Experimental Identification of Spatially-Interconnected Parameter-Invariant and LPV Models for Actuated Beams

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Abstract—This paper presents an experimental case study on system identification for spatially interconnected systems, considering both parameter-invariant and parameter-varying models. Considered here is the problem of vibration control for actuated beams - beams of different lengths are each equipped with an array of collocated actuators and sensors. Here we show how a spatially-interconnected model - a lumped approximation of a distributed system that lends itself to efficient, LMI-based synthesis of distributed control - for such actuated beams can be obtained experimentally. Experimental results show that a long beam with equally distributed actuators/sensors can be satisfactorily represented by a spatially invariant model. This is not possible for a short beam due to the stronger effect of boundary conditions, whereas it is demonstrated experimentally that a spatial LPV model captures the dynamic behaviour, thus enabling the synthesis of distributed spatial LPV controllers.

I. INTRODUCTION

Modelling and control of parameter-distributed systems such as continuous structures has been intensively studied especially in structural engineering and applied widely in various areas. Numerous methods have been developed analytically, experimentally, or in combination of both, to determine the inherent dynamic characteristics, e.g. finite element methods (FEM) [1] as a computer modelling approach, together with experimental modal analysis [2]. Nevertheless, several difficulties are encountered when centralized control schemes are applied. Optimal control technique cannot handle systems with too many inputs and outputs. Centralized control is not practical for such applications due to a high level of connectivity, computational burden and so on.

The development of light-weight piezoelectric material and microelectromechanical systems, enables the sensing and control of distributed systems without significantly changing the dynamics of the original structure. Over the last decade, several methods have been developed for distributed identification and control in the framework of spatially interconnected systems, which was initially developed for spatially invariant and infinitely extended systems, and later generalized to include spatially varying systems in the form of spatial linear parameter varying (LPV) systems. For systems governed by partial differential equations (PDEs), lumped approximations can be represented as spatially interconnected systems, and efficient LMI-based synthesis tools are available for distributed control schemes. [3] introduces a novel approach for spatially invariant systems to describe the dynamics of each subsystem interacting with its nearest neighbours in a structured Roesser state space form [4], which allows the distributed control scheme to inherit the interconnection structure of the plant. [5] extends these analysis and synthesis results to heterogeneous systems.

In many practical cases where a distributed control scheme for a distributed system is to be designed, it will be necessary to experimentally identify a spatially interconnected model of the system. [6] and [7] extend a least-squares-based black-box identification technique for lumped systems to spatially interconnected systems for invariant and spatially varying parameters, respectively. By restructuring the identified input/output model into Roesser state space form, the controller design method proposed in [3] can be applied. Instead of identifying input/output models, [8] presents a decomposable-system framework and a subspace identification technique based on solving bilinear matrix inequalities. Controllers inheriting the same decomposable nature are proposed in [9].

In this paper we present an experimental case study on vibration control involving actuated beams. Aluminium beams of different length are equipped each with an array of collocated actuators and sensors. Lumped approximations of these beams - where each segment corresponds to an actuator/sensor unit - are represented as spatially interconnected systems. Experimental results presented in this paper on actuated beams which are homogeneous and have constant profile but different length, show that the dynamic behaviour of a long beam with evenly distributed actuators/sensors can be satisfactorily captured by a spatially invariant model, whereas this is not possible for a short beam, which can however be satisfactorily modelled by a spatial LPV model through spatial scheduling. This makes it possible to synthesize distributed spatial LPV control schemes (see [10], [11]).

This paper is structured as follows: Section II shortly recaps some necessary preliminaries for later use. Section III uses an analytical model to state the problems we aim to tackle in this paper. Section IV briefly reviews the identification techniques, proposed in earlier work, for linear time- and space-invariant systems and for parameter-varying systems. Section V demonstrates their performance with experimental results. This paper ends with a conclusion in section VI.

