Irreducible collineation groups with two orbits forming an oval ✪

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Abstract

Let G be a collineation group of a finite projective plane π of odd order fixing an oval Ω. We investigate the case in which G has even order, has two orbits Ω0 and Ω1 on Ω, and the action of G on Ω0 is primitive. We show that if G is irreducible, then π has a G-invariant desarguesian subplane π0 and Ω0 is a conic of π0.

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1. Introduction

The famous theorem by Segre states that in the desarguesian projective plane PG(2, q), q odd, every oval consists of all points of a conic, see [20], [13, Theorem 8.2.4] and [14, Theorem 12.9]. Therefore the full collineation group of PG(2, q), q odd, fixing an oval is isomorphic to PΓL(2, q) and it acts on the points of the oval as in its 3-transitive permutation representation.

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In this paper we deal with ovals $\Omega$ and their collineation groups $G$ in any projective plane of odd order $n$. From earlier work by Cofman, Kantor, Lüneburg and Korchmáros it emerged that certain conditions on the structure and the action of a collineation group on an invariant oval imply that the plane must be desarguesian. This initiated a still ongoing research activity aimed at characterizing the desarguesian plane by means of the collineation group fixing an oval. The deepest result so far states that if $G$ acts primitively on $\Omega$, then $\pi$ is a desarguesian plane, $\Omega$ is a conic and, up to one exception, $G$ is doubly transitive on $\Omega$; see [2], and also Theorem 2. It has been conjectured that even transitivity on $\Omega$ implies that $\pi$ is desarguesian, see [17].

Ovals with nice collineation groups are also known to exist in some non-desarguesian planes, notably in the Hughes planes, in the generalized Hughes planes of order 25 and 49, see [5], as well as in the Figueroa planes, see [7], and also Section 3. The collineation groups of these ovals have the following common properties:

1. $G$ has two orbits on $\Omega$, say $\Omega_0$ and $\Omega_1$;
2. $G$ is 2-transitive on $\Omega_0$.

This raises the problem of determining all collineation groups $G$ whose action on an invariant oval $\Omega$ satisfies both conditions (1) and (2).

In investigating this problem, the key idea is to prove the existence of an involutory homology in $G$ and use Hering’s work on collineation groups containing homologies. As a matter of fact, this can be done even if (2) is weakened to require only that $G$ has even order and acts primitively on $\Omega_0$.

The cases where $G$ either fixes a triangle with vertices off the oval, or a point outside $\Omega$ (or, by duality an external line to $\Omega$) have been already investigated, see [1,6]. In both situations, examples are known to exist.

In the remaining case, $G$ is irreducible on $\pi$, that is, it fixes no point, line or triangle of the plane. Our main result is the following theorem.

**Theorem 1.** Let $G$ be a collineation group of a finite projective plane $\pi$ of odd order $n$ such that:

(a) $G$ fixes an oval $\Omega$ of $\pi$ and the action of $G$ on $\Omega$ yields precisely two orbits $\Omega_0$ and $\Omega_1$;
(b) $G$ has even order and acts primitively on $\Omega_0$;
(c) $G$ is an irreducible collineation group.

Then the following holds:

(i) $G$ contains some involutory homologies;
(ii) if $\pi_0$ denotes the substructure of $\pi$ generated by the centers and axes of the non-trivial homologies in $G$ while $K_0$ denotes the kernel of the action of $G$ on $\pi_0$, then $G/K_0$ contains a normal subgroup which is isomorphic to $\text{PSL}(2, q)$ for some odd prime power $q$;
(iii) $\pi_0$ is a desarguesian subplane of odd order $m$, $\Omega_0$ is a conic in $\pi_0$ and either $m = q$ and $G/K_0$ is 2-transitive on $\Omega_0$ or $m = 9$, and $G/K_0$ acts on $\Omega_0$ as $A_5$ or $S_5$ in their primitive representation of degree 10.

Finally we remark that the above mentioned ovals in the Hughes, generalized Hughes and Figueroa planes are the known examples for Theorem 1.
2. Preliminary results

Basic facts on finite projective planes and their ovals can be found in [8] whereas we refer to [15] for what concerns finite groups. In particular $O(G)$ stands for the largest normal subgroup of odd order of a finite group $G$.

Let $\pi$ be a projective plane of odd order $n$, $\Omega$ an oval of $\pi$ and $G$ a collineation group of $\pi$. We collect some previous results that will play a role in the proof of Theorem 1.

Proposition 1. (See [2, Proposition 2.1].) Any perspectivity of $\pi$ fixing $\Omega$ is an involutory homology and either the center of $\sigma$ is an internal point and the axis is an external line of $\Omega$, or the center of $\sigma$ is an external point and the axis is a secant of $\Omega$. Furthermore any two distinct involutory homologies of $\pi$ fixing $\Omega$ have both distinct centers and distinct axes.

