GENERALIZED TWISTED COHOM OBJECTS

SERGIO D. GRILLO

Abstract. A generalization of the concept of twisted internal coHom object in the category of conic quantum spaces (c.f. [2]) was outlined in [3]. The aim of this article is to develope in more detail this generalization.

1. Introduction

Given a couple $\mathcal{A},\mathcal{B}$ of conic quantum spaces, i.e. $\mathcal{A},\mathcal{B}\in \mathrm{FGA_G}$ (the monoidal category of finitely generated graded algebras [4][5][2]), their symmetric twisted tensor products $\mathcal{A}\circ_\tau\mathcal{B}$ [1][2] can also be seen as (2nd admissible) counital 2-cocycle twisting of the quantum space $\mathcal{A}\circ\mathcal{B}$ [3]. Then, in the same sense as in [2], we can study, instead of the maps $\mathcal{A}\rightarrow\mathcal{H}\circ\mathcal{B}$ (which define a comma category whose initial objects are the proper internal coHom objects of $\mathrm{FGA_G}$), certain subclasses of arrows $\mathcal{A}\rightarrow(\mathcal{H}\circ\mathcal{B})_\omega$, where ω is a counital 2-cocycle defining a twist transformation on $\mathcal{H}\circ\mathcal{B}$. The aim of this paper is to show that, in certain circumstances, these classes give rise for each pair $\mathcal{A},\mathcal{B}\in\mathrm{FGA_G}$ to a category $\Omega^{\mathcal{A},\mathcal{B}}$ with initial object, namely $\underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}]$, such that the disjoint union $\Omega^{\cdot}=\bigvee_{\mathcal{A},\mathcal{B}}\Omega^{\mathcal{A},\mathcal{B}}$ has a semigroupoid structure together with a related embedding $\Omega^{\cdot}\hookrightarrow\mathrm{FGA_G}$ that preserves the involved (partial) products. Consequently, $(\mathcal{B},\mathcal{A})\mapsto\underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}]$ defines an $\mathrm{FGA_G}$-cobased category with an additional notion of evaluation given by arrows

$$\mathcal{A}\rightarrow\left(\underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}]\circ\mathcal{B}\right)_\omega .$$

The categories $\Upsilon^{\cdot}$, obtained in [2], are particular cases of the categories $\Omega^{\cdot}$. In this way we generalize the idea of twisted coHom objects in the more general framework of twisting of quantum spaces. This setting enable us, in turn, to a better understanding of the results obtained in the mentioned paper.

This article is based on the contents of [2] and [3] and references therein, thus we shall frequently refer the reader to them. Notation and terminology also follow those papers.

2. The categories $\Omega^{\cdot}$

In order to built up the categories $\Omega^{\mathcal{A},\mathcal{B}}$, let us first make a couple of observations.

1. Consider on the category GrVct (of graded vector spaces) the monoid

$$\mathbf{V}\circ\mathbf{W}=\bigoplus_{n\in\mathbb{N}_0}(\mathbf{V}_n\otimes\mathbf{W}_n),$$

for $\mathbf{V}=\bigoplus_{n\in\mathbb{N}_0}\mathbf{V}_n$ and $\mathbf{W}=\bigoplus_{n\in\mathbb{N}_0}\mathbf{W}_n$. The arrows $\mathbf{V}\rightarrow\mathbf{W}$ in GrVct are homogeneous linear maps; i.e. its restrictions to each $\mathbf{V}_n$ define maps $\mathbf{V}_n\rightarrow\mathbf{W}_n$. It is clear that the forgetful functor $\mathfrak{H}:\mathrm{FGA_G}\hookrightarrow\mathrm{GrVct}$ turns into a monoidal one, and $\mathfrak{H}(\mathcal{A}_\psi)=\mathfrak{H}(\mathcal{A})=\mathbf{A}=\bigoplus_{n\in\mathbb{N}_0}\mathbf{A}_n$ holds for every twist transformation $\psi\in\mathfrak{Z}^2[\mathbf{A}_1]$.

2. Let us construct the comma categories $(\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$, where the functor $\mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B})$ is the composition of $\mathrm{FGA_G} \circ \mathcal{B}$ and $\mathfrak{H}$. Its objects are pairs $\langle \varphi, \mathcal{H} \rangle$ where $\mathcal{H} \in \mathrm{FGA_G}$ and φ is an arrow in GrVct,

$$\varphi : \mathfrak{H}(\mathcal{A}) \to \mathfrak{H}(\mathcal{H} \circ \mathcal{B}) = \mathfrak{H}(\mathcal{H}) \circ \mathfrak{H}(\mathcal{B}).$$

To every $\langle \varphi, \mathcal{H} \rangle \in (\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$, the surjection $\pi^{\varphi} : \mathbf{B}_1^* \otimes \mathbf{A}_1 \twoheadrightarrow \mathbf{H}_1^{\varphi} : b^j \otimes a_i \mapsto h_i^j$ can be related, being h_i^j the elements of $\mathbf{H}_1$ defining the restriction of φ to $\mathbf{A}_1$, i.e. $\varphi(a_i) = h_i^j \otimes b_j$. This linear surjection gives rise to a functor

$$\begin{gathered} \mathfrak{F} : (\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B})) \to \mathrm{FGA_G}, \\ \langle \varphi, \mathcal{H} \rangle \mapsto \mathcal{H}^{\varphi} = (\mathbf{H}_1^{\varphi}, \mathbf{H}^{\varphi}); \quad \alpha \mapsto \alpha|_{\mathbf{H}^{\varphi}}, \end{gathered} \tag{2.1}$$

where $\mathbf{H}^{\varphi}$ is the subalgebra of $\mathbf{H}$ generated by $\mathbf{H}_1^{\varphi}$ (an analogous functor is used in [2] to built up the categories $\Upsilon^{\mathcal{A},\mathcal{B}}$).

