\text{max}}}{\bar{x}_1} \lambda_2(\bar{x}_1) \quad (17)$$

$$\lambda_2(\bar{x}_1) = \left(\frac{\phi_{\text{max}}}{\bar{x}_1}\right) \lambda_2(\bar{x}_1) \quad (18)$$

with $\lambda_2(\bar{x}_1) := \frac{d\lambda_2}{dx_1} \bar{x}_1$ and $I(\bar{x}_1) := \int_{\bar{x}_1}^{x_1} \left| \phi_{\text{max}}(\xi) \right|^{-1} d\xi$. Furthermore, the two boundary conditions on which the solutions depend are evaluated at the equilibrium position of the system (14): $\lambda_2(\bar{x}_1) = \lambda_2(\bar{x}_1) = 0$. The time derivative of the second costate $\lambda_2$, on which the

Note that $\bar{x}_1$ is always well-defined for positive $E$.

3Take the time derivative of the first row of (4) and use its second row to obtain this relation.

4When using equations (17)-(19) to construct the costates, $x_1$ is to be substituted by $\phi_{\text{max}} \cdot \bar{x}_1$, see (16).
optimal motor velocity depends (see (8) and (9)), can now also be computed by noting that \( \dot{\lambda}_2 = \lambda_2' \frac{\dot{x}_1}{\phi_{\max}} \).

\[
\dot{\lambda}_2(\bar{x}_1) = -\frac{\tau J_1(x_1)}{M \phi_{\max}^2} \text{sgn}(\dot{x}_1) \left( \lambda_{20} + C(\bar{x}_1) \lambda_{20}' \right),
\]

(19)

where \( C(\bar{x}_1) := \hat{I}(\bar{x}_1) - \frac{M \phi_{\max}^2}{\tau J_1(x_1) \phi_{\max} |x_1|} \).

It is important to first notice that these three solutions remain only valid as long as \( u \) is constant. Furthermore, when \( x_1 = 0 \) or equivalently when \( \bar{x}_1 = \pm 1 \) holds, the derivative \( \lambda_2' \) is not well-defined and the solutions have singularities at these positions. Making use of the continuity of \( \lambda_2 \) and \( \dot{\lambda}_2 \) at these singularities, it can be shown that for constant \( u \), a jump in the boundary condition \( \lambda_{20} \) will in general follow after reaching the singularity, whereas \( \lambda_{20}' \) remains constant [11]:

\[
\lambda_{20}^+ = -\lambda_{20}^- \pm \Delta \cdot \lambda_{20}',
\]

(20)

\[
\lambda_{20}' = \lambda_{20}'^-.
\]

(21)

The subscripts \((\pm)\) in these two equations are used to denote the boundary conditions before and after the singularity is reached. In addition, the constant \( \Delta := 2 \lim_{x_1 \to 1} - C(\bar{x}_2) \) is to be subtracted from \( \lambda_{20}^- \) once the positive singularity \( \bar{x}_1 = 1 \) is reached and added if the negative singularity \( \bar{x}_1 = -1 \) is reached.

Figure 2 visualises the trajectory of the costates along \( \bar{x}_1 \) for constant \( u \) and for four different torque profiles (\( M = 2kgm^2, \lambda_{20}^- = 1, \lambda_2(0) = \frac{1}{3}, \dot{x}_1(0) = 12\frac{kg}{s} \)). The parameters of the torque profiles are chosen such that the potential energy stored in the spring of each system is equal to each other at their maximum deflection \( \phi_{\max} = 1 \text{rad} \) and all the joints have the same positive initial angular velocity \( \dot{x}_1(0) \). Furthermore, the angular velocities of all four systems are allowed to change their signs only once at \( \bar{x}_1 = 1 \) in \( \bar{x}_1 \in [-1,1] \) so that there is only one jump in \( \lambda_{20} \). The expressions needed to construct these costates using (17)-(19) can be computed analytically and are summarized in Table I. Note that after reaching the singularity, \( \lambda_2 \) only changes its sign if the torque profile is linear \( (\Delta = 0) \) whereas for the other nonlinear torque-profiles the magnitude \( |\lambda_{20}| \) also changes, see Fig. 2 (Left). Furthermore, since \( \lambda_{20}^- \) and \( \phi_{\max} \) are not affected by the singularity we see that all four graphs for \( \lambda_2 \) in Fig. 2 (Right) intersect at \( \bar{x}_1 = 0 \). In order to better understand the behaviour of these costates, we will now shortly analyse the torque profiles for which analytical solutions could be found.

