

Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements

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Introduction

The interplay between geometry and topology on complex algebraic varieties is a classical theme that goes back to Lefschetz [L] and Zariski [Z] and is always present on the scene; see for instance the work by Libgober [Li]. In this paper we study complements of hypersurfaces, with special attention to the case of hyperplane arrangements as discussed in Orlik-Terao’s book [OT1].

Theorem 1 expresses the degree of the gradient map associated to any homogeneous polynomial h as the number of n -cells that have to be added to a generic hyperplane section $D(h) \cap H$ to obtain the complement in \mathbb{P}^n , $D(h)$, of the projective hypersurface $V(h)$. Alternatively, by results of Lê [Le2] one knows that the affine piece $V(h)_a = V(h) \setminus H$ of $V(h)$ has the homotopy type of a bouquet of $(n - 1)$ -spheres. Theorem 1 can then be restated by saying that the degree of the gradient map coincides with the number of these $(n - 1)$ -spheres. In this form, our result is reminiscent of Milnor’s equality between the degree of the local gradient map and the number of spheres in the Milnor fiber associated to an isolated hypersurface singularity [M].

This topological description of the degree of the gradient map has as a direct consequence a positive answer to a conjecture by Dolgachev [Do] on polar Cremona transformations; see Corollary 2. Corollary 4 and the end of Section 3 contain stronger versions of some of the results in [Do] and some related matters.

Corollary 6 (obtained independently by Randell [R2,3]) reveals a striking feature of complements of hyperplane arrangements. They possess a *minimal* CW-structure, i. e., a CW-decomposition with exactly as many k -cells as the k -th Betti number, for all k . Minimality may be viewed as an improvement of the Morse inequalities for twisted homology (the main result of Daniel Cohen in [C]), from homology to the level of cells; see Remark 12 (ii).

In the second part of our paper, we investigate the higher homotopy groups of complements of complex hyperplane arrangements (as π_1 -modules). By the classical work of Brieskorn [B] and Deligne [De], it is known that such a complement is often aspherical. The first explicit computation of nontrivial homotopy groups of this type has been performed by Hattori [Hat], in 1975. This remained the only example of this kind, until [PS] was published.

Hattori proved that, up to homotopy, the complement of a general position arrangement is a skeleton of the standard minimal CW-structure of a torus. From this, he derived a free resolution of the first nontrivial higher homotopy group. We use the techniques developed in the first part of our paper to generalize Hattori's homotopy type formula, for all sufficiently generic sections of aspherical arrangements (a framework inspired from the stratified Morse theory of Goresky-MacPherson [GM]); see Proposition 14. Using the approach by minimality from [PS], we can to generalize the Hattori presentation in Theorem 16, and the Hattori resolution in Theorem 18. The above framework provides a unified treatment of all explicit computations related to nonzero higher homotopy groups of arrangements available in the literature, to the best of our knowledge. It also gives examples exhibiting a nontrivial homotopy group, π_q , for *all* q ; see the end of Section 5.

The associated *combinatorics* plays an important role in arrangement theory. By 'combinatorics' we mean the pattern of intersection of the hyperplanes, encoded by the associated intersection lattice. For instance, one knows, by the work of Orlik-Solomon [OS], that the cohomology ring of the complement is determined by the combinatorics. On the other hand, the examples of Rybnikov [Ry] show that π_1 is not combinatorially determined, in general. One of the most basic questions in the field is to identify the precise amount of topological information on the complement that is determined by the combinatorics.

In Corollary 21 we consider the associated graded chain complex, with respect to the I -adic filtration of the group ring $\mathbb{Z}\pi_1$, of the π_1 -equivariant chain complex of the universal cover, constructed from an arbitrary minimal CW-structure of any arrangement complement. We prove that the associated graded is always combinatorially determined, with \mathbb{Q} -coefficients, and that this actually holds over \mathbb{Z} , for the class of *hypersolvable* arrangements introduced in [JP1]. We deduce these properties from a general result, namely Theorem 20, where we show that the associated graded equivariant chain complex of the universal cover of a minimal CW-complex, whose cohomology ring is generated in degree one, is determined by π_1 and the cohomology ring.

There is a rich supply of examples which fit into our framework of generic sections of aspherical arrangements. Among them, we present in Theorem 23 a large class of combinatorially defined hypersolvable examples, for which the associated graded module of the first higher nontrivial homotopy group of the complement is also combinatorially determined.

1. The main results

There is a gradient map associated to any nonconstant homogeneous polynomial $h \in \mathbb{C}[x_0, \dots, x_n]$ of degree d , namely

$$\text{grad}(h) : D(h) \rightarrow \mathbb{P}^n, \quad (x_0 : \dots : x_n) \mapsto (h_0(x) : \dots : h_n(x))$$

where $D(h) = \{x \in \mathbb{P}^n | h(x) \neq 0\}$ is the principal open set associated to h and $h_i = \frac{\partial h}{\partial x_i}$. This map corresponds to the polar Cremona transformations considered by Dolgachev in [Do]. Our first result is the following topological description of the degree of the gradient map $\text{grad}(h)$.