Using last functor we shall construct each $\Omega^{\mathcal{A},\mathcal{B}}$ as a full subcategory of the corresponding comma category $(\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$. Given a couple of conic quantum spaces $\mathcal{A}$ and $\mathcal{B}$, consider a counital element

$$\omega : \left([\mathbf{B}_1^* \otimes \mathbf{A}_1 \otimes \mathbf{B}_1]^{\otimes}\right)^{\otimes 2} \backsimeq \left([\mathbf{B}_1^* \otimes \mathbf{A}_1 \otimes \mathbf{B}_1]^{\otimes}\right)^{\otimes 2} \tag{2.2}$$

of $\mathfrak{Z}^2[\mathbf{B}_1^* \otimes \mathbf{A}_1 \otimes \mathbf{B}_1]$. Eventually, for $\langle \varphi, \mathcal{H} \rangle \in (\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$, we can translate ω to $[\mathbf{H}_1^{\varphi} \otimes \mathbf{B}_1]^{\otimes}$ through $\pi^{\varphi} : \mathbf{B}_1^* \otimes \mathbf{A}_1 \twoheadrightarrow \mathbf{H}_1^{\varphi}$ in such a way that the diagram

$$\begin{array}{ccc} \mathbf{B}_1 \otimes \mathbf{B}_1^* \otimes \mathbf{A}_1 & \twoheadrightarrow & \mathbf{B}_1 \otimes \mathbf{H}_1^{\varphi} \\ \updownarrow \backsimeq & & \downarrow \\ \mathbf{B}_1 \otimes \mathbf{B}_1^* \otimes \mathbf{A}_1 & \twoheadrightarrow & \mathbf{B}_1 \otimes \mathbf{H}_1^{\varphi} \end{array}$$

be commutative, defining in this way a counital 2-cochain in $\mathfrak{Z}^2[\mathbf{H}_1^{\varphi} \otimes \mathbf{B}_1]$. Last affirmation lies on the results given in **Prop. 4** of [3], applied to the injection $\mathbf{H}_1^{\varphi} \hookrightarrow \mathbf{B}_1^* \otimes \mathbf{A}_1$. Then, if the resulting automorphism is admissible, we can define with it a twist transformation on $\mathcal{H}^{\varphi} \circ \mathcal{B} = \mathfrak{F}\langle \varphi, \mathcal{H} \rangle \circ \mathcal{B}$.

Definition 1. *For every pair $\mathcal{A}, \mathcal{B} \in \mathrm{FGA_G}$ and $\omega \in \mathfrak{Z}^2[\mathbf{B}_1^* \otimes \mathbf{A}_1 \otimes \mathbf{B}_1]$, we define $\Omega^{\mathcal{A},\mathcal{B}}$ as the full subcategory of $(\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$ formed out by diagrams $\langle \varphi, \mathcal{H} \rangle$ such that ω defines a 2-cocycle twisting on $\mathfrak{F}\langle \varphi, \mathcal{H} \rangle \circ \mathcal{B}$, and the homogeneous linear map φ is a morphism of quantum spaces $\mathcal{A} \to (\mathfrak{F}\langle \varphi, \mathcal{H} \rangle \circ \mathcal{B})_{\omega}$.* ■

Given now a collection of cochains $\{\omega_{\mathcal{A},\mathcal{B}}\}_{\mathcal{A},\mathcal{B} \in \mathrm{FGA_G}} \subset \mathfrak{Z}^2[\mathbf{B}_1^* \otimes \mathbf{A}_1 \otimes \mathbf{B}_1]$, we name $\Omega^{\cdot}$ the disjoint union of the categories $\Omega^{\mathcal{A},\mathcal{B}}$ just defined. Clearly, $\mathrm{FGA_G^{\circ}}$ (see [2], or §**1.2** of [3] for a brief review) is a category $\Omega^{\cdot}$ with an associated collection given by identity maps.

Calling $\mathfrak{H}\mathrm{FGA_G^{\circ}}$ the disjoint union of $(\mathfrak{H}(\mathcal{A}) \downarrow \mathfrak{H}(\mathrm{FGA_G} \circ \mathcal{B}))$, it follows that every $\Omega^{\cdot}$ is a full subcategory of $\mathfrak{H}\mathrm{FGA_G^{\circ}}$. On the other hand, let us observe that $\mathfrak{H}\mathrm{FGA_G^{\circ}}$ has a semigroupoid structure given by the functor

$$\langle \varphi, \mathcal{H} \rangle \times \langle \chi, \mathcal{G} \rangle \mapsto \langle (I_H \circ \chi)\, \varphi, \mathcal{H} \circ \mathcal{G} \rangle; \quad \alpha \times \beta \mapsto \alpha \circ \beta, \tag{2.3}$$

and $\mathrm{FGA}_{\mathrm{G}}^{\circ} \subset \mathfrak{H}\mathrm{FGA}_{\mathrm{G}}^{\circ}$ is a sub-semigroupoid. In fact, this map is a partial product functor with domain

