Finite-SNR Diversity-Multiplexing Tradeoff for Two-Way Multi-Antenna Relay Fading Channels

Xiaochen Lin, Meixia Tao and Youyun Xu
Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, 200240, China
Email: {xclin_comm, mxtao}@sjtu.edu.cn

Abstract—In this paper, we study the diversity-multiplexing tradeoff (DMT) of two-way relay channels (TWRC) with multi-antenna relay at finite signal-to-noise ratio (SNR). A two-phase decode-and-forward (DF) relay protocol and Rayleigh fading environment are considered. We first derive upper and lower bounds on the outage probability. These bounds are very tight for all practical SNR regions. Based on these bounds, the estimates of finite-SNR DMT (f-DMT) are then obtained. Our analysis shows that the multiplexing gain approaches one when SNR decreases to zero. On the other hand, the diversity gain approaches the number of relay antennas when SNR increases to infinity. Furthermore, the impact of time sharing, rate allocation and relay location on the outage and f-DMT is also discussed through numerical examples.

I. INTRODUCTION

The two-way relay channel (TWRC) is a fundamental and useful model in wireless relay communication systems. In this model, two source nodes communicate with each other with the aid of a relay node. Traditionally, four time-slots are used to accomplish one round of information exchange for the interference avoidance. Applying network coding, Wu and Chou [1] propose a three-time-slot scheme in which two source nodes transmit messages to the relay node in the first two time-slots, and then the relay broadcasts the mixed signal to both sources. The number of time slots is further reduced to two when applying analog network coding (ANC) or physical-layer network coding (PNC) [2] [3], where the first time slot is for the two source nodes to simultaneously transmit to the relay and the second time slot is for the relay to broadcast.

Based on asymptotic diversity-multiplexing tradeoff (DMT) proposed by Zheng and Tse [4], Narasimhan [5] presents new definitions of diversity gain and multiplexing gain at finite SNR, due to that some practical communication systems would rather operate in low to moderate SNR region. In this new framework, the multiplexing gain is defined as a ratio of the target data rate to the capacity of an AWGN channel at finite SNRs. It indicates the sensitivity to finite SNR of the rate-adaptive strategy. Due to capacity-achieving codes applied in per packet, the packet error rate (PER) is equal to the outage probability. Under a fixed multiplexing gain and finite SNR, diversity gain is defined as a negative slope of the log-log of outage probability versus SNR. The diversity gain at a particular SNR can determine the additional SNR required to decrease a given amount of outage probability for a fixed multiplexing gain. Very recently, Yi and Kim in [6] use the finite-SNR DMT (f-DMT) to characterize the performance of TWRC with amplify-and-forward (AF) relay protocol. In our previous work, we have analyzed outage probability and f-DMT for decode-and-forward (DF) protocol [7]. Note that, these works only consider TWRC with single-antenna.

In this paper, we study the f-DMT for TWRC when the relay has multiple antennas. A two-phase DF protocol is considered. Upper and lower bounds on outage probability are derived and verified be quite close to the actual performance by Monte Carlo simulations. Then based on these bounds, two estimates of f-DMT are obtained. It is shown that the limits of f-DMT estimates are the same as the DMT at asymptotically high SNR. When SNR decreases to zero, the maximal multiplexing gain tends to one. On the other hand, with SNR increasing to infinity, diversity gain becomes equal to the number of relay antennas. Finally, the impact of time sharing, rate allocation, and relay location on the outage and f-DMT performances is discussed by simulation results.

II. SYSTEM MODEL

As shown in Fig. 1, two source nodes \( S_1 \) and \( S_2 \), each equipped with single antenna, exchange information through a relay node \( R \), which is equipped with \( M \) antennas. Assume that every node has the same transmission power \( E \). The links between \( S_1 \), \( S_2 \) and \( R \) are reciprocal, and the channel coefficients are denoted by \( h_{1R} = h_{T1} = h \), \( f_{1R} = f_{T2} = f \). The vectors of channel gain \( h = [h_1, \ldots, h_M]^T \) and \( f = [f_1, \ldots, f_M]^T \) are assumed to be zero-mean Gaussian random vectors with independent and identically distributed (i.i.d) Rayleigh fading, \( h_i \sim \mathcal{CN}(0, \beta_h) \) and \( f_j \sim \mathcal{CN}(0, \beta_f) \). Furthermore, let \( \|h\|^2 \) and \( \|f\|^2 \) denote Frobenius norms of channel vector \( h \) and \( f \), which have gamma distribution. By defining \( X \triangleq \|h\|^2 \) and \( Y \triangleq \|f\|^2 \), the probability density