1) \( \tau J(\phi) = k_0 \phi^n, \quad n \in \{1,3,5,\ldots\} \): By using (15) and noting that \( E = \frac{1}{2} M \phi_{\max}^2 = \frac{n+1}{n+2} k_0 \phi_{\max}^2 \) holds, we can show that the integral \( \hat{I}(\bar{x}_1) \) in (17) can equivalently be written as

\[
I(\bar{x}_1) = \int_0^{\bar{x}_1} (1 - \xi^{n+1})^{-\frac{3}{2}} d\xi,
\]

(22)

so that the integral is explicitly described in terms of hypergeometric functions [11], see Table I (Third Column). Describing the costates (17)-(18) using this analytical expression for \( I(\bar{x}_1) \) is straightforward and omitted for brevity. It is however important to mention here that for these particular torque profiles the normalized angular deflection, which relates the system’s current potential energy to the system’s total energy, uniquely describes the way the costates behave if the boundary conditions are known.

2) \( \tau J(\phi) = k_1 \phi + k_3 \phi^3 \): When the torque profile consists of one linear and one cubic part in terms of the angular deflection, an analytical solution can also be obtained. By introducing the constant \( K_1 := \frac{k_0 \phi_{\max}^2}{2} \), which relates the maximum potential energy stored in the nonlinear part of the spring to the linear part and using (15), the integral \( I(\bar{x}_1) \) can be shown to depend on both \( \bar{x}_1 \) and \( K_1 \), see Table I (Fourth Column). The description of the costates is thus now not only a function of \( \bar{x}_1 \) but also of \( K_1 \), a quantity that depends on the energy. Similarly, the constant \( \Delta \) and thus the change of \( \lambda_{20} \) at the singularities also depend on the energy as we can observe from Table I. Finally, note that the integral \( I(\bar{x}_1) \) can be written explicitly in terms of incomplete and complete elliptic integrals of the first and second kind [1].

3) \( \tau J(\phi) = K_2 \sin(k_0 \phi) \): If the torque in the elastic joint spring is described by a hyperbolic sine function, we can similarly show that the integral \( I(\bar{x}_1) \) depends not only on \( \bar{x}_1 \), but also on the constant \( K_2 = k_0 \phi_{\max} \) (see Table I, Last Column). Consequently, the behaviour at the singularities depends again on the system’s current energy. Furthermore, the integral \( I(\bar{x}_1) \) can be written explicitly using hyperbolic cosine functions and elliptic integrals of the first, second and third kind [1].

Having discussed the solution of the costates for different torque profiles, we focus now on finding the positions at which the control \( u^* \) will switch.

B. Optimal Switching Structure

So far, we have talked about the trajectory of the costates assuming \( u^* \) stays constant. According to (8), we know however that \( u^* \) will in general change depending on the sign of the angular deflection and the costates. In order to fully construct the optimal control trajectory, we need to first determine the positions at which these changes occur. Furthermore, once these positions are obtained we need to find out how changing \( u^* \) affects the trajectory of the states and costates. We will now discuss the solution to these two problems.

1) Switching Curves and the Optimal Control \( u^* \): In the previous section, we showed how both costates can be described in terms of the angular deflection. Since the necessary conditions (8) only depend on these costates and the angular deflection, using equations (17)-(19), we can actually rewrite these conditions such that they are only functions of the angular deflection. Indeed, introducing the ratio \( r := \frac{\lambda_2}{\lambda_{20}} \) the necessary condition for the optimal motor

![Fig. 2: The Costates along the Angular Deflection](image-url)
velocity becomes

\[ u_1^* = \begin{cases} 
\tau_{1,\text{max}} \text{sgn} \left( x_1 x_2 \lambda_{20} \right) & , \quad \lambda_{20} \neq 0 \\
\tau_{\text{min},20} \text{sgn} \left( \lambda_{20} x_1 x_1 \right) , & , \quad \lambda_{20} = 0
\end{cases} \]

(23)

whereas for \( u_2^* \) we have

\[ u_2^* = \begin{cases} 
-\text{sgn} \left( x_1 x_2 x_2 \lambda_{20} \right) & , \quad \lambda_{20} \neq 0 \\
-\text{sgn} (x_1 x_2) , & , \quad \lambda_{20} = 0
\end{cases} \]

(24)

In order to better understand how to make use of these two equations, we will now discuss some properties of the integral \( I(\bar{x}_1) \) and the function \( C(\bar{x}_1) \).