$$\bigvee_{\mathcal{A},\mathcal{B},\mathcal{C}\in\mathrm{FGA}_{\mathrm{G}}} \left(\mathfrak{H}\left(\mathcal{A}\right) \downarrow \mathfrak{H}\left(\mathrm{FGA}_{\mathrm{G}} \circ \mathcal{C}\right)\right) \times \left(\mathfrak{H}\left(\mathcal{C}\right) \downarrow \mathfrak{H}\left(\mathrm{FGA}_{\mathrm{G}} \circ \mathcal{B}\right)\right)$$

and codomain $\mathfrak{H}\mathrm{FGA}_{\mathrm{G}}^{\circ}$, such that

$$\left(\mathfrak{H}\left(\mathcal{A}\right) \downarrow \mathfrak{H}\left(\mathrm{FGA}_{\mathrm{G}} \circ \mathcal{C}\right)\right) \times \left(\mathfrak{H}\left(\mathcal{C}\right) \downarrow \mathfrak{H}\left(\mathrm{FGA}_{\mathrm{G}} \circ \mathcal{B}\right)\right) \to \left(\mathfrak{H}\left(\mathcal{A}\right) \downarrow \mathfrak{H}\left(\mathrm{FGA}_{\mathrm{G}} \circ \mathcal{B}\right)\right)$$

Its associativity comes from that of $\circ$, and the unit elements are given by the diagrams $\langle \ell_{\mathcal{A}}, \mathcal{K}\rangle$, where $\ell_{\mathcal{A}}$ is the homogeneous isomorphism $\mathbf{A} \backsimeq \Bbbk\left[e\right] \otimes \mathbf{A}$, such that $a \mapsto e^n \otimes a$ if $a \in \mathbf{A}_n$. Nevertheless, for a generic collection $\left\{\omega_{\mathcal{A},\mathcal{B}}\right\}_{\mathcal{A},\mathcal{B}\in\mathrm{FGA}_{\mathrm{G}}}$ of cochains, $\Omega^{\cdot}$ fails to be a semigroupoid. Furthermore, in the generic case, each $\Omega^{\mathcal{A},\mathcal{B}}$ fails to have initial objects. To address this problem, we shall consider particular cases.

3. The semigroupoid structure of $\Omega^{\cdot}$

In what follows, all references to sections and theorems correspond to [3]. Recall the monics (c.f. §**2.3.2**)

$$\mathsf{j} : \mathfrak{C}^{\bullet}\left[\mathbf{B}_1\right]^{!} \times \mathfrak{C}^{\bullet}\left[\mathbf{A}_1\right] \times \mathfrak{C}^{\bullet}\left[\mathbf{B}_1\right] \hookrightarrow \mathfrak{C}^{\bullet}\left[\mathbf{B}_1^{*} \otimes \mathbf{A}_1 \otimes \mathbf{B}_1\right]. \tag{3.1}$$

Definition 2. *A collection* $\left\{\omega_{\mathcal{A},\mathcal{B}}\right\}_{\mathcal{A},\mathcal{B}\in\mathrm{FGA}_{\mathrm{G}}}$ *is* ***factorizable*** *if there exists another collection*

$$\left\{\psi_{\mathcal{A}}\right\}_{\mathcal{A}\in\mathrm{FGA}_{\mathrm{G}}}, \;\; \psi_{\mathcal{A}} \in \mathfrak{Z}^2\left[\mathbf{A}_1\right],$$

such that $\omega_{\mathcal{A},\mathcal{B}} = \mathsf{j}\left(\psi_{\mathcal{B}}^{!}, \psi_{\mathcal{A}}, \mathbb{I}^{\otimes 2}\right)$. ■

Since Eq. (3.1), such cochains $\omega_{\mathcal{A},\mathcal{B}}$ are in $\mathfrak{Z}^2\left[\mathbf{B}_1^{*} \otimes \mathbf{A}_1 \otimes \mathbf{B}_1\right]$. To give an example, in the TTP case with $\widehat{\tau}_{\mathcal{A},\mathcal{B}} = id \otimes \sigma_{\mathcal{B}}^{!} \otimes \sigma_{\mathcal{A}}$, the cochain $\psi_{\mathcal{A}}$ would be given by the assignment

$$a_{k_1}...a_{k_r} \otimes a_{k_{r+1}}...a_{k_{r+s}} \mapsto a_{k_1}...a_{k_r} \otimes \left(\sigma_{\mathcal{A}}^{-r}\right)_{k_{r+1}}^{j_1} ... \left(\sigma_{\mathcal{A}}^{-r}\right)_{k_{r+s}}^{j_s} a_{j_1}...a_{j_s}. \tag{3.2}$$

From the injection

$$\mathfrak{Z}^2\left[\mathbf{B}_1\right]^{!} \times \mathfrak{Z}^2\left[\mathbf{A}_1\right] \times \mathfrak{Z}^2\left[\mathbf{B}_1\right] \hookrightarrow \mathfrak{Z}^2\left[\mathbf{B}_1^{*} \otimes \mathbf{A}_1\right] \times \mathfrak{Z}^2\left[\mathbf{B}_1\right], \tag{3.3}$$

we shall also regard $\omega_{\mathcal{A},\mathcal{B}}$ as a cochain belonging to the latter set, depending on our convenience.