Theorem 2. (See [2, Main Theorem].) Let $G$ be a collineation group of $\pi$ leaving $\Omega$ invariant and acting primitively on its points. Then $\pi$ is desarguesian, $\Omega$ is a conic and either $n = q$ and $G$ contains a normal subgroup acting on $\Omega$ as $PSL(2, q)$ in its doubly transitive representation, or $n = 9$ and $G$ acts on $\Omega$ as $A_5$ or $S_5$ in their primitive representation of degree 10.

Theorem 3. (See [3, Theorem A].) Let $G$ be a collineation group of $\pi$ leaving $\Omega$ invariant. If $G$ is non-abelian simple then $G \cong PSL(2, q)$ for $q \geq 5$ odd, and all involutions are homologies.

Proposition 2. (See [11].) If $G$ contains non-trivial perspectivities, then the substructure $\pi_0$ of $\pi$ generated by the centers and axes of non-trivial perspectivities of $G$ is contained in each $G$-invariant subplane of $\pi$. Furthermore if $G$ is an irreducible group, then $\pi_0$ is a subplane of $\pi$.

Let $\pi_0$ be as in Proposition 2. If $\pi_0 \neq \emptyset$ and $G$ is irreducible we call $\pi_0$ the minimal Hering subplane and denote by $K_0$ the kernel of the action of $G$ on $\pi_0$.

Proposition 3. (See [11].) Let $G$ be an irreducible collineation group of $\pi$ containing non-trivial perspectivities. Then $G/K_0$ contains exactly one minimal normal subgroup $M$, and $M$ is either non-abelian simple or an elementary abelian of order 9. Furthermore $M$ contains its centralizer in $G/K_0$.

Theorem 4. (See [4, Theorem 5.1].) Let $G$ be an irreducible collineation group of $\pi$ which preserves $\Omega$ and contains no involutory homologies. Then

$$G = O(G) \rtimes S,$$

with $S \cong Z_{2^h}$.

Theorem 5. (See [4, Theorem 5.5].) Let $G$ be an irreducible collineation group of $\pi$ which preserves $\Omega$ and contains non-trivial homologies. Then $G/K_0$ contains a unique minimal normal subgroup $M_0$ which is non-abelian simple and

$$K_0 = O(G) \rtimes S,$$

where $S \cong Z_{2^h}$.

The following lemmas come from [22] and [1], respectively.
Lemma 1. Let $H$ be a collineation group of a finite projective plane $\pi$ and let $\Delta$ be one of its point-orbits. Let $N$ be a normal subgroup of $H$. If $N$ fixes a point of $\Delta$ then $\Delta$ is pointwise fixed by $N$. If $H$ acts primitively on $\Delta$ then either $N$ fixes $\Delta$ pointwise or $N$ is transitive on $\Delta$. In the latter case if $N$ is a minimal normal abelian subgroup of $H$ then it is regular on $\Delta$.

Lemma 2. Let $H$ be a collineation group of a finite projective plane $\pi$ fixing an oval $\Omega$. Assume that $H$ fixes at least three points on $\Omega$. Then $\text{Fix}(H)$ is a subplane of odd order and the fixed points of $H$ on $\Omega$ form an oval in $\pi$. Here $\text{Fix}(H)$ denotes the substructure of $\pi$ consisting of the points and lines which are fixed by every collineation in $H$.

We are going to prove the following result.

Proposition 4. Let $H$ be a collineation group of a finite projective plane $\pi$ fixing an oval $\Omega$ and having two orbits on it. Let $\Delta$ be one of its two point-orbits on $\Omega$ with $|\Delta| \geq 3$. Let $N$ be a non-trivial normal subgroup of $H$ fixing $\Delta$ pointwise. If $H$ acts primitively on $\Delta$, then $\Delta$ is a conic in the odd order desarguesian subplane $\text{Fix}(N)$ of $\pi$.

Proof. Since $|\Delta| \geq 3$, Lemma 2 yields that $\pi = \text{Fix}(N)$ is a subplane of odd order in $\pi$. If $N$ had a fixed point on $\Omega \setminus \Delta$, then by Lemma 1 it should fix $\Omega \setminus \Delta$ pointwise, whence $\Omega$ should also be pointwise fixed by $N$, that is $N = \{\text{id}\}$, a contradiction. We conclude that $N$ has no fixed point on $\Omega \setminus \Delta$. The set of fixed points of $N$ on $\Omega$ is precisely $\Delta$, which is thus an oval in $\pi$. If $K$ denotes the kernel of the action of $H$ on $\pi$ we get that $H/K$ induces a collineation group of $\pi$ fixing an oval $\Delta$ and acting primitively on its points. By Theorem 2 the subplane $\pi$ is desarguesian of odd order $q$, and $\Delta$ is a conic. \(\square\)

3. The case where $\Omega_0$ is an oval in a subplane

Throughout this section $G$ is a collineation group of a finite projective plane of odd order $n$ fixing an oval $\Omega$ of $\pi$ and satisfying properties (1) and (2) stated in the Introduction.