THEOREM 1. *For any nonconstant homogeneous polynomial $h \in \mathbb{C}[x_0, \dots, x_n]$, the complement $D(h)$ is homotopy equivalent to a CW complex obtained from $D(h) \cap H$ by attaching $\text{deg}(\text{grad}(h))$ cells of dimension n , where H is a generic hyperplane in \mathbb{P}^n . In particular, one has*

$$\text{deg}(\text{grad}(h)) = (-1)^n \chi(D(h) \setminus H).$$

Note that the meaning of ‘generic’ here is quite explicit: the hyperplane H has to be transversal to a stratification of the projective hypersurface $V(h)$ defined by $h = 0$ in \mathbb{P}^n .

The Euler characteristic in the above statement can be replaced by a Betti number as follows. As noted in the introduction, the affine part $V(h)_a = V(h) \setminus H$ of $V(h)$ has the homotopy type of a bouquet of $(n - 1)$ -spheres. Using the additivity of the Euler characteristic with respect to constructible partitions we get

$$\text{deg}(\text{grad}(h)) = b_{n-1}(V(h)_a).$$

We use this form of Theorem 1 at the end of Section 3 to give a topological easy proof of Theorem 4 in [Do].

Theorem 1 looks similar to a conjecture on hyperplane arrangements by Varchenko [V] proved by Orlik and Terao [OT3] and by Damon [Da2], but in fact it is not; see our discussion after Lemma 5.

On the other hand, with some transversality conditions for the irreducible factors of h , Damon has obtained a local form of Theorem 1 in which

$$D(h) \setminus H = \{x \in \mathbb{C}^{n+1} | \ell(x) = 1\} \setminus \{x \in \mathbb{C}^{n+1} | h(x) = 0\}$$

($\ell(x) = 0$ being an equation for H) is replaced by

$$V_t \setminus \{x \in \mathbb{C}^{n+1} | h(x) = 0\}$$

with V_t the Milnor fiber of an isolated complete intersection singularity V at the origin of \mathbb{C}^{n+1} ; see [Da2, Th. 1]. In such a situation the corresponding

Euler number is explicitly computed as a sum of the Milnor number $\mu(V)$ and a “singular Milnor number”; see [Da2, Th. 1] and [Da3, Th. 1 or Cor. 2].

COROLLARY 2. *The degree of the gradient map $\text{grad}(h)$ depends only on the reduced polynomial h_r associated to h .*

This gives a positive answer to Dolgachev’s conjecture at the end of Section 3 in [Do], and it follows directly from Theorem 1, since $D(h) = D(h_r)$.

Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $e > 0$ with global Milnor fiber $F = \{x \in \mathbb{C}^{n+1} | f(x) = 1\}$; see for instance [D1] for more on such varieties. Let $g : F \setminus N \rightarrow \mathbb{R}$ be the function $g(x) = h(x)\bar{h}(x)$, where $N = \{x \in \mathbb{C}^{n+1} | h(x) = 0\}$. Then we have the following:

THEOREM 3. *For any reduced homogeneous polynomial $h \in \mathbb{C}[x_0, \dots, x_n]$ there is a Zariski open and dense subset \mathcal{U} in the space of homogeneous polynomials of degree $e > 0$ such that for any $f \in \mathcal{U}$ one has the following:*

- (i) *the function g is a Morse function;*
- (ii) *the Milnor fiber F is homotopy equivalent to a CW complex obtained from $F \cap N$ by attaching $|C(g)|$ cells of dimension n , where $C(g)$ is the critical set of the Morse function g ;*
- (iii) *the intersection $F \cap N$ is homotopy equivalent to a bouquet of $|C(g)| - (e - 1)^{n+1}$ spheres S^{n-1} .*

In some cases the open set \mathcal{U} can be explicitly described, as in Corollary 7 below. In general this task is a difficult one in view of the proof of Theorem 3. The claim (iii) above, in the special case $e = 1$, gives a new proof for Lê’s result mentioned in the introduction.

LEFSCHETZ THEOREM ON GENERIC HYPERPLANE COMPLEMENTS IN HYPERSURFACES. *For any projective hypersurface $V(h) : h = 0$ in \mathbb{P}^n and any generic hyperplane H in \mathbb{P}^n the affine hypersurface given by the complement $V(h) \setminus H$ is homotopy equivalent to a bouquet of spheres S^{n-1} .*

We point out that both Theorem 1 and Theorem 3 follow from the results by Hamm in [H]. In the case of Theorem 1, the homotopy-type claim is a direct consequence of [H, Th. 5], the new part being the relation between the number of n -cells and the degree of the gradient map $\text{grad}(h)$. We establish this equality by using polar curves and complex Morse theory; see Section 2.

On the other hand, in Theorem 3 the main claim is that concerning the homotopy-type and this follows from [H, Prop. 3], by a geometric argument described in Section 3 and involving a key result by Hironaka. An alternative proof may also be given using Damon’s work [Da1, Prop. 9.14], a result which extends previous results by Siersma [Si] and Looijenga [Lo].

Our results above have interesting implications for the topology of hyperplane arrangements which were our initial motivation in this study. Let \mathcal{A} be a hyperplane arrangement in the complex projective space \mathbb{P}^n , with $n > 0$. Let $d > 0$ be the number of hyperplanes in this arrangement and choose a linear equation $\ell_i(x) = 0$ for each hyperplane H_i in \mathcal{A} , for $i = 1, \dots, d$.