Theorem 1. *If a category* $\Omega^{\cdot}$ *is associated to a factorizable collection, then* $\Omega^{\cdot}$ *is a sub-semigroupoid of* $\mathfrak{H}\mathrm{FGA}_{\mathrm{G}}^{\circ}$.

Proof. Consider the quantum spaces $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, and diagrams $\langle\varphi, \mathcal{H}\rangle \in \Omega^{\mathcal{A},\mathcal{B}}$ and $\langle\psi, \mathcal{G}\rangle \in \Omega^{\mathcal{B},\mathcal{C}}$, with associated linear spaces (*via* the functor $\mathfrak{F}$)

$$\mathbf{H}_1^{\varphi} = span\left[h_i^j\right]_{i,j=1}^{\mathfrak{n},\mathfrak{m}} \;\; and \;\; \mathbf{G}_1^{\psi} = span\left[g_i^j\right]_{i,j=1}^{\mathfrak{m},\mathfrak{p}}.$$

We are denoting by $\dim \mathbf{A}_1 = \mathfrak{n}$, $\dim \mathbf{B}_1 = \mathfrak{m}$ and $\dim \mathbf{C}_1 = \mathfrak{p}$ the dimensions of the generator spaces defining $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. We must show that

$\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle$ (see Eq. (2.3)) is an object of $\Omega^{\mathcal{A},\mathcal{C}}$, and that the objects $\langle \ell_{\mathcal{A}}, \mathcal{K}\rangle$ are in $\Omega^{\cdot}$. That means the quantum space $\mathfrak{F}\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle$, generated by

$$span\left[\sum_{j=1\ldots\mathfrak{m}} h_i^j \otimes g_j^k\right]_{i,k=1}^{\mathfrak{n},\mathfrak{p}} \subset \mathbf{H}_1^{\varphi}\otimes\mathbf{G}_1^{\psi} \subset \mathbf{H}^{\varphi}\circ\mathbf{G}^{\psi},$$

is such that $(I_H \circ \chi)\ \varphi$ defines an arrow $\mathcal{A}\rightarrow(\mathfrak{F}\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle\circ\mathcal{C})_{\omega}$ in $\mathrm{FGA_G}$. To this end, let us introduce some notation.

Denote by $\mathbf{h}$ and $\mathbf{g}$ the matrices with entries $h_i^j \in \mathbf{H}_1$ and $g_i^j \in \mathbf{G}_1$, and by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the vectors whose components are $a_i \in \mathbf{A}_1$, $b_i \in \mathbf{B}_1$ and $c_i \in \mathbf{C}_1$. Since $\langle \varphi, \mathcal{H}\rangle$ and $\langle \psi, \mathcal{G}\rangle$ are elements of $\Omega^{\cdot}$, $\omega_{\mathcal{A},\mathcal{B}}$ and $\omega_{\mathcal{B},\mathcal{C}}$ defines cochains in $\mathfrak{Z}^2[\mathbf{H}_1^{\varphi}]\times\mathfrak{Z}^2[\mathbf{B}_1]$ and $\mathfrak{Z}^2\left[\mathbf{G}_1^{\psi}\right]\times\mathfrak{Z}^2[\mathbf{C}_1]$ (see Eq. (3.3)), respectively. The latter are given by

$$\omega_{\mathcal{A},\mathcal{B}}(\mathbf{h}_{r,s}\otimes\mathbf{b}_{r,s}) = [\psi_{\mathcal{A}}]_{r,s}\cdot\mathbf{h}_{r,s}\cdot\left[\psi_{\mathcal{B}}^{-1}\right]_{r,s}\otimes\mathbf{b}_{r,s} \tag{3.4}$$

and

$$\omega_{\mathcal{B},\mathcal{C}}(\mathbf{g}_{r,s}\otimes\mathbf{c}_{r,s}) = [\psi_{\mathcal{B}}]_{r,s}\cdot\mathbf{g}_{r,s}\cdot\left[\psi_{\mathcal{C}}^{-1}\right]_{r,s}\otimes\mathbf{c}_{r,s} \tag{3.5}$$

where the symbols $\mathbf{h}_{r,s}=\mathbf{h}_r\otimes\mathbf{h}_s$ and $[\psi_{\mathcal{A}}]_{r,s}\cdot\mathbf{h}_{r,s}\cdot\left[\psi_{\mathcal{B}}^{-1}\right]_{r,s}$ denote elements of the form

$$h_{i_1}^{j_1}\ldots h_{i_r}^{j_r}\otimes h_{k_1}^{l_1}\ldots h_{k_s}^{l_s}\in(\mathbf{H}_1^{\varphi})^{\otimes r}\otimes(\mathbf{H}_1^{\varphi})^{\otimes s}$$

and

$$(\psi_{\mathcal{A}})_{n_1\ldots n_r,m_1\ldots m_s}^{i_1\ldots i_r,k_1\ldots k_s}\ h_{i_1}^{j_1}\ldots h_{i_r}^{j_r}\otimes h_{k_1}^{l_1}\ldots h_{k_s}^{l_s}\ \left(\psi_{\mathcal{B}}^{-1}\right)_{j_1\ldots j_r,l_1\ldots l_s}^{p_1\ldots p_r,q_1\ldots q_s},$$

respectively. Now, $\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle\in\Omega^{\mathcal{A},\mathcal{C}}$ if and only if $(I_H \circ \chi)\ \varphi$ defines the mentioned arrow in $\mathrm{FGA_G}$, with ω given by

$$\omega_{\mathcal{A},\mathcal{C}}\left(\left(\mathbf{h}\dot{\otimes}\mathbf{g}\right)_{r,s}\otimes\mathbf{c}_{r,s}\right) = [\psi_{\mathcal{A}}]_{r,s}\cdot\left(\mathbf{h}\dot{\otimes}\mathbf{g}\right)_{r,s}\cdot\left[\psi_{\mathcal{C}}^{-1}\right]_{r,s}\otimes\mathbf{c}_{r,s}.$$