![Fig. 1. System model of TWRC with multi-antenna relay](image-url)
The mutual informations are given by

\[ I(x;Y) = \frac{1}{\Gamma(M)(\beta_f)^M} y^{M-1}e^{-\frac{y}{\beta_f}}. \]  

In this system model, we consider two-phase DF protocol. The transmission time of each round of information exchange is normalized to one. In the first phase, called multiple-access (MA) phase, \( S_1 \) and \( S_2 \) simultaneously transmit towards \( R \) for a time fraction of \( t \), which is a time-sharing parameter. In the second phase, named as broadcast (BC) phase, \( R \) transmits to \( S_1 \) and \( S_2 \) for the rest time of \( 1 - t \). Due to half-duplex constraint, the direct link between \( S_1 \) and \( S_2 \) is naturally ignored in the two-phase scheme. The relay first jointly decodes the two signals, applies ideal random binning, and then transmits the re-encoded signals to the destinations. It is supposed that the noise in all links is additive white Gaussian with zero mean and unit variance.

Let \( R \) denote the target sum-rate of TWRC and \( \alpha \) be a rate allocation parameter. Hence, the target rates of \( S_1 \) and \( S_2 \) are \( R_1 = \alpha R \) and \( R_2 = (1 - \alpha)R \), respectively. The outage probability of TWRC is defined as the probability that the target rate pair \( R = [R_1, R_2]^T \) lies outside the achievable rate region conditioned on \( h \) and \( f \). Mathematically, the outage occurs when any of the following inequalities does not hold

\[ R_1 \leq \min\left(t I(x_1; y_1|x_2, h, f), (1-t)I(x_2; y_2|h, f)\right) \]
\[ R_2 \leq \min\left(t I(x_2; y_1|x_1, h, f), (1-t)I(x_1; y_2|h, f)\right) \]
\[ R_1 + R_2 \leq t I(x_1, x_2; y_1, y_2|h, f) \]

then we define the following events

\[ O_1 = \{ t I(x_1; y_1|x_2, h, f) < R_1, \ \ \ (1-t)I(x_2; y_1|h, f) < R_2 \} \]
\[ O_2 = \{ t I(x_2; y_1|x_1, h, f) < R_2, \ \ (1-t)I(x_1; y_2|h, f) < R_1 \} \]
\[ O_3 = \{ t I(x_1, x_2; y_1, y_2|h, f) < R_1 + R_2 \} \]

It is easily seen that the outage probability of TWRC can be expressed as

\[ P_{out} = P(O) = P(O_1 \cup O_2 \cup O_3). \]  

### III. Outage Probability Bounds

In order to calculate outage probability given in (14), we first define joint outage event \( O_{12} = \{ O_1 \cup O_2 \} \), then outage event of TWRC is \( O = \{ O_{12} \cup O_3 \} \). Because \( O_1 \) and \( O_2 \) are independent by definition, so

\[ P_{O_{12}} = P(O_{12} \cup O_3) = 1 - P_{O_{12}} - P_{O_3}. \]  

where \( O_{12} \) and \( O_3 \) are the complements of \( O_1 \) and \( O_2 \), respectively. Substituting the expressions (10) and (12) into the definition of \( O_1 \) and given that \( ||h||^2 \) follows a chi-square distribution given in (1), we can easily obtain the probability of \( O_1 \) as

\[ P_{O_1} = P\left(||h||^2 < C\right) = 1 - e^{-\frac{C}{\beta_h}} \sum_{k=0}^{M-1} \frac{C^k}{k!} \]

where we have used \( R = r \log_2(1 + E) \) in Section IV (26),

\[ \gamma(n, x) = \int_0^\infty t^{n-1}e^{-t}dt = (n-1)!e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!} \]  

and \( C = \max\left\{(1+E)^\frac{2\alpha}{1-\alpha}, (1+E)^\frac{2\rho}{3}\right\} \). The probability of \( O_2 \) is similarly obtained as

\[ P_{O_2} = 1 - e^{-\frac{D}{\beta_f}} \sum_{l=0}^{M-1} \frac{D^l}{l!} \]

where \( D = \max\left\{(1+E)^\frac{2\alpha}{1-\alpha}, (1+E)^\frac{2\rho}{3}\right\} \).