II. PRELIMINARIES

A. Spatially Interconnected Systems

Spatially distributed systems can be spatially discretized into interconnected subsystems [3] as depicted in Fig. 1. The
dynamics of each subsystem $G_s$ is represented by a state space model, the so-called Rosser state space form

$$
\begin{bmatrix}
  x^t(k+1,s) \\
  x^s(k+1,s) \\
  y(k,s)
\end{bmatrix} =
\begin{bmatrix}
  A^t & A^s & B^1 \\
  A^s & A^{ss} & B^1 + B^2 \\
  C^t & C^{ss} & D^1 \\
  C^s & C^{ss} & D^2
\end{bmatrix}
\begin{bmatrix}
  x^t(k,s) \\
  x^s(k,s) \\
  d(k,s)
\end{bmatrix} +
\begin{bmatrix}
  B^3 \\
  B^{s+1} \\
  D^3 \\
  D^{s+1}
\end{bmatrix}
\begin{bmatrix}
  u(k,s) \\
  u(k,s+1)
\end{bmatrix}
$$

where $x^t \in \mathbb{R}^{m_0}$ denotes temporal states, $x^s \in \mathbb{R}^{m_+}$ and $x^s \in \mathbb{R}^{m-}$ spatial states in positive and negative directions, respectively, $d \in \mathbb{R}^{m+d}$ and $z \in \mathbb{R}^{m_z}$ the performance channel, $y \in \mathbb{R}^{m_y}$ and $u \in \mathbb{R}^{m_u}$ the external input and output channel, respectively. All signals are multi-dimensional with respect to time $k$ and space $s$. In this paper we focus on systems with a single spatial dimension.

For spatially invariant systems, all subsystems are identical, i.e. $G_{s-1} = G_s = G_{s+1}$, etc. This spatially invariant model can be augmented by local feedback in the form of linear fractional transformation, as shown in Fig. 2, to represent spatial LPV models with the same interconnection structure, where $G_{s-1} = G_s = G_{s+1}$ still holds. The spatially varying properties are captured by the spatially varying blocks $\Delta(s) \in \Delta$, where

$$
\Delta = \{ \Delta : \text{diag}\{\delta_{i_1}I_{m_{i_1}}, \ldots, \delta_{i_N}I_{m_{N_s}}\}, |\delta_i| \leq 1, i = 1, \ldots, N_s \} \quad (2)
$$

and $m_{i_1}, \ldots, m_{N_s}$ indicate the multiplicity of each scheduling parameter. For the corresponding Roesser state space model see [11].

![Fig. 2. Subsystems with spatially varying parameters](image)

**B. Model Structure for Identification**

Even though a subspace identification of distributed systems is possible [8], for the application considered here the identification of input/output models turns out to be more efficient, in particular when identifying spatial LPV models.

Let $u(k, s)$ be a 2-D discrete input signal applied to a 2-D discrete SISO data generating system. Here we identify models with ARX structure (AutoRegressive with eXogeneous input) as shown in Fig. 3. The input/output representations of the plant model and the noise model take the form:

$$
A(q_t, q_s) y(k, s) = B(q_t, q_s) u(k, s),
$$

$$
A(q_t, q_s) v(k, s) = e(k, s),
$$

where $e(k, s)$ represents a 2-D Gaussian white noise signal with zero mean.

**III. SPATIALLY DIScretized MODEL of an Actuated Euler-Bernoulli BEAM**

Before the LPV identification problem is addressed, we first consider an analytical approach to model the dynamics of an spatially invariant Euler-Bernoulli beam.

It is well-established that the dynamics of an uniform Euler-Bernoulli beam is governed by the PDE

$$
EI \frac{\partial^4 y(t,x)}{\partial x^4} + \rho A \frac{\partial^2 y(t,x)}{\partial t^2} = u(t,x),
$$

where $u(t,x)$ is the input, $y(t,x)$ is the deflection of the beam, $E$ is the Young modulus, $I$ is the area moment of inertia, $A$ is the cross-sectional area, and $\rho$ is the mass density.