Here $\dot{\otimes}$ is denoting matrix contraction between $\mathbf{h}$ and $\mathbf{g}$. It follows from straightforward calculations that, if $\omega_{\mathcal{A},\mathcal{C}}$ is well defined on $\mathfrak{F}\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle\circ\mathcal{C}$, then $(I_H \circ \chi)\ \varphi$ is an homogeneous linear map $\mathbf{A}\rightarrow\mathbf{H}\otimes\mathbf{G}\otimes\mathbf{C}$ defining the wanted morphism. So, let us first show that. To this end, extend $\omega_{\mathcal{A},\mathcal{C}}$ to $\mathbf{H}^{\varphi}\circ\mathbf{G}^{\psi}\circ\mathbf{C}$ by putting $\omega_{\mathcal{A},\mathcal{C}}(\mathbf{h}_{r,s}\otimes\mathbf{g}_{r,s}\otimes\mathbf{c}_{r,s})$ equal to

$$[\psi_{\mathcal{A}}]_{r,s}\cdot\mathbf{h}_{r,s}\otimes\mathbf{g}_{r,s}\cdot\left[\psi_{\mathcal{C}}^{-1}\right]_{r,s}\otimes\mathbf{c}_{r,s} =$$

$$= [\psi_{\mathcal{A}}]_{r,s}\cdot\mathbf{h}_{r,s}\cdot\left[\psi_{\mathcal{B}}^{-1}\right]_{r,s}\dot{\otimes}[\psi_{\mathcal{B}}]_{r,s}\cdot\mathbf{g}_{r,s}\cdot\left[\psi_{\mathcal{C}}^{-1}\right]_{r,s}\otimes\mathbf{c}_{r,s}.$$

From the last expression, and recalling that $\omega_{\mathcal{A},\mathcal{B}}$ and $\omega_{\mathcal{B},\mathcal{C}}$ (given in (3.4) and (3.5)) are admissible, it follows that $\omega_{\mathcal{A},\mathcal{C}}$ is admissible for $\mathcal{H}^{\varphi}\circ\mathcal{G}^{\psi}\circ\mathcal{C}$, and also for the subspace $\mathfrak{F}\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle\circ\mathcal{C}$. Then, $\langle (I_H \circ \chi)\ \varphi, \mathcal{H}\circ\mathcal{G}\rangle\in\Omega^{\mathcal{A},\mathcal{C}}$.

Finally, we have to show the units $\langle \ell_{\mathcal{A}}, \mathcal{K}\rangle$ are objects of the corresponding categories $\Omega^{\mathcal{A}}$. We know that $\ell_{\mathcal{A}}$ is an homogeneous linear map such that $\ell_{\mathcal{A}}(a) = e^n\otimes a$, if $a\in\mathbf{A}_n$. In particular, we can write $\ell_{\mathcal{A}}(a_i) = e\delta_i^j\otimes a_j$. Then, the cochain $\omega_{\mathcal{A}}$ for $\langle \ell_{\mathcal{A}}, \mathcal{K}\rangle$ is given by

$$\omega_{\mathcal{A}}(\mathbf{k}_{r,s}\otimes\mathbf{a}_{r,s}) = [\psi_{\mathcal{A}}]_{r,s}\cdot\mathbf{k}_{r,s}\cdot\left[\psi_{\mathcal{A}}^{-1}\right]_{r,s}\otimes\mathbf{a}_{r,s}$$

with

$$\mathbf{k}_{r,s} = e^r\, \delta_{i_1}^{j_1} ... \delta_{i_r}^{j_r} \otimes e^s\, \delta_{k_1}^{l_1} ... \delta_{k_s}^{l_s}.$$

Accordingly $\omega_{\mathcal{A}}$ for $\langle \ell_{\mathcal{A}}, \mathcal{K} \rangle$ is the identity map, and $(\mathcal{K} \circ \mathcal{A})_\omega = \mathcal{K} \circ \mathcal{A}$. ∎

The following result is immediate.

Proposition 1. *Let us call* $\circ_\Omega$ *the partial product associated to above mentioned semigroupoid structure of* $\Omega^{\cdot}$. *The embedding* $\mathfrak{P}^\Omega : \Omega^{\cdot} \hookrightarrow \mathrm{FGA}_{\mathrm{G}} : \langle \varphi, \mathcal{H} \rangle \mapsto \mathcal{H}$ *preserves the respective (partial) products and units, i.e.*

$$\mathfrak{P}^\Omega \circ_\Omega = \circ \left(\mathfrak{P}^\Omega \times \mathfrak{P}^\Omega \right) \quad \text{and} \quad \mathfrak{P}^\Omega \langle \ell_{\mathcal{A}}, \mathcal{K} \rangle = \mathcal{K}. \quad ∎$$

4. The generalized coHom objects

For $\Omega^{\mathcal{A},\mathcal{B}}$ to have initial objects we need an additional condition on $\omega_{\mathcal{A},\mathcal{B}}$.