We further suppose that $\Omega_0$ is an oval in a subplane $\pi$ of odd order $m$. The group $G$ preserves $\pi$ and induces a collineation group $\overline{G}$ on $\pi$. Since $\overline{G}$ fixes $\Omega_0$ and acts 2-transitively on its points, Theorem 2 in [16] implies that $\pi$ is desarguesian, say $\pi = PG(2, p^r)$ for some odd prime $p$, and that the relation $PSL(2, p^r) \leq G \leq P\Gamma L(2, p^r)$ holds.

If we further assume that $G$ acts faithfully on $\pi$ or, equivalently, on $\Omega_0$, then $\overline{G} = G$ and we even have $PSL(2, p^r) \leq G \leq P\Gamma L(2, p^r)$.

Since $G$ is transitive on $\Omega_1$, we have that $|\Omega_1|$ is a divisor of $|G|$ and so, in particular, $|\Omega_1| \leq |G| \leq |P\Gamma L(2, p^r)| = rp^r(p^{2r} - 1)$. We obtain the estimate

\[ n + 1 = |\Omega_2| = |\Omega_1| + |\Omega_0| \leq rp^r(p^{2r} - 1) + p^r + 1 \]

whence,

\[ n \leq rp^{3r} - (r - 1)p^r. \]

Since $r \geq 1$, this gives $n \leq rp^{3r}$ whence $n < p^{4r}$. 

3.1. The desarguesian case

Let $p$ be an odd prime and let $r, k$ be positive integers. Assume $\pi = PG(2, p^r)$ to be the canonical subplane of $\pi = PG(2, p^{rk})$. Let $\Omega_0$ be the irreducible conic of equation $x_0^2 - x_1x_2 = 0$ of $\pi$. The same equation defines an irreducible conic $\Omega$ of $\pi$ whose points in $\pi$ are those of $\Omega_0$.

The setwise stabilizer of $\Omega$ in the full collineation group of $\pi$ is isomorphic to $P\Gamma L(2, p^{rk})$ in its natural 3-transitive permutation representation and contains $PGL(2, p^{rk})$ as a normal subgroup.

The semilinear transformations whose coefficients lie in the subfield $GF(p^r)$ and whose companion field-automorphisms fix $GF(p^r)$ setwise form a subgroup isomorphic to $P\Gamma L(2, p^r)$, the linear part of which is $G = PGL(2, p^r)$.

One orbit of $G$ on $\Omega$ is $\Omega_0$ and the action of $G$ on $\Omega_0$ is the natural action of $PGL(2, p^r)$ on $GF(p^r) \cup \{\infty\}$. Observe also that $G$ acts faithfully on $\pi$.

When does $G$ act transitively on $\Omega_1 = \Omega \setminus \Omega_0$?

A necessary condition is that $|\Omega_1| = (p^{rk} + 1) - (p^r + 1) = p^r(p^{rk-1} - 1)$ be a divisor of $|PGL(2, p^r)| = p^r(p^{2r} - 1)$, which is the case if and only if $k$ is 2 or 3.

We want to show that in either case $G$ is indeed transitive on $\Omega_1$. To this purpose we investigate the fixed points on $\Omega_1$ of a non-identity transformation $g$ in $G$.

If $g$ has exactly one fixed point on $\Omega_0$, then $g$ is a transformation of order $p$ and so, $g$ has a unique fixed point on the whole of $\Omega$ and this fixed point is in $\Omega_0$ already.

If $g$ has two different fixed points on $\Omega_0$, then since each non-identity transformation in $G$ has at most two fixed points, these are the two fixed points of $g$ on the whole of $\Omega$.

If $g$ is fixed-point-free on $\Omega_0$, then its characteristic polynomial $a(x)$ is an irreducible quadratic polynomial in $GF(p^r)[x]$. The splitting field of $a(x)$ is $GF(p^{2r})$, which is not a subfield of $GF(p^{3r})$. Therefore if $k = 3$, then $g$ is also fixed-point-free on $\Omega_1$ while if $k = 2$, then $g$ has two fixed points on $\Omega_1$.

In the former case the stabilizer in $G$ of a point in $\Omega_1$ is trivial and so $G$ acts semiregularly on $\Omega_1$. Therefore, $|G| = |\Omega_1| = p^r(p^{2r} - 1)$ and $G$ is also transitive on $\Omega_1$.

In the latter case the two fixed points of $G$ are also fixed by the unique Singer cyclic subgroup $S$ of $G$ of order $p^r + 1$, containing $g$; the normalizer $N$ of $S$ in $G$ exchanges the two fixed points of $S$ and so, since $N$ is the unique proper subgroup of $G$ properly containing $S$, we have that the stabilizer in $G$ of one such fixed point is precisely $S$. The $G$-orbit of this point has thus size $|G : S| = p^r(p^{2r} - 1)/(p^r + 1) = p^r(p^r - 1) = |\Omega_1|$ and we have that $G$ is transitive on $\Omega_1$.