Consider the homogeneous polynomial $Q(x) = \prod_{i=1}^d \ell_i(x) \in \mathbb{C}[x_0, \dots, x_n]$ and the corresponding principal open set $M = M(\mathcal{A}) = D(Q) = \mathbb{P}^n \setminus \cup_{i=1}^d H_i$. The topology of the hyperplane arrangement complement M is a central object of study in the theory of hyperplane arrangements, see Orlik-Terao [OT1]. As a consequence of Theorem 1 we prove the following:

COROLLARY 4. (1) *For any projective arrangement \mathcal{A} as above one has*

$$b_n(D(Q)) = \text{deg}(\text{grad}(Q)).$$

(2) *In particular:*

(a) *The following are equivalent:*

- (i) *the morphism $\text{grad}(Q)$ is dominant;*
- (ii) *$b_n(D(Q)) > 0$;*
- (iii) *the projective arrangement \mathcal{A} is essential; i.e., the intersection $\cap_{i=1}^d H_i$ is empty.*

(b) *If $b_n(D(Q)) > 0$ then $d \leq n + b_n(D(Q))$. As special cases:*

- (b1) *$b_n(D(Q)) = 1$ if and only if $d = n + 1$ and up to a linear coordinate change we have $\ell_i(x) = x_{i-1}$ for all $i = 1, \dots, n + 1$;*
- (b2) *$b_n(D(Q)) = 2$ if and only if $d = n + 2$ and up to a linear coordinate change and re-ordering of the hyperplanes, $\ell_i(x) = x_{i-1}$ for all $i = 1, \dots, n + 1$ and $\ell_{n+2}(x) = x_0 + x_1$.*

Note that the equivalence of (i) and (iii) is a generalization of Lemma 7 in [Do], and (b1) is a generalization of Theorem 5 in [Do].

To obtain Corollary 4 from Theorem 1 all we need is the following:

LEMMA 5. *For any arrangement \mathcal{A} as above, $(-1)^n \chi(D(Q) \setminus H) = b_n(D(Q))$.*

Let $\mathcal{A}' = \{H'_i\}_{i \in I}$ be an affine hyperplane arrangement in \mathbb{C}^n with complement $M(\mathcal{A}')$ and let $\ell'_i = 0$ be an equation for the hyperplane H'_i . Consider the multivalued function

$$\phi_a : M(\mathcal{A}') \rightarrow \mathbb{C}, \quad \phi_a(x) = \prod_i \ell'_i(x)^{a_i}$$

with $a_i \in \mathbb{C}$. Varchenko conjectured in [V] that for an essential arrangement \mathcal{A}' and for generic complex exponents a_i the function ϕ_a has only nondegenerate

critical points and their number is precisely $|\chi(M(\mathcal{A}'))|$. This conjecture was proved in more general forms by Orlik-Terao [OT3] via algebraic methods and by Damon [Da2] via topological methods based on [DaM] and [Da1].

In particular Damon shows in Theorem 1 in [Da2] that the function ϕ_1 obtained by taking $a_i = 1$ for all $i \in I$ has only isolated singularities and the sum of the corresponding Milnor numbers equals $|\chi(M(\mathcal{A}'))|$. Consider the morsification

$$\psi(x) = \phi_1(x) - \sum_{j=1}^n b_j x_j$$

where $b_j \in \mathbb{C}$ are generic and small. Then one may think that by the general property of a morsification, ψ has only nondegenerate critical points and their number is precisely $|\chi(M(\mathcal{A}'))|$. In fact, as a look at the simple example $n = 3$ and $\phi_1 = xyz$ shows, there are new nondegenerate singularities occurring along the hyperplanes. This can be restated by saying that in general one has

$$\deg(\text{grad}\phi_1) \geq |\chi(M(\mathcal{A}'))|$$

and not an equality similar to our Corollary 4 (1). Note that here $\text{grad}\phi_1 : M(\mathcal{A}') \rightarrow \mathbb{C}^n$.

The classification of arrangements for which $|\chi(M(\mathcal{A}'))| = 1$ is much more complicated than the one from Corollary 4(b1) and the interested reader is referred to [JL].

Theorem 1, in conjunction with Corollary 4, Part (1), has very interesting consequences. We say that a topological space Z is *minimal* if Z has the homotopy type of a connected CW-complex K of finite type, whose number of k -cells equals $b_k(K)$ for all $k \in \mathbb{N}$. It is clear that a minimal space has integral torsion-free homology. The converse is true for 1-connected spaces; see [PS, Rem. 2.14].

The importance of this notion for the topology of spaces which look homologically like complements of hyperplane arrangements was recently noticed in [PS]. Previously, the minimality property was known only for generic arrangements (Hattori [Hat]) and fiber-type arrangements (Cohen-Suciu [CS]). Our next result establishes this property, in full generality. It was independently obtained by Randell [R2,3], using similar techniques. (See, however, Example 13.) The minimality property below should be compared with the main result from [GM, Part III], where the existence of a *homologically perfect* Morse function is established, for complements of (arbitrary) arrangements of real affine subspaces; see [GM, p. 236].

COROLLARY 6. *Both complements, $M(\mathcal{A}) \subset \mathbb{P}^n$ and its cone, $M'(\mathcal{A}) \subset \mathbb{C}^{n+1}$, are minimal spaces.*

It is easy to see that for $n > 1$, the open set $D(f)$ is not minimal for f generic of degree $d > 1$ (just use $\pi_1(D(f)) = H_1(D(f), \mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$), but the Milnor fiber F defined by f is clearly minimal. We do not know whether the Milnor fiber $\{Q = 1\}$ associated to an arrangement is minimal in general.