**Fig. 3. ARX model structure**

The polynomials $A$ and $B$ are parametrized as

$$
A(q_t, q_s) = 1 + \sum_{(i_k, s_i) \in M^u} a_{(i_k, s_i)} q_t^{-i_k} q_s^{-i_s},
$$

$$
B(q_t, q_s) = \sum_{(i_k, s_i) \in M^u} b_{(i_k, s_i)} q_t^{-i_k} q_s^{-i_s},
$$

where $q_t$ and $q_s$ represent temporal- and spatial-shift operators, respectively, e.g. $q_t^{-1} u(k, s) = u(k-1, s)$. Coefficients $a_{(i_k, s_i)}$ and $b_{(i_k, s_i)}$ can in general be either temporally/spatially invariant or varying; however, temporally varying coefficients are not considered here.

$M^u$ and $M^v$ describe the input and output masks, or the so-called support regions in multidimensional digital signal processing [12]. The two masks lie in a 2-D plane and indicate which temporally and spatially shifted inputs and outputs contribute to the current output. Because the spatially distributed system is causal in time and non-causal in space, the support region must be a subset of the first and forth quadrant, as shown in Fig. 5. Thus, the output of subsystem $s$ at time $k$, i.e. $y(k, s)$, is determined by the difference equation

$$
y(k, s) = - \sum_{(i_k, s_i) \in M^v} a_{(i_k, s_i)} y(k - i_k, s - i_s)
+ \sum_{(i_k, s_i) \in M^u} b_{(i_k, s_i)} u(k - i_k, s - i_s) + e(k, s).
$$

**Fig. 1. Part of a spatially interconnected system**

**Fig. 2. Subsystems with spatially varying parameters**

**Fig. 3. ARX model structure**
with free-free boundary condition at both ends $x = 0, l$:

$$
EI \frac{\partial^2 y(t, x)}{\partial x^2} |_{x=0,l} = 0, \quad EI \frac{\partial^3 y(t, x)}{\partial x^3} |_{x=0,l} = 0,
$$

(7)

where $y(t, x)$ denotes the temporarily and spatially continuous transverse displacement of the beam, $u(t, x)$ the distributed force (see Fig. 4), $E$ Young’s module, $I$ the moment of inertia, $\rho$ the density, $A$ the area of cross section.

Then a Roesser state space model is obtained (11) as

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-a(2,0) & a(1,0) - a(1,2) & a(1,1) - a(1,-2) & a(1,1) - a(1,-1) & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

IV. LEAST-SQUARES-BASED IDENTIFICATION

This section briefly reviews the identification techniques for spatially invariant models proposed in [6] and for spatial LPV models in [7].

A. Identification of Spatially Invariant Models

Restructure the difference equation (5) into a regressor form

$$
y(k, s) = \phi^T(k, s)\theta + e(k, s),
$$

(8)

where the unknown parameter vector $\theta \in \mathbb{R}^p$

$$
\theta = \left[ \begin{array}{c}
\text{cat}_{i_k}\text{cat}_{i_s} a(i_k, i_s) \\
\text{cat}_{i_k} \text{cat}_{i_s} b(i_k, i_s)
\end{array} \right]_{(i_k, i_s) \in M^u}
$$

(9)

takes constant value for a spatially invariant system, $\text{cat}_{i_k}\text{cat}_{i_s}$ denotes the concatenation of the coefficients in $A(q_t, q_s)$ and $B(q_t, q_s)$, respectively. The data vector $\phi(k, s) \in \mathbb{R}^p$

$$
\phi(k, s) = \left[ \begin{array}{c}
-\text{cat}_{i_k}\text{cat}_{i_s} y(k - i_k, s - i_s) \\
\text{cat}_{i_k} \text{cat}_{i_s} u(k - i_k, s - i_s)
\end{array} \right]_{(i_k, i_s) \in M^u}
$$