Remark. In the previous construction, let $\sigma$ be the automorphism defined as follows $GF(p^{rk}) \rightarrow GF(p^{3rk}), x \mapsto x^{p^r}$, and consider the subgroup $\tilde{G}$ generated by $G$ and $\sigma$. Clearly $\tilde{G}$ has the same orbits on $\Omega$ as $G$; the subgroup $N = \langle \sigma \rangle$ is normal in $\tilde{G}$ and fixes $\Omega_0$ pointwise. The group $\tilde{G}$ with the normal subgroup $N$ yields thus an instance of the situation presented in Proposition 4.

3.2. The non-desarguesian cases

The above construction also works in some non-desarguesian planes.

Case (a). Let $q$ be an odd prime power and let $\pi$ denote the Figueroa plane of order $q^3$, see [10] and [12]. Let $P = PG(2, q^3)$ be the desarguesian plane of order $q^3$ from which $\pi$ arises using Grundhöfer’s construction [10], as outlined below.
Let $\alpha$ be a planar collineation of order 3 of $\mathcal{P}$ and let $I$ denote the incidence relation of $\mathcal{P}$. The points and lines of $\mathcal{P}$ fall into three disjoint classes each, depending upon the orbit structure under the group generated by $\alpha$, namely:

$$\mathcal{P}_1 = \{ P \in \mathcal{P} \mid P^\alpha = P \},$$

$$\mathcal{P}_2 = \{ P \in \mathcal{P} \setminus \mathcal{P}_1 \mid P, P^\alpha, P^{\alpha^2} \text{ are collinear} \},$$

$$\mathcal{P}_3 = \{ P \in \mathcal{P} \mid P, P^\alpha, P^{\alpha^2} \text{ form a triangle} \}.$$

The lines classes $L_1$, $L_2$, and $L_3$ are defined dually. Note that the points of class $\mathcal{P}_2$ are precisely the non-fixed points on a fixed line (a line of class $L_1$) and so these lines contain no points of class $\mathcal{P}_3$. We define an involutory bijection $\mu : \mathcal{P}_3 \rightarrow L_3$ by $P^\mu = P^\alpha P^{\alpha^2}$ and $\ell^\mu = \ell^\alpha \cap \ell^{\alpha^2}$ for $P \in \mathcal{P}_3$ and $\ell \in L_3$.

Let $I_{33} = I \cap (\mathcal{P}_3 \times L_3)$. To construct the Figueroa plane, we modify the incidences in this portion; in particular we define a new incidence relation $I_\alpha = (I \setminus I_{33}) \cup I^*$ where $I^* \subseteq \mathcal{P}_3 \times L_3$ and is defined by $P I^* \ell$ if and only if $\ell^\mu I P^\mu$. The resulting incidence structure $\pi$ is a non-desarguesian projective plane.

Now, let $\mathcal{C}$ be a conic which is invariant under $\alpha$. Observe that such a conic can contain neither a point from class $\mathcal{P}_2$, nor a tangent from class $L_2$. Let $\pi_\alpha$ be the subplane fixed by $\alpha$, and let $\Omega_0 = \mathcal{C} \cap \pi_\alpha$. If $\Omega_0$ is not empty then it is a conic in $\pi_\alpha$. Define $\Omega = \Omega_0 \cup \Omega_1$ with $\Omega_1 = \{ \ell^\mu \mid \ell \text{ is a tangent to } \mathcal{C} \text{ of class } L_3 \}$.

**Theorem 6.** The point set $\Omega$ is an oval in the Figueroa plane $\pi$.

Let $\pi_0$ be a subplane of $PG(2, q^3)$ of order $q$. Then $\pi_0$ is a subplane of $\pi$ as well. Furthermore, every collineation $\beta$ of $PG(2, q^3)$ fixing $\pi_0$ defines a collineation of $\pi$, that is, $\beta$ is an inherited collineation of $\pi$. In particular, the linear collineation group $PGL(2, q)$ of $\pi_0$ which preserves $\Omega_0$ (and $\mathcal{C}$) gives rise to an (inherited) collineation group $G \cong PGL(2, q)$ of $\pi$. By the construction of Cherowitzo’s oval (called “ovali di Roma” in [7]), $G$ preserves $\Omega$. Furthermore, $G$ acts on $\Omega_0$ as $PGL(2, q)$ in its sharply 3-transitive permutation representation, while the action of $G$ on $\Omega_1$ is the same as the action of $PGL(2, q)$ on $\mathcal{C} \setminus \Omega_0$. In particular, $G$ is transitive on $\Omega_1$.

**Case (b).** Let $q$ be an odd prime power and let $\pi$ be a projective plane of order $q^2$ which has a collineation group $\Gamma \cong PGL(2, q)$. Suppose that $\Gamma$ satisfies the following conditions:

(i) every involution in $G$ is a homology;

(ii) no two distinct involutions of $G$ have the same center, or the same axis.

By [5, Theorem 1], $\Gamma$ preserves a desarguesian subplane $\pi_0$, and one of the three orbits of $G$ in $\pi_0$ is an irreducible conic $\Omega_0$ of $\pi_0$. Suppose that $G$ has also the following property:

(*) a cyclic subgroup of maximal order of $G$ fixes a point $L$ in $\pi \setminus \pi_0$.