From Theorem 3 we get a substantial strengthening of some of the main results by Orlik and Terao in [OT2]. Let \mathcal{A}' be the central hyperplane arrangement in \mathbb{C}^{n+1} associated to the projective arrangement \mathcal{A} . Note that $Q(x) = 0$ is a reduced equation for the union N of all the hyperplanes in \mathcal{A}' . Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $e > 0$ with global Milnor fiber $F = \{x \in \mathbb{C}^{n+1} | f(x) = 1\}$ and let $g : F \setminus N \rightarrow \mathbb{R}$ be the function $g(x) = Q(x)\overline{Q}(x)$ associated to the arrangement. The polynomial f is called \mathcal{A}' -generic if

(GEN1) the restriction of f to any intersection L of hyperplanes in \mathcal{A}' is nondegenerate, in the sense that the associated projective hypersurface in $\mathbb{P}(L)$ is smooth, and

(GEN2) the function g is a Morse function.

Orlik and Terao have shown in [OT2] that for an *essential* arrangement \mathcal{A}' , the set of \mathcal{A}' -generic functions f is dense in the set of homogeneous polynomials of degree e , and, as soon as we have an \mathcal{A}' -generic function f , the following basic properties hold for any arrangement.

(P1) $b_q(F, F \cap N) = 0$ for $q \neq n$ and

(P2) $b_n(F, F \cap N) \leq |C(g)|$, where $C(g)$ is the critical set of the Morse function g .

An explicit formula for the number $|C(g)|$ is given in [OT2] in terms of the lattice associated to the arrangement \mathcal{A}' . Moreover, for a special class of arrangements called *pure* arrangements it is shown in [OT2] that (P2) is actually an equality. In fact, the proof of (P2) in [OT2] uses Morse theory on noncompact manifolds, but we are unable to see the details behind the proof of Corollary (3.5); compare to our discussion in Example 13.

With this notation the following is a direct consequence of Theorem 3.

COROLLARY 7. *For any arrangement \mathcal{A} the following hold:*

- (i) *the set of \mathcal{A}' -generic functions f is dense in the set of homogeneous polynomials of degree $e > 0$;*
- (ii) *the Milnor fiber F is homotopy equivalent to a CW complex obtained from $F \cap N$ by attaching $|C(g)|$ cells of dimension n , where $C(g)$ is the critical set of the Morse function g . In particular $b_n(F, F \cap N) = |C(g)|$ and the intersection $F \cap N$ is homotopy equivalent to a bouquet of $|C(g)| - (e-1)^{n+1}$ spheres S^{n-1} .*

Similar results for nonlinear arrangements on complete intersections have been obtained by Damon in [Da3] where explicit formulas for $|C(g)|$ are given.

The aforementioned results represent a strengthening of those in [D2] (in which the homological version of Theorems 1 and 3 above was proved).

The investigation of higher homotopy groups of complements of complex hypersurfaces (as π_1 -modules) is a very difficult problem. In the irreducible case, see [Li] for various results on the first nontrivial higher homotopy group. The arrangements of hyperplanes provide the simplest nonirreducible situation (where π_1 is never trivial, but at the same time rather well understood). This is the topic of the second part of our paper.

Our results here use the general approach by minimality from [PS], and significantly extend the homotopy computations therefrom. In Section 5, we present a unifying framework for all known explicit descriptions of nontrivial higher homotopy groups of arrangement complements, together with a numerical $K(\pi, 1)$ -test. We give specific examples, in Section 6, with emphasis on combinatorial determination. A general survey of Sections 5 and 6 follows. (To avoid overloading the exposition, formulas will be systematically skipped.)

Our first main result in Sections 5 and 6 is Theorem 16. It applies to arrangements \mathcal{A} which are *k-generic sections*, $k \geq 2$, of aspherical arrangements, $\widehat{\mathcal{A}}$. Here '*k-generic*' means, roughly speaking, that \mathcal{A} and $\widehat{\mathcal{A}}$ have the same intersection lattice, up to rank $k + 1$; see Section 5(1) for the precise definition.

The *general position* arrangements from [Hat] and the *fiber-type* aspherical ones from [FR] belong to the *hypersolvable* class from [JP1]. Consequently ([JP2]), they all are 2-generic sections of fiber-type arrangements. At the same time, the *iterated generic hyperplane sections*, \mathcal{A} , of essential aspherical arrangements, $\widehat{\mathcal{A}}$, from [R1], are also particular cases of *k-generic sections*, with $k = \text{rank}(\mathcal{A}) - 1$.

For such a *k-generic section* \mathcal{A} , Theorem 16 firstly says that the complement $M(\mathcal{A})$ ($M'(\mathcal{A})$) is aspherical if and only if $p = \infty$, where p is a topological invariant introduced in [PS]. Secondly (if $p < \infty$), one can write down a $\mathbb{Z}\pi_1$ -module presentation for π_p , the first higher nontrivial homotopy group of the complement (see §5(8), (9) for details). Both results essentially follow from Propositions 14 and 15, which together imply that $M(\mathcal{A})$ and $M(\widehat{\mathcal{A}})$ share the same p -skeleton.