Because \( O_{12} \) and \( O_3 \) are neither mutually independent nor disjoint, exact computation of the outage probability given in (9) is not tractable except for a few special cases. Thus in order to obtain insight into outage and f-DMT performances, upper and lower bounds on (9) are computed. A simple upper bound is given by the union bound

\[ P_{out} = P(O_{12} \cup O_3) \leq P_{O_{12}} + P_{O_3} \]
\[ \leq P_{O_{12}} + P_{O_{3,ub}} = P_{outub} \]  

And a lower bound is computed as follows

\[ P_{out} = P(O_{12} \cup O_3) \geq \max\{P_{O_{12}}, P_{O_{12}h}\} = P_{outlb} \]

We now compute the upper and lower bounds of probability of \( O_3 \) in (21) shown at the bottom of the next page. Since \( Z \) is a sum of two independent gamma random variables with parameters \( (M, \beta_h) \) and \( (M, \beta_f) \), applying Moschopoulos’s theorem [9] about the distribution of sum of independent Gamma variables, we obtain the pdf of \( Z \) as

\[ p_Z(z) = a \sum_{k=0}^{\infty} \frac{\delta_k z^{p+k+1} e^{-\frac{z}{\beta}}}{\Gamma(p+k)\beta^{p+k}} \]  

then we define the following events
where $\beta = \min (\beta_h, \beta_f)$, $a = \left( \frac{\beta^2}{\beta_h \beta_f} \right)^M$, $\rho = 2M$, and

$$
\delta_k = \begin{cases} 
\delta_0 = 1 \\
\delta_{k+1} = \frac{M}{k+1} \sum_{l=1}^{k+1} \left( \frac{1}{\beta_h} \right)^l + \left( \frac{1}{\beta_f} \right)^l \delta_{k+1-l} & k = 0, 1, 2, \ldots
\end{cases}
$$

Then the upper bound of $P_{O_3}$ is straightforwardly computed as

$$
P_{O_{3ub}} = P \left[ ||h||^2 + ||f||^2 < \frac{2M-1}{E} \right] = \frac{(1+e^{\gamma}) - 1}{E}
$$

$$
a \sum_{k=0}^{\infty} \delta_k \left( 1 - e^{-\frac{G}{k!}} \sum_{l=0}^{k} \left( \frac{G}{k!} \right)^l \right).
$$

(24)

where $G = \left( \frac{1+e^{\gamma}}{E} \right)^{t-1}$. The lower bound of $O_3$ can be derived using bivariate probability distribution as

$$
P_{O_{3lb}} = \sum_{l=0}^{\infty} \delta_k \left( 1 - e^{-\frac{G}{k!}} \sum_{l=0}^{k} \left( \frac{G}{k!} \right)^l \right).$$

(25)

Finally, substituting (18) and (24) into (19), we obtain the closed-form expression for the upper bound of the outage probability of TWRC. Likewise, the closed-form expression for the lower bound of the outage probability of TWRC is obtained by substituting (18) and (25) into (20).