Let $\Delta$ be the $G$-orbit of such a fixed point $L$. Then by [5, Theorem 2], the set $\Omega = \Omega_0 \cup \Delta$ is an oval in $\pi$. Clearly, $G$ preserves $\Omega$ such that $\Omega_0$ is a 2-transitive orbit while $\Delta$ is a transitive orbit of $G$.
The above construction is known to provide ovals with properties (1) and (2) in the desarguesian plane and in the Hughes plane, as well as, in the generalized Hughes planes of order 25 and 49.

Let $\pi$ be the Hughes plane of order $q^2$ constructed from the Dickson near-field $R$, see [14]. Denote by $\pi_0$ the desarguesian subplane of $\pi$ of order $q$ arising from the kernel $F = GF(q)$ of $R$.

Let $\Gamma$ denote the collineation group of $\pi$. We know that $\Gamma$ is the product $\Sigma \rtimes \Theta$, where $\Sigma$ contains the extension to $\pi$ of the projectivities of $\pi_0$ and $\Theta$ contains the collineations of $\pi$ which are induced by the automorphisms of $R$. We have $\Sigma \cong PGL(3, q)$ and $\Theta \cong \text{Aut}(R)$, see [19].

Let $\Omega$ be the Room oval obtained from the conic $C$ in $\pi_0$ of equation $xy = z^2$, see [18]. The oval $\Omega$ is invariant under the group which contains the extensions to $\pi$ of the projectivities of $\pi_0$ fixing the conic $C$. Hence, there exists a subgroup $G$ of $\Gamma$ such that:

(I) $G \cong PGL(2, q)$;

(II) $G$ fixes $\Omega$ acting on $C$ in its natural 2-transitive permutation representation.

The group $G$ was proved to act transitively on $\Omega \setminus C$ in [5]. Therefore, $G$ has each of the properties (i), (ii) and (i∗). As it was pointed out in [5], the generalized Hughes plane of order $q^2$ has a collineation group $\Gamma \cong PGL(2, q)$ satisfying conditions (i) and (ii). Since the values of $q$ for which a generalized Hughes plane exists are 5, 7, 11, 23, 29, 59, a computer aided exhaustive search seems possible to decide whether $\Gamma$ also satisfies the condition (i∗). This has been done so far for $q = 5, 7, 11$, see Section 3 in [5]. In the two smallest cases $q = 5, 7$, but not for $q = 11$, the answer has been affirmative. Actually, the generalized Hughes plane of order 121 has a collineation group $G \cong PSL(2, 11)$ satisfying all three conditions (i), (ii) and (i∗). Nevertheless, the $G$-invariant conic $\Omega_0$ of $\pi_0$ does not extend to an oval of $\pi$.

4. Proof of Theorem 1

The first step is to prove (i) of Theorem 1.

**Proposition 5.** $G$ contains involutory homologies.

**Proof.** We distinguish two cases according to whether the group action on $\Omega_0$ is faithful or not. In the former case the assertion follows from [1, Proposition 1]. Thus, suppose that the kernel $K$ of the group action on $\Omega_0$ is not trivial.

Since $G$ is an irreducible group, $|\Omega_0| \geq 4$ and hence, Proposition 4 implies that $\Omega_0$ is a conic in the odd order desarguesian subplane $\text{Fix}(K)$ of $\pi$. In particular, we have that $|\Omega_1|$ and $|\Omega_0|$ are both even. If all involutions in $G$ are Baer involutions, then applying Theorem 4 it follows that $G = O(G) \rtimes S$, where $S \cong Z_2^s$, $s \geq 1$. Since $|O(G)|$ is odd, $O(G)$ cannot be transitive on $\Omega_0$ and Lemma 1 implies that $O(G)$ must fix $\Omega_0$.

If $O(G) \neq \{\text{id}\}$, then $O(G)$ fixes no point on $\Omega_1$ and again by Proposition 4, $\text{Fix}(O(G))$ is a desarguesian subplane of $\pi$. Hence Theorem 2 implies that the group $\overline{G}$ induced by $G$ on $\text{Fix}(O(G))$ contains either $PSL(2, q)$ or $A_5$. But this leads to a contradiction as $\overline{G}$ is isomorphic to a subgroup of the cyclic group $Z_2^s$.

Therefore $O(G)$ is trivial and $G$ is a cyclic group of order $2^s$. On the other hand $G$ primitive on $\Omega_0$ implies $s = 1$ and hence $|\Omega_0| = 2$, which contradicts the irreducibility of $G$.  □
Therefore the substructure $\pi_0$ generated by the centers and axes of the non-trivial homologies of $G$ is non-empty. Furthermore since $G$ is irreducible, Proposition 2 implies that $\pi_0$ is a subplane of $\pi$.