In Theorem 18, we substantially extend and improve results from [Hat] and [R1] (see also Remark 19). Here we examine \mathcal{A} , an iterated generic hyperplane section of rank ≥ 3 , of an essential aspherical arrangement, $\widehat{\mathcal{A}}$. Set $M = M(\mathcal{A})$. In this case, $p = \text{rank}(\mathcal{A}) - 1$ [PS]. We show that the $\mathbb{Z}\pi_1(M)$ -presentation of $\pi_p(M)$ from Theorem 16 extends to a finite, minimal, free $\mathbb{Z}\pi_1(M)$ -resolution. We infer that $\pi_p(M)$ cannot be a projective $\mathbb{Z}\pi_1(M)$ -module, unless $\text{rank}(\mathcal{A}) = \text{rank}(\widehat{\mathcal{A}}) - 1$, when it is actually $\mathbb{Z}\pi_1(M)$ -free.

In Theorem 18 (v), we go beyond the first nontrivial higher homotopy group. We obtain a complete description of *all* higher rational homotopy groups, $L_* := \bigoplus_{q \geq 1} \pi_{q+1}(M) \otimes \mathbb{Q}$, including both the graded Lie algebra structure of L_* induced by the Whitehead product, and the graded $\mathbb{Q}\pi_1(M)$ -module structure.

The computational difficulties related to the twisted homology of a connected CW-complex (in particular, to the first nonzero higher homotopy group) stem from the fact that the $\mathbb{Z}\pi_1$ -chain complex of the universal cover is very difficult to describe, in general. As explained in the introduction, we have two results in this direction, at the I -adic associated graded level: Theorem 20 and Corollary 21.

Corollary 21 belongs to a recurrent theme of our paper: exploration of new phenomena of combinatorial determination in the homotopy theory of arrangements. Our combinatorial determination property from Corollary 21 should be compared with a fundamental result of Kohno [K], which says that the rational graded Lie algebra associated to the lower central series filtration of π_1 of a projective hypersurface complement is determined by the cohomology ring.

In Theorem 23, we examine the hypersolvable arrangements for which $p = \text{rank}(\mathcal{A}) - 1$. We establish the combinatorial determination property of the I -adic associated graded module (over \mathbb{Z}) of the first higher nontrivial homotopy group of the complement, π_p , in Theorem 23 (i). The proof uses in an essential way the ubiquitous Koszul property from homological algebra.

We also infer from Koszulness, in Theorem 23 (ii), that the successive quotients of the I -adic filtration on π_p are finitely generated free abelian groups, with ranks given by the combinatorial *I -adic filtration formula* (22). This resembles the *lower central series (LCS) formula*, which expresses the ranks of the quotients of the lower central series of π_1 of certain arrangements, in combinatorial terms. The LCS formula for pure braid groups was discovered by Kohno, starting from his pioneering work in [K]. It was established for all fiber-type arrangements in [FR], and then extended to the hypersolvable class in [JP1].

Another new example of combinatorial determination is the fact that the generic affine part of a union of hyperplanes has the homotopy type of the Folkman complex, associated to the intersection lattice. This follows from Theorem 1 and Corollary 4; see the discussion after the proof of Theorem 3.

2. Polar curves, affine Lefschetz theory and degree of gradient maps

The use of the local polar varieties in the study of singular spaces is already a classical subject; see Lê [Le1], Lê-Teissier [LT] and the references

therein. Global polar curves in the study of the topology of polynomials is a topic under intense investigations; see for instance Cassou-Noguès and Dimca [CD], Hamm [H], Némethi [N1,2], Siersma and Tibăr [ST]. For all the proofs in this paper, the classical theory is sufficient: indeed, all the objects being homogeneous, one can localize at the origin of \mathbb{C}^{n+1} in the standard way, see [D1]. However, using geometric intuition, we find it easier to work with global objects, and hence we adopt this viewpoint in the sequel.

We recall briefly the notation and the results from [CD], [N1,2]. Let $h \in \mathbb{C}[x_0, \dots, x_n]$ be a polynomial (even nonhomogeneous to start with) and assume that the fiber $F_t = h^{-1}(t)$ is smooth, for some fixed $t \in \mathbb{C}$.

For any hyperplane in \mathbb{P}^n , $H : \ell = 0$ where $\ell(x) = c_0x_0 + c_1x_1 + \dots + c_nx_n$, we define the corresponding polar variety Γ_H to be the union of the irreducible components of the variety

$$\{x \in \mathbb{C}^{n+1} \mid \text{rank}(dh(x), d\ell(x)) = 1\}$$

which are not contained in the critical set $S(h) = \{x \in \mathbb{C}^{n+1} \mid dh(x) = 0\}$ of h .

LEMMA 8 (see [CD], [ST]). *For a generic hyperplane H ,*

- (i) *The polar variety Γ_H is either empty or a curve; i.e., each irreducible component of Γ_H has dimension 1.*
- (ii) *$\dim(F_t \cap \Gamma_H) \leq 0$ and the intersection multiplicity (F_t, Γ_H) is independent of H .*
- (iii) *The multiplicity (F_t, Γ_H) is equal to the number of tangent hyperplanes to F_t parallel to the hyperplane H . For each such tangent hyperplane H_a , the intersection $F_t \cap H_a$ has precisely one singularity, which is an ordinary double point.*

The nonnegative integer (F_t, Γ_H) is called the polar invariant of the hypersurface F_t and is denoted by $P(F_t)$. Note that $P(F_t)$ corresponds exactly to the classical notion of class of a projective hypersurface; see [L].