Figure 3 illustrates \( I(\bar{x}_1) \) and \( C(\bar{x}_1) \) for different torque profiles, where the solid and dashed lines represent \( C(\bar{x}_1) \) and \( I(\bar{x}_1) \), respectively. It can be shown that the integral \( I(\bar{x}_1) \) is a monotonously increasing function, which is symmetrical with respect to the origin and tends to infinity at the singularities \( \bar{x}_1 = \pm 1 \), see [11]. Similarly, \( C(\bar{x}_1) \) is also symmetrical with respect to the origin, since \( C(-\bar{x}_1) = -C(\bar{x}_1) \) holds, see Table I (Fourth Row). In addition, \( C(\bar{x}_1) \) is a monotonously increasing function in \( \bar{x}_1 \in (1, 0) \) and \( \bar{x}_1 \in (0, 1) \), since \( \frac{dC}{dx_1} = \frac{\partial C_{\text{max}}}{\partial x_2} \cdot \frac{dx_1}{dx_1} = \geq 0 \) holds in these intervals. Finally, it tends to infinity at \( \bar{x}_1 = 0 \), where it also has a discontinuity and at the singularities we have \( \lim_{\bar{x}_1 \to \pm 1} C(\bar{x}_1) = \pm \frac{x_1}{x_1} \). Notice that for the linear and cubic torque profiles in Fig. 3, both \( I(\bar{x}_1) \) and \( C(\bar{x}_1) \) are merely functions of \( x_1 \), whereas for the other two profiles \( \phi_{\text{max}} \) has also an additional influence on these functions (see \( K_1 \) and \( K_2 \) in Table I). Furthermore, \( I(\bar{x}_1) \) and \( C(\bar{x}_1) \) do not have any intersection point, since \( I(\bar{x}_1) - C(\bar{x}_1) = \frac{\partial C_{\text{max}}}{\partial x_2} \cdot \frac{dx_1}{dx_1} \geq 0 \) holds.

Using equations (23)-(24) and the discussed properties for \( I(\bar{x}_1) \) and \( C(\bar{x}_1) \), which are valid for any torque profile satisfying our initial assumptions, we can now graphically determine the optimal control \( u^* \) along the optimal trajectory. Indeed, \( u^* \) is uniquely determined in terms of \( \bar{x}_1 \), the boundary conditions \( \lambda_{20}, \lambda_{20} \) with the corresponding ratio \( r \) and finally the direction of the trajectory. This dependence can be illustrated qualitatively in one graph as done in Figure 4. For \( \lambda_{20} \neq 0 \), the solid blue lines in Fig. 4 correspond to \( -C(\bar{x}_1) \), whereas the dashed lines to the y-axis and to \( -I(\bar{x}_1) \). In addition, the black lines correspond to curves with constant \( r \) if \( \lambda_{20} \neq 0 \) and to curves with constant \( \lambda_{20} \), otherwise. Note that for constant \( u \), \( \lambda_{20} \), \( \lambda_{20} \) and the corresponding ratio \( r \) are all uniquely determined by the system’s boundary conditions and only change at \( \bar{x}_1 = \pm 1 \). The solid and dashed blue lines in Figure 4 can thus be regarded as switching curves, since a switching of the control \( u^* \) will almost always occur, if the black curve corresponding to the ratio \( r^* \) (Fig. 4, Left and Right) or \( \lambda_{20} \) (Fig. 4, Middle) along the optimal trajectory, intersect with one of these lines. Indeed, if none of the controls change \( \bar{x}_1 \) is known to keep oscillating and according to (23)-(24) with (20) the optimality conditions will then not be satisfied. In other words, the intersection points reveal the switching positions at which \( u^* \) switches. In order to fully construct the OC trajectory, we need to now only determine the behaviour of the OC trajectory at these switching positions.