Now, applying Theorem 5 shows that the kernel $K_0$ of the group action on $\pi_0$ is

$$K_0 = O(G) \ltimes S,$$

where $S$ is a cyclic group of order $2^h$, $h \geq 0$. In addition the factor group $G_0 = G/K_0$ can be viewed as a strongly irreducible collineation group of $\pi_0$. Therefore, taking into account Proposition 3 and Theorem 5, it follows that $G_0$ contains a unique minimal normal subgroup $M_0$ which is non-abelian simple, and the centralizer $C_{G_0}(M_0)$ is trivial.

**Proposition 6.** If $K_0$ is trivial then $\Omega_0$ is the full intersection of $\pi_0$ and $\Omega$. Furthermore $\pi_0$ is a desarguesian subplane of odd order $m$, $\Omega_0$ is a conic in $\pi_0$, and $\text{PSL}(2,q) \leq G \leq \text{PGL}(2,q)$ up to isomorphism.

**Proof.** If $K_0$ is trivial then $G_0 = G$ and thus by Theorem 3, $G$ contains a minimal normal subgroup $M = M_0$ which is isomorphic to $\text{PSL}(2,q)$, with $q \geq 5$ odd. Furthermore, since $C_G(M) = \{\text{id}\}$, $M \trianglelefteq G \trianglelefteq \text{Aut}(M)$, that is, $\text{PSL}(2,q) \trianglelefteq G \leq \text{PGL}(2,q)$ up to isomorphism.

Next step is to show that $\Omega_0$ is contained in $\pi_0$. As the group $G$ preserves $\pi_0$, we have just to prove that $\pi_0 \cap \Omega_0 \neq \emptyset$.

First we observe that the action of $G$ on $\Omega_0$ is faithful, that is, $G$ is a permutation group on $\Omega_0$. In fact, if the kernel $K$ of the action of $G$ on $\Omega_0$ were not trivial, then by Proposition 4, $\text{Fix}(K)$ would be a $G$-invariant subplane of $\pi$. In particular the axes and the centers of the involutory homologies of $G$ would belong to $\text{Fix}(K)$, that is, $K$ would fix $\pi_0$. But then, $K$ would be a subgroup of $K_0$ which implies $K = \{\text{id}\}$, a contradiction.

We consider the action of $M$ on $\Omega_0$, and $G$ is a primitive permutation group on $\Omega_0$, it turns out that $M$ acts transitively on $\Omega_0$. Furthermore, to show that $M$ is not regular on $\Omega_0$, assume on the contrary that $G_A \cap M = \{\text{id}\}$ for every $A \in \Omega_0$. Then $G = G_A \ltimes M$. Since the factor group $\text{PGL}(2,q)/\text{PSL}(2,q)$ is commutative, see [9], $G_A \cong (G_A \ltimes M)/M$ is also commutative.

By [9, §3.4.5], the primitivity of $G$ on $\Omega_0$ implies that $G$ is a Frobenius group with Frobenius kernel $M$. But this yields that $M$ is solvable, see [9, §3.4.7], which is impossible.

Now, take a point $A \in \Omega_0$. Since the stabilizer $M_A$ in $M$ is not trivial, two cases are distinguished depending on the parity of $|M_A|$.

**Case (I).** $M_A$ has even order. We shall show that $M_A$ contains at least two involutory homologies. Assume on the contrary that $h = (C, \ell)$ is the unique involutory homology in $M_A$. Clearly, the axis $\ell$ is a secant to the oval $\Omega$ through the point $A$.

Suppose that $\ell$ is a secant to $\Omega_0$ as well. Then the secants to $\Omega_0$ which are images of $\ell$ under $M$ (and also under $G$), cut out on $\Omega_0$ pairwise disjoint pairs of points. Since $G$ is transitive on $\Omega_0$, these pairs define a $G$-invariant partition of $\Omega_0$, contradicting the primitivity of $G$ on $\Omega_0$.

Suppose that $\ell$ is not a secant to $\Omega_0$. Then $A$ is the unique common point of $\ell$ and $\Omega_0$. Choose an involution $g \in C_M(h)$ distinct from $h$. Then $gh = hg$, and hence $g$ also fixes $A$, a contradiction.

Therefore, let $s_1 = (C_1, \ell_1)$ and $s_2 = (C_2, \ell_2)$ be two distinct involutory homologies in $M_A$. Since $\ell_1$ and $\ell_2$ are lines in $\pi_0$, their common point $A$ is also in $\pi_0$, that is, $\Omega_0 \subseteq \pi_0$. 


Case (II). \( M_A \) has odd order. From the classification of subgroups of \( \text{PSL}(2,q) \), \( M_A \) is isomorphic to one of the following groups (see [21]):

(a) cyclic groups of order \( n \), with \( n \mid \frac{q+1}{2} \);
(b) semidirect products of elementary abelian \( p \)-groups of order \( p^f \) with cyclic groups of order \( n \), with \( f \leq m, n \mid p^f - 1 \) and \( n \mid \frac{q-1}{2} \);
(c) elementary abelian \( p \)-groups of order \( p^f \), \( f \leq m \).