Theorem 2. *If* $\Omega^{\cdot}$ *is associated to a factorizable collection given by* $\{\psi_{\mathcal{A}}\}_{\mathcal{A} \in \mathrm{FGA}_{\mathrm{G}}}$ *such that each* $\mathfrak{i}\psi_{\mathcal{A}} = \mathfrak{i}\psi$ *is **2nd*** $\mathcal{A}$***-admissible**, *then each* $\Omega^{\mathcal{A},\mathcal{B}}$ *have initial object*

$$\underline{hom}^\Omega [\mathcal{B}, \mathcal{A}] = \mathcal{B}_{\mathfrak{i}\psi} \triangleright \mathcal{A}_{\mathfrak{i}\psi} = (\mathcal{B} \triangleright \mathcal{A})_{\mathfrak{j}\left(\mathfrak{i}\psi^!, \mathfrak{i}\psi\right)}.$$

In particular, $\underline{hom}^\Omega [\mathcal{K}, \mathcal{A}] = \mathcal{A}_{\mathfrak{i}\psi}$ *and* $\underline{hom}^\Omega [\mathcal{K}, \mathcal{K}] = \mathcal{K}$; *thus,*

$$\underline{hom}^\Omega [\mathcal{B}, \mathcal{A}] = \underline{hom}^\Omega [\mathcal{K}, \mathcal{B}] \triangleright \underline{hom}^\Omega [\mathcal{K}, \mathcal{A}]. \quad ∎$$

Before going to the proof, let us make some remarks. Since $\psi_{\mathcal{A}} \in \mathfrak{Z}^2 [\mathbf{A}_1]$, there exists a primitive $\theta_{\mathcal{A}} \in \mathfrak{P}^1 [\mathbf{A}_1]$ such that $\psi_{\mathcal{A}} = \partial \theta_{\mathcal{A}}$, $\mathfrak{i}\psi_{\mathcal{A}} = \partial \theta_{\mathcal{A}}^{-1}$ and $\psi_{\mathcal{A}}^! = \partial^! \theta_{\mathcal{A}}^!$ (see §**3.2.1**, §**3.3.1** and §**4.2.1**, respectively). In addition, if $\mathbf{I} = \bigoplus_{n \geq 2} \mathbf{I}_n$ is the graded ideal related to $\mathcal{A}$, we have from §**3.2.2** that (provided $\mathfrak{i}\psi$ is admissible)

$$\mathbf{I}_{\mathfrak{i}\psi,n} = \theta(\mathbf{I}_n), \quad \mathbf{I}_{\mathfrak{i}\psi,n}^\perp = \theta(\mathbf{I}_n)^\perp = \theta^! \left(\mathbf{I}_n^\perp\right). \tag{4.1}$$

Proof. **(of theorem)** We shall show $\mathcal{B}_{\mathfrak{i}\psi} \triangleright \mathcal{A}_{\mathfrak{i}\psi}$ defines an object of $\Omega^{\mathcal{A},\mathcal{B}}$ and then that is initial.

Let us note that, given $\psi, \varphi \in \mathfrak{Z}^2$, with $\psi = \partial\theta$ and $\varphi = \partial\chi$, it follows that

$$\begin{aligned} \mathfrak{j}(\mathfrak{i}\psi, \mathfrak{i}\varphi) &= \mathfrak{j}\left(\partial\left(\theta^{-1}\right), \partial\left(\chi^{-1}\right)\right) = \mathfrak{j}\left(\partial\left(\theta^{-1}, \chi^{-1}\right)\right) = \\ &= \partial\left(\mathfrak{j}(\theta, \chi)^{-1}\right) = \mathfrak{i}\mathfrak{j}(\psi, \varphi). \end{aligned}$$

Also recall, if $\mathfrak{i}\psi$ is (2nd) $\mathcal{A}$-admissible, then ψ is (2nd) $\mathcal{A}_{\mathfrak{i}\psi}$-admissible (see **Prop. 14** of §**3.3.1**).

By **Theor. 16** of §**4.2.2**, using the 2nd $\mathcal{A}$-admissibility of $\mathfrak{i}\psi_{\mathcal{A}} = \mathfrak{i}\psi$,

$$(\mathcal{B}_{\mathfrak{i}\psi} \triangleright \mathcal{A}_{\mathfrak{i}\psi}) \circ \mathcal{B} = (\mathcal{B} \triangleright \mathcal{A})_{\mathfrak{j}\left(\mathfrak{i}\psi^!, \mathfrak{i}\psi\right)} \circ \mathcal{B} = (\mathcal{B} \triangleright \mathcal{A})_{\mathfrak{i}\mathfrak{j}\left(\psi^!, \psi\right)} \circ \mathcal{B}$$

$$= ((\mathcal{B} \triangleright \mathcal{A}) \circ \mathcal{B})_{\mathfrak{i}\mathfrak{j}\left(\psi^!, \psi, \mathbb{I}^{\otimes 2}\right)} = ((\mathcal{B} \triangleright \mathcal{A}) \circ \mathcal{B})_{\mathfrak{i}\omega},$$