2) Behaviour at the Switching Positions: In this subsection we will provide the equations which describe the change of the OC trajectory after a switching. In particular, we show how the normalized angular deflection \( \bar{x}_1 \), the sign of the angular velocity \( \bar{x}_1 \) and the boundary conditions \( \lambda_{20} \) and \( \lambda_{20} \) are influenced by a switching, since they uniquely determine the optimal control. In order to simplify our discussion, we will use an upper left superscript, which corresponds to the number of the switchings the control has done along the OC trajectory after a switching.

7The only exception is the situation when we have \( r^* = 0 \) and \( \lambda_{20} \neq 0 \). In this case no change in the controls will occur when passing through the origin, which is the intersection point of the two dashed lines (see Fig. 4).

Note that \( \lambda_1 = \lambda_2 \) does not only depend on \( \bar{x}_1 \) but also on \( \text{sgn}(\bar{x}_1) \), see (19).
optimal trajectory. Consequently, we introduce the following variables

\[ i \bar{r}, \quad i \bar{E} := u \quad \text{and} \quad E \quad \text{after the} \quad i \text{-th switch of} \quad u \]
\[ i \bar{x}_1, \quad i \bar{x}_{10} := x_1 \quad \text{and} \quad \bar{x}_1 \quad \text{at the} \quad i \text{-th switch} \]
\[ \bar{r}, \quad \bar{x}_1 := x_1 \quad \text{and} \quad \tilde{x}_1 \quad \text{at the} \quad i \text{-th switch}, \]
\[ \bar{r}, \quad \bar{r} := r \quad \text{after the} \quad i \text{-th switch}, \]

with the corresponding normalized variables \( i \bar{x}_{10} := \frac{i \bar{x}_{10}}{\phi_{\text{max}}} \)

and \( i \bar{x}_1 := \frac{i \bar{x}_1}{\phi_{\text{max}}} \), where \( \phi_{\text{max}} \) is the maximum deflection corresponding to \( i \bar{E} \) and \( i \in \{1, 2, \ldots \} \).

From the continuity of the displacement \( x_1 \), we can immediately see how \( i \bar{x}_1 \) and \( i+1 \bar{x}_{10} \) are related to each other:

\[ i \bar{x}_1 = i+1 \bar{x}_{10} \Rightarrow i+1 \bar{x}_{10} = i \bar{x}_1 \frac{\phi_{\text{max}}}{\phi_{\text{max}}}. \]

According to (25), we can conclude that \( \bar{x}_1 \) will jump at the switching position \( i \bar{x}_1 \), unless \( \phi_{\text{max}} = i+1 \phi_{\text{max}} \) or \( i \bar{x}_1 \) is zero. We have already shown that the optimal control \( u^* \) will switch between constant values unless a singular trajectory exists. The energy \( i+1 \bar{E} > 0 \) will thus always stay constant in a finite time interval after the switching of \( u^* \), if we ignore singular trajectories. Consequently, we can still apply the coordinate transformation we used in Section IV-A and all the resulting equations (17)-(24) remain valid. In particular, the solution for \( \lambda_{20}(\bar{x}_1) \) and \( \lambda_{20}(\bar{x}_1) \) can still be written as a linear combination of the same two linearly independent functions given in (17) and (18). As already mentioned, the constants \( \lambda_{20}, \lambda_{20} \) in these equations will however in general change, since \( \lambda_{20} \) and \( \lambda_{20} \) are continuous while \( \bar{x}_1 \) is not. In addition, remember that the description of the costates does not only change after a switching of \( u^* \). It also changes, when a singularity \( \bar{x}_1 = \pm 1 \) is reached as already discussed in Section IV-A and illustrated in Figure 2.

The change of the boundary conditions \( \lambda_{20}, \lambda_{20} \) and the sign of \( \bar{x}_1 \) can now be found similar to (25) by making use of the continuity of the costates \( \lambda_{2}, \lambda_{2} \) and the link velocity \( x_2 \) at the switching positions. Table II summarizes the changes of these four quantities depending on the switching position and the control that switches\(^8\). One important property we can conclude from the results in Table II is that if \( \lambda_{20} \) equals to zero, it always remains at this value. Along the optimal trajectory, both controls will then switch simultaneously at \( x_1 = 0 \) and the stiffness controller switches additionally whenever \( \phi_{\text{max}} \) corresponding to the current energy is obtained (See Fig. 4, Middle)\(^9\). Furthermore, if \( \lambda_{20} \neq 0 \) holds, the sign of \( \lambda_{20} \) never changes since \( \phi_{\text{max}} \) and \( E \) are always positive.