We think of a projective hyperplane H as the direction of an affine hyperplane $H' = \{x \in \mathbb{C}^{n+1} \mid \ell(x) = s\}$ for $s \in \mathbb{C}$. All the affine hyperplanes with the same direction form a pencil, and it is precisely this type of pencil that is used in the affine Lefschetz theory; see [N1,2]. Némethi considers only connected affine varieties, but his results clearly extend to the case of any pure dimensional smooth variety.

PROPOSITION 9 (see [CD], [ST]). *For a generic hyperplane H' in the pencil of all hyperplanes in \mathbb{C}^{n+1} with a fixed generic direction H , the fiber F_t is homotopy equivalent to a CW-complex obtained from the section $F_t \cap H'$ by attaching $P(F_t)$ cells of dimension n . In particular*

$$P(F_t) = (-1)^n(\chi(F_t) - \chi(F_t \cap H')) = (-1)^n\chi(F_t \setminus H').$$

Moreover, in this statement ‘generic’ means that the affine hyperplane H' has to verify the following two conditions:

- (g1) its direction in \mathbb{P}^n has to be generic, and
- (g2) the intersection $F_t \cap H'$ has to be smooth.

These two conditions are not stated in [CD], but the reader should have no problem in checking them by using Theorem 3' in [CD] and the fact proved by Némethi in [N1,2] that the only bad sections in a good pencil are the singular sections. Completely similar results hold for generic pencils with respect to a closed smooth subvariety Y in some affine space \mathbb{C}^N ; see [N1,2], but note that the polar curves are not mentioned there.

Proof of Theorem 1. In view of Hamm’s affine Lefschetz theory, see [H, Th. 5], the only thing to prove is the equality between the number k_n of n -cells attached and the degree of the gradient.

Assume from now on that the polynomial h is homogeneous of degree d and that $t = 1$. It follows from (g1) and (g2) above that we may choose the generic hyperplane H' passing through the origin.

Moreover, in this case, the polar curve Γ_H , being defined by homogeneous equations, is a union of lines L_j passing through the origin. For each such line we choose a parametrization $t \mapsto a_j t$ for some $a_j \in \mathbb{C}^{n+1}, a_j \neq 0$. It is easy to see that the intersection $F_1 \cap L_j$ is either empty (if $h(a_j) = 0$) or consists of exactly d distinct points with multiplicity one (if $h(a_j) \neq 0$). The lines of the second type are in bijection with the points in $\text{grad}(h)^{-1}(D_{H'})$, where $D_{H'} \in \mathbb{P}^n$ is the point corresponding to the direction of the hyperplane H' . It follows that

$$d \cdot \text{deg}(\text{grad}(h)) = P(F_1).$$

The d -sheeted unramified coverings $F_1 \rightarrow D(h)$ and $F_1 \cap H' \rightarrow D(h) \cap H'$ give the result, where H is the projective hyperplane corresponding to the affine hyperplane (passing through the origin) H' . Indeed, they imply the equalities: $\chi(F_1) = d \cdot \chi(D(h))$ and $\chi(F_1 \cap H') = d \cdot \chi(D(h) \cap H')$. Hence we have $\text{deg}(\text{grad}(h)) = (-1)^n \chi(F_1, F_1 \cap H')/d = (-1)^n \chi(D(h), D(h) \cap H) = k_n$.

Remark 10. The gradient map $\text{grad}(h)$ has a natural extension to the larger open set $D'(h)$ where at least one of the partial derivatives of h does not vanish. It is obvious (by a dimension argument) that this extension has the same degree as the map $\text{grad}(h)$.

3. Nonproper Morse theory

For the convenience of the reader we recall, in the special case needed, a basic result of Hamm, see [H, Prop. 3], with our addition concerning the condition (c0) in [DP, Lemma 3 and Ex. 2]. The final claim on the number of cells to be attached is also standard, see for instance [Le1].

PROPOSITION 11. *Let A be a smooth algebraic subvariety in \mathbb{C}^p with $\dim A = m$. Let f_1, \dots, f_p be polynomials in $\mathbb{C}[x_1, \dots, x_p]$. For $1 \leq j \leq p$, denote by Σ_j the set of critical points of the mapping $(f_1, \dots, f_j) : A \setminus \{z \in A \mid f_1(z) = 0\} \rightarrow \mathbb{C}^j$ and let Σ'_j denote the closure of Σ_j in A . Assume that the following conditions hold.*

- (c0) *The set $\{z \in A \mid |f_1(z)| \leq a_1, \dots, |f_p(z)| \leq a_p\}$ is compact for any positive numbers $a_j, j = 1, \dots, p$.*
- (c1) *The critical set Σ_1 is finite.*
- (cj) (for $j = 2, \dots, p$) *The map $(f_1, \dots, f_{j-1}) : \Sigma'_j \rightarrow \mathbb{C}^{j-1}$ is proper.*

Then A has the homotopy type of a space obtained from $A_1 = \{z \in A \mid f_1(z) = 0\}$ by attaching m -cells and the number of these cells is the sum of the Milnor numbers $\mu(f_1, z)$ for $z \in \Sigma_1$.