because $\omega = \omega_{\mathcal{A},\mathcal{B}} = \mathfrak{j}\left(\psi_{\mathcal{B}}^!, \psi_{\mathcal{A}}, \mathbb{I}^{\otimes 2}\right)$. Then,

$$((\mathcal{B}_{\mathfrak{i}\psi} \triangleright \mathcal{A}_{\mathfrak{i}\psi}) \circ \mathcal{B})_\omega = (((\mathcal{B} \triangleright \mathcal{A}) \circ \mathcal{B})_{\mathfrak{i}\omega})_\omega = (\mathcal{B} \triangleright \mathcal{A}) \circ \mathcal{B},$$

and consequently the map $\delta : \mathcal{A} \to (\mathcal{B} \triangleright \mathcal{A}) \circ \mathcal{B} : a_i \mapsto z_i^j \otimes b_j$ (with $z_i^j = b^j \otimes a_i$), defining the coevaluation of the proper coHom object $\underline{hom}[\mathcal{B}, \mathcal{A}] = \mathcal{B} \triangleright \mathcal{A}$, gives also a morphism $\delta : \mathcal{A} \to ((\mathcal{B}_{\mathfrak{i}\psi} \triangleright \mathcal{A}_{\mathfrak{i}\psi}) \circ \mathcal{B})_\omega$. Moreover, since the equality

$\mathfrak{F}\langle\delta,\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}\rangle=\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}$, the pair $\langle\delta,\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}\rangle$ is in $\Omega^{\mathcal{A},\mathcal{B}}$. Let us show such a pair is an initial object.

Suppose $\mathcal{A}$ and $\mathcal{B}$ have related dimensions $\dim\mathbf{A}_1=\mathfrak{n}$ and $\dim\mathbf{B}_1=\mathfrak{m}$, and ideals $\mathbf{I}$ and $\mathbf{J}$, respectively. Let us consider the vector space

$$\mathbf{D}_1\doteq span\left[z_i^j\right]_{i,j=1}^{\mathfrak{n},\mathfrak{m}}$$

and the linear map $\delta_1:a_i\mapsto z_i^j\otimes b_j$. Under the identification $b^j\otimes a_i=z_i^j$, the cochain $\omega_{\mathcal{A},\mathcal{B}}=\omega$ defines a counital 2-cocycle in $\mathfrak{C}^2\left[\mathbf{D}_1\otimes\mathbf{B}_1\right]$, and δ_1 can be extended to an algebra homomorphism

$$\delta_1^{\otimes}:\mathbf{A}_1^{\otimes}\rightarrow\left(\mathbf{D}_1^{\otimes}\circ\mathbf{B}_1^{\otimes}\right)_{\omega}.$$

Analogous calculations that enable us to arrive at Eq. (3.4) of [3], show that

$$\delta_1^{\otimes}\left(\mathbf{a}_r\right)=\left[\theta_{\mathcal{A}}\right]_r\cdot\mathbf{z}_r\cdot\left[\theta_{\mathcal{B}}^{-1}\right]_r\dot{\otimes}\,\mathbf{b}_r,$$

(where we are using notation of previous theorem), or in coordinates,

$$\delta_1^{\otimes}\left(a_{i_1}...a_{i_r}\right)=\left(\theta_{\mathcal{A}}\right)_{i_1...i_r}^{j_1...j_r}\;z_{j_1}^{l_1}...z_{j_r}^{l_r}\;\left(\theta_{\mathcal{B}}^{-1}\right)_{l_1...l_r}^{k_1...k_r}\otimes b_{k_1}...b_{k_s}.$$

In fact, identifying $\mathbf{D}_1^{\otimes}$ with $\left[\mathbf{B}_1^*\otimes\mathbf{A}_1\right]^{\widehat{\otimes}}$, we have

$$\left[\theta_{\mathcal{A}}\right]_r\cdot\mathbf{z}_r\cdot\left[\theta_{\mathcal{B}}^{-1}\right]_r=\theta_{\mathcal{B}}^{!}\left(\mathbf{b}_r^*\right)\otimes\theta_{\mathcal{A}}\left(\mathbf{a}_r\right);\;\;\mathbf{b}_r^*=b^{k_1}...b^{k_r}.$$

In this notation, the generators of $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{J}^{\perp}$ can be written

$$\left[R_{\lambda_r}\right]_r\cdot\mathbf{a}_r,\quad\left[S_{\mu_r}\right]_r\cdot\mathbf{b}_r\quad and\quad\mathbf{b}_r^*\cdot\left[S^{\perp\omega_r}\right]_r,$$

respectively. To have a well defined map δ from $\delta_1^{\otimes}$ when $\mathbf{A}_1^{\otimes}$ and $\mathbf{B}_1^{\otimes}$ are quotient by their corresponding ideals, $\mathbf{D}_1^{\otimes}$ must be quotient by the elements

$$\left[R_{\lambda_r}\right]_r\cdot\left[\theta_{\mathcal{A}}\right]_r\cdot\mathbf{z}_r\cdot\left[\theta_{\mathcal{B}}^{-1}\right]_r\cdot\left[S^{\perp\omega_r}\right]_r.\tag{4.2}$$