IV. Finite-SNR DMT Estimates

In this section, we present the estimates of f-DMT based on the bounds derived in the previous section. The multiplexing gain is defined as the ratio of $R$ to the capacity of an AWGN channel at finite-SNR $\gamma$ [5]

$$
r = \frac{R}{\log_2(1+g\gamma)}.
$$

(26)

with array gain $g = 1$ due to two source nodes with only single-antenna. Furthermore, for a given $r$, diversity gain is defined as the negative slope of log-log plot of outage probability versus SNR [5]

$$
d(r, \gamma) = -\frac{\partial \ln P_{out}(r, \gamma)}{\partial \ln \gamma}.
$$

(27)

In this paper, since the noise is assumed to have unit variance and the gains of the two channels between the two source nodes and the relay node can be different, we simply replace the SNR $\gamma$ in (26) and (27) by the transmit power $E$. Then diversity gain $d$ can be re-expressed as

$$
d(r, E) = -\frac{\partial \ln P_{out}(r, E)}{\partial \ln E} = -\frac{E}{P_{out}(r, E)} \frac{\partial P_{out}(r, E)}{\partial E}.
$$

(28)

The estimates of f-DMT can be obtained by substituting the results of (19) or (20) into (28). The f-DMT estimate $\hat{d}_1(r, E)$ based the the upper bound of (19) is derived by

$$
\hat{d}_1(r, E) = -\frac{E}{P_{out ub}} \frac{\partial P_{out ub}(r, E)}{\partial E} = \frac{E}{P_{out ub}} \left( \frac{\partial A_1}{\partial E} A_2 A_3 + \frac{\partial A_2}{\partial E} A_3 + \frac{\partial A_3}{\partial E} + \frac{\partial B_1}{\partial E} B_2 + \frac{\partial B_2}{\partial E} \right)
$$

(29)

where $A_1 = e^{-\frac{G}{k!}}$, $A_2 = \sum_{k=0}^{M-1} \frac{(\frac{G}{k!})^k}{k!}$, $A_3 = \sum_{k=0}^{M-1} \frac{(\frac{G}{k!})^k}{k!}$, $B_1 = e^{-\frac{G}{k!}}$, and $B_2 = \sum_{k=0}^{M-1} \frac{(\frac{G}{k!})^k}{k!}$. Moreover, write their partial derivatives as

$$
\frac{\partial A_1}{\partial E} = \sum_{k=0}^{M-1} \frac{1}{(k+1)!} \frac{G^{k+1}}{(\beta_h)^{k+1}} \frac{\partial C}{\partial E}, \quad \frac{\partial A_2}{\partial E} = \sum_{k=0}^{M-1} \frac{1}{(k+1)!} \frac{G^{k+1}}{(\beta_f)^{k+1}} \frac{\partial D}{\partial E}, \quad \frac{\partial B_1}{\partial E} = -\frac{1}{\beta_e} \frac{G^{k+1}}{(\beta_f)^{k+1}} \frac{\partial C}{\partial E}, \quad \frac{\partial B_2}{\partial E} = \sum_{k=0}^{M-1} \frac{1}{(k+1)!} \frac{G^{k+1}}{(\beta_f)^{k+1}} \frac{\partial D}{\partial E}.
$$

(30)

In the above equations, we have $\frac{\partial C}{\partial E} = \frac{t}{E^2} \left( (1+e^{\gamma})^{t-1} - (1+e^{\gamma})^{t-1} \right)$, and $\frac{\partial D}{\partial E} = \frac{t}{E^2} \left( (1+e^{\gamma})^{t-1} - (1+e^{\gamma})^{t-1} \right)$.
and the partial derivatives of $C$ and $D$ are presented as (30) and (31) at the bottom of this page.

The other estimate of f-DMT using the lower bound of outage probability is computed as

$$\hat{d}_2(r, E) = -\frac{E}{P_{out \ lb}(r, E)} \frac{\partial P_{out \ lb}(r, E)}{\partial E}$$

$$= \frac{E}{P_{out \ lb}} \left\{ \frac{\partial A_1 A_2 A_3}{\partial E}, or \frac{\partial B_3 B_4}{\partial E} \right\}$$

$$+ \frac{1}{\Gamma(M)(\beta_h)^M} \sum_{k=0}^{M-1} \frac{\partial f^G}{\partial E} f(E, x) dx \right\} (32)$$

where $B_3 = e^{-\frac{G}{\beta_h}}$, $B_4 = \sum_{k=0}^{\infty} \frac{(G)^k}{(\beta_h)^k}$, $\frac{\partial B_3}{\partial E} = -\frac{1}{\beta_h} e^{-\frac{G}{\beta_h}} \frac{\partial G}{\partial E}$, and $\frac{\partial B_4}{\partial E} = \sum_{k=0}^{M-1} \frac{G^{k-1}}{(\beta_h)^k} \frac{\partial G}{\partial E}$. Applying uncertain limit integral functions derivative method, we have

$$\frac{\partial f^G}{\partial E} f(E, x) dx = f(E, G) \frac{\partial G}{\partial E} + \int_0^G \frac{\partial f(E, x)}{\partial E} dx.$$