To sum up, we have provided all the necessary equations needed to construct the optimal trajectory and the corresponding optimal control along the singular arcs in Section III and along the nonsingular arcs in Section IV. In the following section, we will make use of our results to analyse the OC strategy for maximizing the link velocity of elastic joints.

V. MAXIMIZING THE LINK VELOCITY OF ELASTIC JOINTS

In this section, we want to analyse the maximum link velocity which elastic joints can obtain in a given final time \( t_f \), when they are initially at rest: \( x(0) = 0 \). In particular, we want to compute this maximum velocity for the torque profiles we analysed in Section IV-A and discuss the difference between their OC strategies. The OC problem we need to solve for this analysis consists of the terminal cost:

\[ J(u) = -\beta_2 x_2(t_f), \]

where \( \beta_2 \) can be any positive constant scalar. The boundary conditions for \( \lambda^* \) at the final time can then be obtained from the Minimum Principle as (see (10)):

\[ \lambda^*(t_f) = \left( \begin{array}{c} 0 \\ -\beta_2 \end{array} \right). \]

According to (27), the first costate \( \lambda_1^*(t_f) \) is equal to zero, whereas the value of \( \lambda_2^*(t_f) \) can be any negative scalar. Furthermore, since the system is initially at rest we can conclude from the symmetry of the system (4) that if the optimal control \( u^* = (u_1^* \quad u_2^*) \) minimizes \(-\beta_2 x_2(t_f)\), the control \( u = (-u_1^* \quad u_2^*) \) will minimize \( \beta_2 x_2(t_f) \) at the terminal time \( t_f \). This means that the sign of \( \lambda_2^*(t_f) \) does not play any role in determining the switching structure. It only affects the initial sign of \( u_2^*(0) \). We have already shown that along non-singular arcs the positions at which \( \lambda_1^* \) equals to zero correspond exactly to positions at which \( u_1^* \) switches.

With this property, we can find the solution to our optimal control problem using our results from Section IV as follows:

Given an initial ratio \( r \in (-\infty, \infty) \) or the initial boundary condition \( \lambda_{20} = 0 \), the optimal switching structure is uniquely determined along the whole optimal trajectory (see Fig. 4 and Table II)\(^10\). The positions \( \bar{x}_1^* \), at which the line of constant ratio \( r^* \) or constant \( \lambda_{20} \) intersects the switching curve for \( u_1^* \) in Fig. 4, correspond then to candidates for the terminal state of the optimal trajectory. Indeed, if we denote the time needed to reach one of these positions by \( t_f \), the link velocity \( \dot{q}(t_f) \) at this position will be a candidate for the maximum link velocity at the final time \( t_f \), since all the

\(^8\)We omit a thorough analysis for brevity. A detailed discussion of these changes at the switching positions of \( u_2 \) is given in [11]. The changes at the other switching positions can be derived similarly.

\(^9\)This particular switching structure makes intuitively sense, since at the switching positions the maximum kinetic and potential energy are obtained, respectively. As we will see in Section V, the optimal control structure is however in general more complicated.

\(^10\)Using (8) with \( x(0) = 0 \), we can conclude that \( \lambda_{20} \leq 0 \) holds since we have \( \lambda_{20}(u_1(0)) = \lambda_{20}(\bar{x}_1(0)) \neq u_1(0) = \frac{\phi_{\text{max}} \lambda_{20}}{\phi_{\text{max}} \lambda_{20}} \leq 0 \). Consequently, only Fig. 4 (Left) and Fig. 4 (Middle) need to be used for determining the switching structure of \( u^* \).
If Velocity Control $u_1^*$

Stiffness Control $u_2^*$

Position $i\bar{x}_1^* = 0$ $i\bar{x}_1^* \in (-1, 1) \backslash \{0\}$ $i\bar{x}_1^* = \pm 1$

Condition If $\lambda_{20} = 0$ If $\lambda_{20} \neq 0$ If $r = \pm \frac{1}{4}$