(10)

contains the measured input and output data indexed by the pre-chosen masks $M^u$ and $M^v$. Taking the simple Euler-Bernoulli beam as an example, we have

$$
\theta = [a(1,0), a(1,1), a(1,2), a(1,-1), a(1,-2), a(2,0), b(1,0)]^T,
$$

$$
\phi(k, s) = [-y(k - 1, s - 1) - y(k - 1, s - 2) - y(k - 1, s + 1) - y(k - 1, s + 2) - y(k - 2, s - 1) - y(k - 2, s - 2) - y(k - 2, s + 1) - y(k - 2, s + 2) - y(k - 3, s - 1) - y(k - 3, s - 2) - y(k - 3, s + 1) - y(k - 3, s + 2) - y(k - 4, s - 1) - y(k - 4, s - 2) - y(k - 4, s + 1) - y(k - 4, s + 2) - y(k - 5, s - 1) - y(k - 5, s - 2) - y(k - 5, s + 1) - y(k - 5, s + 2)]^T.
$$

The output vector $Y$ and the regressor matrix $\Phi$ are filled with measurements from time $k = 1$ to $N_k$ of subsystems $s = 1$ to $N_s$:

$$
Y = [y(1,1), \cdots, y(N_k, 1), \cdots, y(1, N_s), \cdots, y(N_k, N_s)]^T;
$$

$$
\Phi = [\phi(1,1), \cdots, \phi(N_k, 1), \cdots, \phi(1, N_s), \cdots, \phi(N_k, N_s)]^T.
$$

Then the parameter vector $\theta$ that minimizes the cost function

$$
J = \sum_{s=1}^{N_s} \sum_{k=1}^{N_k} (y(k, s) - \phi^T(k, s)\theta)^2
$$

is obtained as

$$
\theta = (\Phi^T\Phi)^{-1}\Phi^TY.
$$

(11)
B. Identification of Spatial LPV Models

The identification of spatial LPV input/output models assumes the coefficients in (5) to be a function of the scheduling variable $\delta$ in (2), which is taken here as the space variable.

We define the support region in terms of the two masks $M_a$ and $M_b$ as in the analytical beam model. The data vector $\phi(k, s) \in \mathbb{R}^p$ is constructed as in (10). We assume that for all subsystems, the coefficients in (5) are polynomial functions of the spatial parameter $\delta$ up to a degree of $m$. Note that in general the dependence can be any smooth function [13]. Here the coefficients are parametrized as

$$a_{(i_k,i_s)}(\delta) = \sum_{j=0}^{m} \alpha_{(i_k,i_s,j)} \delta^j$$

$$b_{(i_k,i_s)}(\delta) = \sum_{j=0}^{m} \beta_{(i_k,i_s,j)} \delta^j,$$

where $\alpha_{(i_k,i_s,j)}$ and $\beta_{(i_k,i_s,j)}$ ($j = 0, \cdots, m$) are real constants to be estimated.

Define the vector

$$P = [1, \delta, \cdots, \delta^m]^T$$

where the degree $m$ of the polynomial function is pre-chosen. Then the regressor form for an LPV model is structured as

$$\hat{Y}(k, s) = (\phi(k, s) \otimes P)^T \tilde{\theta}$$

where $\tilde{\theta} \in \mathbb{R}^{p(m+1)}$ with

$$\tilde{\theta} = \begin{bmatrix} c a t_{i_k} c a t_{i_s} c a t_{j} a_{(i_k,i_s,j)} \end{bmatrix}_{j=0:m,(i_k,i_s) \in M_p}$$

(13)

Introduce a new regressor vector $\psi(k, s) = \phi(k, s) \otimes P$. The measurements from a simultaneous excitation of all $N_s$ subsystems from time $k = 1$ to $N_k$ fill the output vector and the measurement matrix as

$$Y = \begin{bmatrix} y(1, 1), \cdots, y(N_k, 1), \cdots, y(1, N_s), \cdots, y(N_k, N_s) \end{bmatrix}^T$$

$$\Psi = \begin{bmatrix} \psi(1, 1), \cdots, \psi(N_k, 1), \cdots, \psi(1, N_s), \cdots, \psi(N_k, N_s) \end{bmatrix}^T.$$

The parameter vector $\tilde{\theta}$ is then computed as

$$\tilde{\theta} = (\Psi \Psi^T)^{-1} \Psi Y.$$  

(14)

Remarks

- In both the spatially invariant and the LPV cases, based on the initial results obtained by the least-squares method, the instrumental variable method can be sequentially applied to obtain unbiased estimates if the additive output noise in (5) is coloured instead of white, as discussed in [14].