Given

$$g(E, x) = \frac{G - x}{\beta_f (1 + E x)},$$

$$\frac{\partial g(E, x)}{\partial E} = \frac{\frac{\partial G}{\partial E} (1 + E x) - (G - x) x}{\beta_f (1 + E x)^2},$$

we substitute $G$ into

$$f(E, x) = x^{M-1} e^{-\frac{G}{\beta_h} - \frac{x}{\beta_f}} g(E, x),$$

and then we get

$$f(E, G) = G^{M-1} e^{-\frac{G}{\beta_h}},$$

and

$$\frac{\partial f(E, x)}{\partial E} = x^{M-1} e^{-\frac{G}{\beta_h} - \frac{x}{\beta_f}} \frac{\partial g(E, x)}{\partial E} g^{k-1}(E, x) (k - g(E, x)).$$

After obtaining the estimates of f-DMT, we next investigate the asymptotic behavior when $E$ tends to infinity. Since the estimate expressions of f-DMT include the parameters $\alpha$ and $t$, we need to discuss the limiting results under different parameter settings which may be difficult to compute. To simplify the problem, we take a special case of $\alpha = t = 1/2$. It is shown that

$$\lim_{E \to \infty} \hat{d}_1 = \lim_{E \to \infty} \hat{d}_2 = n \left( 1 - 2r \right) \quad 0 \leq r \leq 0.5 \quad (33)$$

This result is consistent with asymptotically high SNR DMT [10]. As another limiting result of interest, the maximal diversity gain as the multiplexing gain $r$ decreases to zero is given by

$$\lim_{r \to 0} \hat{d}_1 = \lim_{r \to 0} \hat{d}_2 = n \left( 1 - \frac{E}{(1 + E) \ln(1 + E)} \right) \quad (34)$$

V. NUMERICAL RESULTS

This section presents some numerical results to illustrate the impact of SNRs, the number of antennas, time sharings, rate allocations, and relay locations on the outage and f-DMT performances for TWRC. For simplicity, the relay lies between $S_1$ and $S_2$. The distance between $S_1$ and $S_2$ is fixed at 1. Let $D$ and $1 - D$ denote the distances from $S_1$ to $R$ and from $S_2$ to $R$, respectively. Let the path loss factor is as in urban areas. Hence, $\beta_h$ and $\beta_f$ can be expressed as a function of $D$: $\beta_h = D^{-4}$ and $\beta_f = (1 - D)^{-4}$.

In Fig. 2 the outage probability is plotted for different numbers of relay antennas. The upper and lower bounds of outage probability are compared to exact Monte Carlo simulations. It is shown that the bounds are very tight. This is because the gaps incurred by the probability of $O_3$ can be ignored compared with the probability of $O_{12}$ when the
number of relay antennas is not very large. Therefore, in the rest of this section, we only focus on the upper bound.

Fig. 3 plots the estimates of f-DMTs with different parameter settings. It is seen that if SNR is small, the maximal multiplexing gain is close to 1. In the contrary, as SNR increases, the number of relay antennas at the relay can significantly improve the outage probability and hence the diversity order but can have little impact on the multiplexing gain.

VI. CONCLUSION

The finite-SNR framework for outage probability and f-DMT is a powerful tool to understand the performance of multi-antenna relay TWRC with two-phase DF protocol. This paper first derives the upper and lower bounds of outage probability, and the bounds approximate very well the true performance obtained by Monte Carlo simulation. Then these bounds are used to obtain the f-DMT estimates as a function of multiplexing gain and SNR. Our analysis shows that having multiple antennas at the relay can significantly improve the outage probability and hence the diversity order but can have little impact on the multiplexing gain.

REFERENCES