$|i\bar{x}_1| < 1$ 0 $|i\bar{x}_1| < 1$ 1

$|i\bar{x}_1| = 0$ $|i\bar{x}_1| - 2 i u_1$ $-2 i u_1$ $0$

$|i\bar{x}_1| = \pm 1$ $|i\bar{x}_1| - 2 i u_1$, $\lambda_{20} = 0$ $i\dot{x}_1^*$ $0$

$|i\bar{x}_1| = \ldots$ $\lambda_{20} = 0$ $|i\bar{x}_1| - 2 i u_1$, $\lambda_{20} = 0$ $i\dot{x}_1^*$ $0$

$|i\bar{x}_1| = \ldots$ $\lambda_{20} = 0$ $|i\bar{x}_1| - 2 i u_1$, $\lambda_{20} = 0$ $i\dot{x}_1^*$ $0$

$|i\bar{x}_1| = \ldots$ $\lambda_{20} = 0$ $|i\bar{x}_1| - 2 i u_1$, $\lambda_{20} = 0$ $i\dot{x}_1^*$ $0$

TABLE II: Behaviour at the Switchings ($i\tau_{J_0} = \tau_f(i\bar{x}_{10}, i\bar{u}_2)$, $i\tau_{J_s} = \tau_f(i\bar{x}_1^*, i\bar{u}_2)$)

optimality conditions are satisfied. In addition, considering all the possible initial ratios or equivalently all possible initial switching positions $\frac{1}{2}x_i^* \in [-1, 1]$ on the switching curves, we can construct the set of all trajectories that satisfy the optimality conditions. The switching structure which yields the maximum $\dot{q}(t_f)$ at $t_f$ will then give us the maximum link velocity as well as the OC strategy\(^{11}\).

In the following two subsections, we will look at the solution of this OC problem for various SEA’s ($\sigma = \text{const.}$) and VSJ’s ($\sigma = u_2$).

A. Serial Elastic Joints ($\sigma = \text{const.}$)

In this subsection, we discuss the maximum link velocity for the four fixed torque profiles we already investigated in Section IV, see Fig. 2-3. Figure 5 illustrates the maximum velocity $\dot{q}(t_f)$ as a function of the terminal time with $u_{1,\text{max}} = \frac{1}{4}\frac{\pi}{r}$, where the solution is found by following the procedure mentioned above. According to this figure, we see first of all that two different solutions can sometimes be found for a given $t_f$, as we see for instance at $t_f \approx 2.7s$ for the SEA with the cubic torque profile\(^{12}\). Note that such a local solution does not exist for SEA’s with linear torque profiles. Nevertheless, for all the torque profiles $\max \dot{q}(t_f)$ is a strictly increasing function of the final time $t_f$. For small final times ($t_f < 1s$) having linear springs is more advantageous than having nonlinear springs, whereas when the final time is increased nonlinear springs are more beneficial. Especially the SEA with the cubic spring outperforms its linear counterpart by obtaining almost twice the final link velocity at $t_f = 4s$. The nonlinearity in the torque profile described by the hyperbolic sine function, on the other hand, does not bring much advantage.

A comparison of the optimal switching structures for this last SEA and the SEA with linear springs is illustrated in Figure 6\(^{13}\). Note that since $\sigma$ is fixed, we can simply ignore the switching curves regarding the stiffness controller and only intersection points of $r^*$ with $-C(\bar{x}_1)$ are computed when constructing the OC trajectory. From Fig. 6 (Left), we can also see that for the SEA with linear springs $u_1^*$ always switches after one half period corresponding to the system’s eigenfrequency $\omega = \sqrt{\frac{M}{J}}$. Indeed, since $C(\bar{x}_1)$ is symmetrical with respect to the origin and $r^*$ only changes its sign at the singularities, we always have $|i\bar{x}_1| = |i\bar{x}_1|$, $i \in \{2, 3, \ldots\}$. In other words, the ratio of the system’s potential energy to the total energy just after the $i-1$th switch and just before the $i$th switch is equal to each other. The duration between these two switching positions corresponds exactly to one half period for linear systems and is independent of the system’s current energy. The optimal excitation is thus periodic with the system’s resonant frequency. If we now look at Fig. 6 (Right), we see that the symmetry between these two positions are lost, since $-C(\bar{x}_1)$ is energy-dependent and $r^*$ does not only change its sign but also its magnitude at the singularities $\bar{x}_1 = \pm 1$. The resulting optimal excitation becomes in this case energy-dependent and in general non-periodic.