- The identification of a more realistic noise model having Box-Jenkins instead of ARX structure is discussed in [15].

V. EXPERIMENTAL RESULTS

A. Description of Test-beds

To study spatially invariant and LPV identification techniques for a spatially distributed model, two experimental set-ups have been constructed for test use - one short and one long aluminium beam with free-free boundary condition, equipped with 5 and 16 collocated piezoelectric actuators and sensors, respectively, as shown in Fig. 6. Their physical parameters are listed in Table I. The free-free boundary condition is realized by suspending the beam with an array of soft springs in parallel.

<table>
<thead>
<tr>
<th>Description</th>
<th>Short Beam</th>
<th>Long Beam</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length, $l$</td>
<td>0.77</td>
<td>4.8</td>
<td>m</td>
</tr>
<tr>
<td>Width, $w$</td>
<td>0.04</td>
<td>0.04</td>
<td>m</td>
</tr>
<tr>
<td>Thickness, $t_h$</td>
<td>0.004</td>
<td>0.003</td>
<td>m</td>
</tr>
<tr>
<td>Density, $\rho$</td>
<td>2710</td>
<td>2710</td>
<td>$kg/m^3$</td>
</tr>
<tr>
<td>Young's Module, $E$</td>
<td>$70 \times 10^9$</td>
<td>$70 \times 10^9$</td>
<td>$N/m^2$</td>
</tr>
</tbody>
</table>

Table I

Parameters of the two aluminium beams used in the experiments

The location of actor/sensor units on the short beam is described by the vector of spatial coordinates

$$s = [0.655, 0.505, 0.355, 0.205, 0.055]m,$$

whereas that of the long beam is

$$s = [0.125, 0.425, 0.725, 1.025, 1.325, 1.625, 1.925, 2.225, 2.525, 2.825, 3.125, 3.425, 3.725, 4.025, 4.325, 4.625]m.$$

Note that the coordinates of 5 actuators/sensors on the short beam do not imply 5 identical subsystems - the 5th pair is closer to the free end than the 1st one, whereas the other 3
subsystems can be considered identical. On the long beam, the distances between any two neighbouring units are the same, as well as the distances from the boundary units to the free ends.

DuraAct Patch Transducers made of lead zirconate titanate (PZT) are used here both as actuators and sensors due to its inverse and direct piezoelectric effect. When working as an actuator, by applying an electric field to the patches, a pure bending moment at both ends of the actuator is generated, which results in a strain on the structure. As a sensor, it is capable of measuring the local curvature of the surface it is attached to.

Apart from its lightweight structure, piezoelectric material exhibits several attractive features: a short response time, a high efficiency and a high mechanical durability. Nevertheless, piezoelectric material exhibits hysteresis and drift. These nonlinear phenomena are observed in experiments.

Fig. 7 shows the collocated pattern of the attached actuators and sensors. Compared with the analytical model discussed in section III, the actuation voltage applied to the experimental set-up is not directly related to the distributed force, but proportional to a torque, whereas the measured voltage is proportional to the curvature, not the transverse deflection. The experimentally obtained black-box model is expected to capture the actuator and sensor dynamics. As for the free-free boundary condition, the exchange of information - the curvature - on the two ends with ‘virtual’ subsystems is zero.

The online computation is executed through a Labview real-time system. The hardware components are listed in Table II.