Assume that \( M_A \) contains a cyclic subgroup \( C \) of order \( n \), with \( n \mid \frac{q+1}{2} \) and let \( h \) denote a generator of \( C \). By the classification of subgroups of \( \text{PSL}(2,q) \) \( M_A \) is contained in a dihedral subgroup of \( M \). Hence, \( h \) is the product of two involutory homologies in \( M \), say \( h_1 = (C_1, \ell_1) \) and \( h_2 = (C_2, \ell_2) \). Then \( L = \ell_1 \cap \ell_2 \) is a fixed point of \( h \) and any other fixed point of \( h \) lies on the line \( \ell \) through \( C_1 \) and \( C_2 \). Note that \( A \neq L \) because \( A \) cannot be fixed by any involution, \( M_A \) being of odd order. Hence \( A \in L \). Furthermore \( \ell \) must contain another point from \( \Omega_0 \) otherwise, \( h_1 \) (and \( h_2 \)) would fix \( A \), again a contradiction with \( 2 \nmid |M_A| \). Therefore \( \ell \) is secant to \( \Omega_0 \).

It follows that the sets \( \text{Fix}(C^s) \cap \Omega_0 \), with \( C^s \) ranging over all conjugates of \( C \) in \( G \), form a \( G \)-invariant partition of \( \Omega_0 \) contradicting the primitivity of \( G \) on \( \Omega_0 \). This rules out both cases (a) and (b).

We are left with the case where \( M_A \) is an elementary abelian \( p \)-group of order \( p^f \). Note that \( M_A \) has at least two fixed points on \( \Omega_0 \) otherwise, the normalizer \( N_M(M_A) \) would also fix the point \( A \) and thus \( N_M(M_A) = M_A \). Therefore \( M_A \) would be a Sylow \( p \)-subgroup and as a direct consequence, \(|\Omega_0| = (q^2 - 1)/2 \) whereas \(|\Omega_0 \setminus \{A\}| = tq \), a contradiction as \( q \geq 5 \).

It follows that the minimum number \( d \) of points in \( \Omega_0 \) which are fixed by a subgroup of \( M_A \) is at least two. Let \( S \) be the set of all subgroups of \( M_A \) each having exactly \( d \) fixed points on \( \Omega_0 \). Take a subgroup \( P \) from \( S \) whose order is as large as possible.

We show that no point in \( \text{Fix}(P) \cap \Omega_0 \) is fixed by a conjugate \( P^g \) of \( P \) in \( G \). Assume on the contrary that \( L \in \text{Fix}(P^g) \cap \text{Fix}(P) \), with \( L \in \Omega_0 \). Then the stabilizer \( M_L \) of \( L \) contains the subgroup \( P' \) generated by \( P \) and \( P^g \). Note that \( P \) is a proper subgroup of \( P' \). Since \( G \) is transitive on \( \Omega_0 \) there is an \( h \in G \) that maps \( L \) to \( A \), then \( P^h \) is a subgroup of \( M_A \). But this contradicts the definition of \( P \) as \(|P^h| > |P| \).

Therefore the sets \( \text{Fix}(P^g) \cap \Omega_0 \) with \( P^g \) ranging over all conjugates of \( P \) in \( G \) form a \( G \)-invariant partition of \( \Omega_0 \), contradicting again the primitivity of \( G \) on \( \Omega_0 \). Thus case (c) does not occur.

Hence, it has been shown that \( \Omega_0 \) is contained in \( \pi_0 \). To complete the proof of the first assertion in Proposition 6 it remains to show that \( \Omega_0 \) is the full intersection of \( \Omega \) and \( \pi_0 \). Actually, it suffices to show that \( \Omega_0 \) is an oval in \( \pi_0 \). For this purpose assume that \( \Omega_0 \) is a \((k+1)\)-arc of the projective plane \( \pi_0 \) of order \( m \), with \( k < m \). If \( A \in \Omega_0 \) then, there are \( m - k \) lines through \( A \) which meet \( \Omega_0 \) in no other point but \( A \). These \( m - k \) lines are secant to the oval \( \Omega \). Since \( A \) varies on \( \Omega_0 \), we get \((k+1)(m-k)\) distinct lines of \( \pi_0 \) which meet \( \Omega_1 \) in distinct points. As \( G \) is transitive on \( \Omega_1 \), there exists exactly one line of \( \pi_0 \) through each point of \( \Omega_1 \), whence

\[(k+1)(m-k) = n-k. \tag{2}\]

On the other hand, \( n \geq m^2 + m \) together with (2) gives \((m-k)^2 + km \leq 0 \), a contradiction. Therefore, \(|\Omega_0| = m+1 \), namely \( \Omega_0 \) is an oval in \( \pi_0 \). \( \square \)

Since \( G \) acts primitively on \( \Omega_0 \), Theorem 2 applies and this completes the proof of Theorem 1 for \( K_0 = \{\text{id}\} \).
Remark. For $K_0 = \{\text{id}\}$, consider the case $m = 9$. The order $n$ of $\pi$ satisfies
\[ n \geq m^2 + m = 90. \] (3)
Suppose $G \cong A_5$. Since $G$ is transitive on $\Omega_1$, the relation $(n - 9) \mid 60$ follows, which, together with (3), gives a contradiction. Therefore, for $m = 9$ the group $G$ must be isomorphic to $S_5$ and hence $(n - 9) \mid 120$. Then (3) gives $n = 129$.