Proof of Theorem 3. We set $X = h^{-1}(1)$. Let $v : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^N$ be the Veronese mapping of degree e sending x to all the monomials of degree e in x and set $Y = v(X)$. Then Y is a smooth closed subvariety in \mathbb{C}^N and $v : X \rightarrow Y$ is an unramified covering of degree c , where $c = \text{g.c.d.}(d, e)$. To see this, use the fact that v is a closed immersion on $\mathbb{C}^N \setminus \{0\}$ and $v(x) = v(x')$ if and only if $x' = u \cdot x$ with $u^c = 1$.

Let H be a generic hyperplane direction in \mathbb{C}^N with respect to the subvariety Y and let $C(H)$ be the finite set of all the points $p \in Y$ such that there is an affine hyperplane H'_p in the pencil determined by H that is tangent to Y at the point p and the intersection $Y \cap H'_p$ has a complex Morse (alias non-degenerate, alias A_1) singularity. Under the Veronese mapping v , the generic hyperplane direction H corresponds to a homogeneous polynomial of degree e which we call from now on f .

To prove the first claim (i) we proceed as follows. It is known that using affine Lefschetz theory for a pencil of hypersurfaces $\{h = t\}$ is equivalent to using (nonproper) Morse theory for the function $|h|$ or, what amounts to the same, for the function $|h|^2$. More explicitly, in view of the last statement at the end of the proof of Lemma (2.5) in [OT2] (which clearly applies to our more general setting since all the computations there are local), g is a Morse function if and only if each critical point of $h : F \setminus N \rightarrow \mathbb{C}$ is an A_1 -singularity. By the homogeneity of both f and h , this last condition on h is equivalent to the fact that each critical point of the function $f : X \rightarrow \mathbb{C}$ is an A_1 -singularity, condition fulfilled in view of the choice of H and since $v : X \rightarrow Y$ is a local isomorphism.

Now we pass on to the proof of the claim (ii) in Theorem 3. One can derive this claim easily from Proposition 9.14 in Damon [Da1]. However since his proof is using previous results by Siersma [Si] and Looijenga [Lo], we think that our original proof given below and based on Proposition 11, longer but more self-contained, retains its interest.

Any polynomial function $h : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ admits a Whitney stratification satisfying the Thom a_h -condition. This is a constructible stratification \mathcal{S} such that the open stratum, say S_0 , coincides with the set of regular points for h and for any other stratum, say $S_1 \subset h^{-1}(0)$, and any sequence of points $q_m \in S_0$ converging to $q \in S_1$ such that the sequence of tangent spaces $T_{q_m}(h)$ has a limit T ; one has $T_q S_1 \subset T$. See Hironaka [Hi, Cor. 1, p. 248]. The requirement of f proper in that corollary is not necessary in our case, as any algebraic map can be compactified. Here and in the sequel, for a map $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ and a point $q \in \mathcal{X}$ we denote by $T_q(\phi)$ the tangent space to the fiber $\phi^{-1}(\phi(q))$ at the point q , assumed to be a smooth point on this fiber.

Since in our case h is a homogeneous polynomial, we can find a stratification \mathcal{S} as above such that all of its strata are \mathbb{C}^* -invariant, with respect to the natural \mathbb{C}^* -action on \mathbb{C}^{n+1} . In this way we obtain an induced Whitney stratification \mathcal{S}' on the projective hypersurface $V(h)$. We select our polynomial f such that the corresponding projective hypersurface $V(f)$ is smooth and transversal to the stratification \mathcal{S}' . In this way we get an induced Whitney stratification \mathcal{S}'_1 on the projective complete intersection $V_1 = V(h) \cap V(f)$.

We use Proposition 11 above with $A = F$ and $f_1 = h$. All we have to show is the existence of polynomials f_2, \dots, f_{n+1} satisfying the conditions listed in Proposition 11.

We will select these polynomials inductively to be generic linear forms as follows. We choose f_2 such that the corresponding hyperplane H_2 is transversal to the stratification \mathcal{S}'_1 . Let \mathcal{S}'_2 denote the induced stratification on $V_2 = V_1 \cap H_2$. Assume that we have constructed f_2, \dots, f_{j-1} , $\mathcal{S}'_1, \dots, \mathcal{S}'_{j-1}$ and V_1, \dots, V_{j-1} . We choose f_j such that the corresponding hyperplane H_j is transversal to the stratification \mathcal{S}'_{j-1} . Let \mathcal{S}'_j denote the induced stratification on $V_j = V_{j-1} \cap H_j$. Do this for $j = 3, \dots, n$ and choose for f_{n+1} any linear form.

With this choice it is clear that for $1 \leq j \leq n$, V_j is a complete intersection of dimension $n - 1 - j$. In particular, $V_n = \emptyset$; i.e.

$$(c0') \quad \{x \in \mathbb{C}^{n+1} \mid f(x) = h(x) = f_2(x) = \dots = f_n(x) = 0\} = \{0\} .$$

Then the map $(f, h, f_2, \dots, f_n) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$ is proper, which clearly implies the condition (c0).

The condition (c1) is fulfilled by our construction of f . Assume that we have already checked that the conditions (ck) are fulfilled for $k = 1, \dots, j - 1$. We explain now why the next condition (cj) is fulfilled.

Assume that the condition (cj) fails. This is equivalent to the existence of a sequence p_m of points in Σ'_j such that

$$(*) \quad |p_m| \rightarrow \infty \text{ and } f_k(p_m) \rightarrow b_k \text{ (finite limits) for } 1 \leq k \leq j - 1.$$

Since Σ_j is dense in Σ'_j , we can even assume that $p_m \in \Sigma_j$.