![Fig. 7. An array of collocated actuators/sensors with a free-free boundary condition](image)

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Type</th>
<th>Description</th>
<th>Producer</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/16</td>
<td>P-876.A11</td>
<td>PZT as sensor</td>
<td>Physik Instrumente</td>
</tr>
<tr>
<td>5/16</td>
<td>P-876.A12</td>
<td>PZT as actuator</td>
<td>Physik Instrumente</td>
</tr>
<tr>
<td>5/16</td>
<td>E-835</td>
<td>PZT driver module</td>
<td>Physik Instrumente</td>
</tr>
<tr>
<td>1</td>
<td>N6353</td>
<td>Analog input card</td>
<td>National Instruments</td>
</tr>
<tr>
<td>1</td>
<td>N6723</td>
<td>Analog output card</td>
<td>National Instruments</td>
</tr>
</tbody>
</table>

**TABLE II**

**HARDWARE DESCRIPTION**

**B. Experimental Results**

The distribution of the actuator/sensor pairs on the long beam makes the interconnection of 16 identical subsystems possible. It is known that for a long beam, the first few eigenmodes accumulate densely within a low frequency range and dominate the vibration behaviour. Therefore, the identification experiments are carried out by exciting 16 subsystems with 16 out-of-phase chirp signals up to 50 Hz with a sampling time \( T = 0.001\text{s} \). Applied here is the identification technique for spatially invariant models with the choice of the input and output masks in Fig. 8. Cross validation results are given in Fig. 9 in terms of amplitude spectrum of outputs 1, 3 and 8 within [25 30] Hz under the excitation of a sum of sinusoids, whose frequencies cover the given bandwidth [0 50] Hz with a very fine grid.

![Fig. 8. Input and output masks for the spatially invariant long aluminium beam](image)

![Fig. 9. Experimental results - long beam: comparison of the output amplitude spectrum of subsystems 1, 3, and 8, excited by a sum of sinusoids - blue: measured, red: spatially invariant model](image)

In contrast to the long beam, the short beam is stronger affected by the boundary units. For this reason, here a spatial LPV structure is employed by defining the spatial coordinate as the scheduling parameter, i.e. \( \delta_i = s_i \). The achieved model accuracy is then compared with that of a spatially invariant identification.

Identification experiments are performed by actuating the 5 patches simultaneously with 5 out-of-phase chirp signals up to 110 Hz, exploiting the prior knowledge that the first two natural frequencies are 35 and 96 Hz. We choose the same sampling time as before, i.e. \( T = 0.001\text{s} \). The polynomial function of \( \delta \) has a degree of 3. The input and output masks are determined by trial and error as shown in Fig. 10. Taking actuator and sensor dynamics into consideration, it can be expected that the experimental identification requires
a higher system order compared with the PDE of an Euler-Bernoulli beam.

\[ M_u \quad i_k \quad M_y \quad i_k \]

Fig. 10. Input and output masks for the short aluminium beam

Fig. 11 shows the comparison of the time responses when actuating the beam with the 2nd natural frequency. With the masks in Fig. 10, the identified model with an LPV structure (red) outperforms the invariant structure (green) at the resonance frequency. Note also the asymmetric response between subsystem 1 and 5 reflecting the different distance of the actuator/sensor units at both ends from the boundary.

![Graph showing experimental results](image)

Fig. 11. Experimental results - short beam: comparison of the response of 5 subsystems in time, excited at the 2nd natural frequency - blue: measured, red: LPV identification, green: spatially invariant identification

VI. CONCLUSIONS

This paper presents the successful experimental identification of spatially distributed systems with invariant and varying parameters. On one hand, a simple least-squares method demonstrates its performance by capturing well the dynamics of the spatially invariant long beam; on the other hand, the LPV identification turns out to be superior for the spatially varying short beam. Both obtained models can be easily restructured into Roesser state space models, for which efficient controller synthesis tools are available - [3] for spatially invariant models, [10] for LPV models using constant Lyapunov functions, [11] using parameter-dependent Lyapunov functions. Controller design and closed-loop experiments are currently being carried out and will be reported separately.

REFERENCES