Now, it is clear that a pair $\langle\varphi,\mathcal{H}\rangle\in\mathfrak{H}\mathrm{FGA}_{\mathrm{G}}^{\circ}$ is in $\Omega^{\mathcal{A},\mathcal{B}}$ if and only if there exist elements $h_i^j\in\mathbf{H}_1\subset\mathfrak{F}\langle\varphi,\mathcal{H}\rangle$ satisfying relations (4.2) (replacing $\mathbf{z}_r$ by $\mathbf{h}_r$). But the elements of (4.2) span precisely the space (from Eq. (4.1))

$$\theta_{\mathcal{B}}^{!}\left(\mathbf{J}_r^{\perp}\right)\otimes\theta_{\mathcal{A}}\left(\mathbf{I}_r\right)=\mathbf{J}_{\mathrm{i}\psi,r}^{\perp}\otimes\mathbf{I}_{\mathrm{i}\psi,r},$$

which generates algebraically the ideal related to $\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}$. Then, the function $z_i^j\mapsto h_i^j$ can be extended to an arrow $\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}\rightarrow\mathfrak{F}\langle\varphi,\mathcal{H}\rangle$. But this is the unique arrow in $\Omega^{\mathcal{A},\mathcal{B}}$ that can be defined between these objects, that is to say, $\langle\delta,\mathcal{B}_{\mathrm{i}\psi}\triangleright\mathcal{A}_{\mathrm{i}\psi}\rangle$ is an initial object of $\Omega^{\mathcal{A},\mathcal{B}}$.

Finally, since the cochains of $\mathfrak{C}^{\bullet}\left[\mathbb{k}\right]$ are always (2nd) $\mathcal{K}$-admissible, in particular the primitive ones $\mathfrak{P}^1\left[\mathbb{k}\right]$, it follows that any twisting of $\mathcal{K}$ is isomorphic to $\mathcal{K}$. Then, the last claim of the theorem follows immediately from the first one. ∎

Note that the 2nd admissibility condition for each $\mathrm{i}\psi_{\mathcal{A}}$ replaces the automorphism property of the related $\sigma_{\mathcal{A}}$ of the TTP case. Such a condition is immediate in the TTP case, because the properties defining a twisting map imply the related 2-cocycles are anti-bicharacters (c.f. §**2.2.2** and §**4.1.2**).

Now, a couple of immediate corollaries.

Corollary 1. *The gauge equivalence (see §3.3)* $\underline{hom}^{\Omega}\left[\mathcal{B},\mathcal{A}\right]\backsim\underline{hom}\left[\mathcal{B},\mathcal{A}\right]$ *is valid for all conic quantum spaces* $\mathcal{B},\mathcal{A}$. ∎

Corollary 2. *In the category of quadratic quantum spaces* QA (*and any* $\mathrm{FGA}_{\mathrm{G}}^{m}$), *the initial objects of* $\Omega^{\mathcal{A},\mathcal{B}}$ *are isomorphic to*

$$\underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}] = (\mathcal{B}_{\mathrm{i}\psi})^{!} \bullet \mathcal{A}_{\mathrm{i}\psi} = \left(\mathcal{B}^{!}\right)_{\mathrm{i}\psi^{!}} \bullet \mathcal{A}_{\mathrm{i}\psi} = \left(\mathcal{B}^{!} \bullet \mathcal{A}\right)_{\mathrm{j}\left(\mathrm{i}\psi^{!},\mathrm{i}\psi\right)}. \quad \blacksquare$$

Under the hypothesis mentioned in previous theorem, and from the semigroupoid structure of $\Omega^{\cdot}$ compatible with the monoid in $\mathrm{FGA}_{\mathrm{G}}$, we have the annunciated result:

Theorem 3. *The assignment* $(\mathcal{B},\mathcal{A}) \mapsto \underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}]$ *define an* $\mathrm{FGA}_{\mathrm{G}}$*-cobased category with arrows*

$$\underline{hom}^{\Omega}[\mathcal{C},\mathcal{A}] \to \underline{hom}^{\Omega}[\mathcal{B},\mathcal{A}] \circ \underline{hom}^{\Omega}[\mathcal{C},\mathcal{B}],$$

the cocomposition, and for $\underline{end}^{\Omega}[\mathcal{A}]$ *the counit epimorphism*

$$\underline{end}^{\Omega}[\mathcal{A}] \twoheadrightarrow \mathcal{K} \quad / \quad z_i^j \mapsto \delta_i^j\, e,$$

and the monomorphic comultiplication

$$\underline{end}^{\Omega}[\mathcal{A}] \hookrightarrow \underline{end}^{\Omega}[\mathcal{A}] \circ \underline{end}^{\Omega}[\mathcal{A}] \quad / \quad z_i^j \mapsto z_i^k \otimes z_k^j. \quad \blacksquare$$

References

[1] A. Čap, H. Schichl, J. Vanžura, On twisted tensor products of algebras, Comm. in Alg. 23 (1995) 4701-4735.

[2] S. Grillo, H. Montani, Twisted internal coHom objects in the category of quantum spaces, math. QA/0112233.

[3] S. Grillo, Twisting of quantum spaces and twisted coHom objects, math. QA/0202205.

[4] Yu. Manin, Some remarks on Koszul algebras and quantum groups, Ann. Inst. Fourier 33 (1987) 191-205.
Yu. Manin, Quantum groups and non commutative geometry, CRM, Université de Montréal, 1988.

[5] Yu. Manin, Topics in non commutative geometry, Princeton University Press, 1991.

Centro Atómico Bariloche and Instituto Balseiro, *8400-S. C. de Bariloche*, *Argentina*
E-mail address: `sergio@cabtep2.cnea.gov.ar`