B. Variable Stiffness Joints ($\sigma = u_2 \in [-1, 1]$)

In this subsection, we discuss the OC strategy of two different VSJ’s. The first one consists of an adjustable linear spring with $\tau_{J,i} = k_{i}\phi$ and the second one an adjustable cubic spring with $\tau_{J,i} = k_{c,i}\phi^3$ ($i \in \{1, 2\}$), $k_{1} = 2 \frac{N_m}{r a_{20}}$, $k_{2} = 8 \frac{N_m}{r a_{20}}$, $k_{c,1} = 4 \frac{N_m}{r a_{20}}$, $k_{c,2} = 16 \frac{N_m}{r a_{20}}$, $M = 2\text{kgm}^2$.

\(^{11}\)Note that the duration needed to reach the final position $\bar{x}_1^*$ can be calculated using (15). Furthermore, singular arcs are hence ignored in this discussion. This can be justified for the joints we analyse by first noting that $\frac{\partial \tau_{J,i}}{\partial x_1}$ is everywhere positive, except possibly at $x_1 = 0$ and that $\max \dot{q}(t_f)$ is a strictly increasing function of the terminal time $t_f$ as we later see in Figures 5 and 7.

\(^{12}\)This is due to the fact that Minimum Principle only provides necessary conditions for the OC\([12]\).

\(^{13}\)The additional subscripts ($\pm$) in $r$ are used similar to (20) and (21), if singularities $\bar{x}_1 = 1$ are reached.
storable potential energy in the springs are equal to each other ($\frac{1}{2}k_{l,i} = \frac{1}{4}k_{c,i}, i \in \{1, 2\}$).

Figure 7 compares the maximum link velocity of both systems as well as their optimal switching structures for $t_f = 2.5s$. According to Fig. 7, we can see that for both systems $\max \dot{q}(t_f)$ is strictly increasing. In addition, similar to SEA’s, joints with adjustable nonlinear springs are only beneficial if $t_f$ is big enough.

If we now look at the switching structure of both systems in Fig. 7, we see directly how additional switching curves for $u_2^*$ influence the switching structure of the optimal control. Nevertheless, taking a closer look at the switching structure of the system with adjustable linear springs and comparing it with the SEA in Fig. 6 (Left), we see that the same physical interpretation regarding the switching positions of the motor control $u_1^*$ holds. Indeed, using the symmetry of $C(\bar{x}_1)$ and the jumps of the ratio at the switching curves for $u_2^*$ and at the singularities $\bar{x}_1 = \pm 1$, we can show that $|\dot{x}_{10}| = |\dot{x}_1|$ always holds between the switchings of the control $u_1^*$, see Fig. 7 (Left)\(^\text{15}\). For the VSJ’s with cubic springs, on the other hand, such a straightforward interpretation does not exist and the switching structure is much more complicated.

<table>
<thead>
<tr>
<th>Linear Spring: $\tau_J(\phi) = 2\alpha \phi^2$</th>
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<tbody>
<tr>
<td>Nonlinear Spring: $\tau_J(\phi) = \frac{\sinh(\phi)}{\cosh(1) - 1}$</td>
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**Fig. 6:** Comparison of Optimal Switching Structures ($t_f = 4s, \sigma = \text{const.}, \lambda_{20} < 0$)

**Fig. 7:** Comparison of $\max \dot{q}(t_f)$ and Optimal Switching Structures ($t_f = 2.5s, \sigma \in [-1, 1], \lambda_{20} < 0$)

**VI. CONCLUSION**

In this paper, we have introduced a general method for variable stiffness joints to construct the OC strategy minimizing a terminal cost. Applying the method to SEA’s and VSJ’s, we managed to extend the well-known principle of excitation of linear systems with resonant frequency to energy-dependent excitations of joints with nonlinear springs. Our current research focuses on experimentally verifying our theoretical results with existing designs. We also want to study the effect of other nonlinearities such as external forces caused by gravity. Finally, we want to investigate how the dynamics of the velocity-controlled motor and the stiffness controller influence the optimal control strategy.

**REFERENCES**