Now, assume that $K_0$ is not trivial.

Lemma 3. Each element of $K_0$ commutes with each involutory homology of $G$.

Proof. Let $\alpha$ be a $(A, \ell)$-homology of $G$ and $g \in K_0$. By [14, Lemma 4.11], $g^{-1}ag$ is a $(A^8, \ell^8)$-homology. Since the center $A$ and the axis $\ell$ of $\alpha$ are contained in $\pi_0$, it follows $A^8 = A$ and $\ell^8 = \ell$. Proposition 1 implies $g^{-1}ag = \alpha$. \(\square\)

We distinguish two cases according to whether $O(G)$ is trivial or not.

Case (i). $O(G) = \{\text{id}\}$. By (1), $S = K_0$ holds. Since $S$ is cyclic, it contains a unique involution $j$. Taking $K_0 \trianglelefteq G$ into account, this implies that $j \in Z(G)$. As $G$ is primitive on $\Omega_0$, it follows that either $j$ generates a transitive group on $\Omega_0$, or $j$ fixes $\Omega_0$ pointwise. The former case cannot actually occur as $|\Omega_0| \geq 2$.

As $j$ fixes $\pi_0$ pointwise, $j$ is a Baer involution of $\pi$ and $\pi^* = \text{Fix}(j)$ is its Baer subplane. Note that $\Omega_0$ is contained in $\pi^*$. Actually, $\Omega_0 = \pi^* \cap \Omega$ because $j$ fixes no point on $\Omega_1$ by Lemma 1.

Now, Theorem 2 applies to $\pi^*$ and $\Omega_0$. Therefore, $\pi^*$ is a desarguesian plane of order $q$, and $\Omega_0$ is a conic of $\pi^*$. Furthermore, if $J$ is the subgroup of $K_0$ fixing $\pi^*$ pointwise and $G^* = G/J$ is regarded as a collineation group of $\pi^*$, then one of the following two cases can occur: either $G^*$ contains a normal subgroup $N^*$ isomorphic to $PSL(2, q)$, or $\pi^*$ has order 9 and $G^* \cong A_5$, or $G^* \cong S_5$.

In the former case, the number $k^*$ of involutions of $N^*$ is either $\frac{1}{2}(q^2 + q)$, or $\frac{1}{2}(q^2 - q)$, according as $q \equiv 1 \pmod{4}$, or $q \equiv 3 \pmod{4}$. Also, such involutions are homologies. Since $\pi_0$ is left invariant by $G^*$, the center of each homology in $N^*$ lies on $\pi_0$. As two distinct homologies preserving the same oval have distinct centers, we have $k^* \leq m^2 + m + 1$. On the other hand, $m^2 \leq q$ when $\pi_0$ is a proper subplane of $\pi^*$, a contradiction. Hence, $\pi^* = \pi_0$.

If $q = 9$, and $G^*$ contains a subgroup $M^* \cong A_5$, then $M^*$ has 15 involutions, each of them is a homology. Again, this implies that $\pi^* = \pi_0$, as no proper subplane of $PG(2, 9)$ contains more than 13 points, and Theorem 1 follows for $O(G) = \{\text{id}\}$.

Case (ii). $O(G) \neq \{\text{id}\}$.

Lemma 4. $O(G)$ fixes $\Omega_0$ pointwise and it has no fixed point on $\Omega_1$.

Proof. Assume on the contrary that the action of $O(G)$ on $\Omega_0$ is not trivial. By Lemma 1, $O(G)$ is transitive on $\Omega_0$ and we get that $|\Omega_1|$ and $|\Omega_0|$ are both odd. Then an involutory homology $\alpha$ fixes one point $A \in \Omega_1$ and one point $B \in \Omega_0$. From Lemma 3, each element of $O(G)$ commutes with $\alpha$, hence the group $O(G)$ must fix the points $A$ and $B$. It follows that $O(G)$ fixes $\Omega$ pointwise, contrary to our assumption. \(\square\)
By Lemma 4, the group $O(G)$ acts trivially on $\Omega_0$, hence Proposition 4 implies that $\Omega_0$ is a conic in the desarguesian subplane $\pi = \text{Fix}(O(G))$ of odd order $q$. Similar arguments as those used in case (i) lead to $\pi = \pi_0$.

Finally, since the group $G/K_0$ also acts primitively on $\Omega_0$ in $\pi_0$, the minimal normal subgroup $M_0$ of $G/K_0$ must be transitive on $\Omega_0$, thus Theorem 1 follows from Theorem 3.

References