Note that $\Sigma_{j-1} \subset \Sigma_j$ and the condition $c(j-1)$ is fulfilled. This implies that we may choose our sequence p_m in the difference $\Sigma_j \setminus \Sigma_{j-1}$. In this case we get

$$(**) \quad f_j \in \text{Span}(df(p_m), dh(p_m), f_2, \dots, f_{j-1})$$

the latter being a j -dimensional vector space.

Let $q_m = \frac{p_m}{|p_m|} \in S^{2n+1}$. Since the sphere S^{2n+1} is compact we can assume that the sequence q_m converges to a limit point q . By passing to the limit in (*) we get $q \in V_{j-1}$. Moreover, we can assume (by passing to a subsequence) that the sequence of $(n-j+1)$ -planes $T_{q_m}(h, f, f_2, \dots, f_{j-1})$ has a limit T . Since $p_m \notin \Sigma_{j-1}$, we have

$$T_{q_m}(h, f, f_2, \dots, f_{j-1}) = T_{q_m}(h) \cap T_{q_m}(f) \cap H_2 \cap \dots \cap H_{j-1} .$$

As above, we can assume that the sequence $T_{q_m}(h)$ has a limit T_1 and, using the a_h -condition for the stratification \mathcal{S} we get $T_q S_i \subset T_1$ if $q \in S_i$. Note that $T_{q_m}(f) \rightarrow T_q(f)$ and hence $T = T_1 \cap T_q(f) \cap H_2 \cap \dots \cap H_{j-1}$. It follows that

$$T_q S_{i,j-1} = T_q S_i \cap T_q(f) \cap H_2 \cap \dots \cap H_{j-1} \subset T ,$$

where $S_{i,j-1} = S_i \cap V(f) \cap H_2 \cap \dots \cap H_{j-1}$ is the stratum corresponding to the stratum S_i in the stratification \mathcal{S}'_{j-1} . On the other hand, the condition (**) implies that $T_q S_{i,j-1} \subset T \subset H_j$, a contradiction to the fact that H_j is transversal to \mathcal{S}'_{j-1} .

To prove (iii) just note that the intersection $F \cap N$ is $(n-2)$ -connected (use the exact homotopy sequence of the pair $(F, F \cap N)$ and the fact that F is $(n-1)$ -connected) and has the homotopy type of a CW-complex of dimension $\leq (n-1)$.

Let us now reformulate slightly Theorem 1 as explained already in Section 1. Note that $\chi(D(h) \setminus H) = \chi(\mathbb{P}^n \setminus (V(h) \cup H)) = \chi(\mathbb{C}^n \setminus V(h)_a) = 1 - (1 + (-1)^{n-1} b_{n-1}(V(h)_a)) = (-1)^n b_{n-1}(V(h)_a)$. Here $V(h)_a = V(h) \setminus H$ is the affine part of the projective hypersurface $V(h)$ with respect to the hyperplane at infinity H and, according to L e's affine Lefschetz Theorem, is homotopy equivalent to a bouquet of $(n-1)$ -spheres. It follows that

$$\text{deg}(\text{grad}(h)) = b_{n-1}(V(h)_a) .$$

Even in the case of an arrangement \mathcal{A} , the corresponding equality, $b_n(M(\mathcal{A})) = b_{n-1}(\mathcal{A}_a)$, derived by using Corollary 4, seems to be new. Here we denote by \mathcal{A} not only the projective arrangement but also the union of all the hyperplanes in \mathcal{A} . Theorem 4.109 in [OT1] implies that, in the case of an essential arrangement, the bouquet of spheres \mathcal{A}_a considered above is homotopy equivalent to the Folkman complex $F(\mathcal{A}')$ associated to the corresponding central arrangement \mathcal{A}' in \mathbb{C}^{n+1} ; see [OT1, pp. 137–142] and [Da1, p. 40].

The main interest in Dolgachev’s paper [Do] is focused on homaloidal polynomials, i.e., homogeneous polynomials h such that $\text{deg}(\text{grad}(h)) = 1$. In view of the above reformulation of Theorem 1 it follows that a polynomial h is homaloidal if and only if the affine hypersurface $V(h)_a$ is homotopy equivalent to an $(n - 1)$ -sphere. There are several direct consequences of this fact.

(i) If $V(h)$ is either a smooth quadric (i.e. $d = 2$) or the union of a smooth quadric Q and a tangent hyperplane H_0 to Q (in this case $d = 3$), then $\text{deg}(\text{grad}(h)) = 1$. Indeed, the first case is obvious (either from the topology or the algebra), and the second case follows from the fact that both $H_0 \setminus H$ and $(Q \cap H_0) \setminus H$ are contractible.

(ii) Using the topological description of an irreducible projective curve as the wedge of a smooth curve of genus g and m circles, we see that for an irreducible plane curve C of degree d ,

$$b_1(C_a) = 2g + m + d - 1.$$

Using this and the Mayer-Vietoris exact sequence for homology one can derive Theorem 4 in [Do], which gives the list of all reduced homaloidal polynomials h in the case $n = 2$. This list is reduced to the two examples in (i) above plus the union of three nonconcurrent lines.

(iii) If the hypersurface $V(h)$ has only isolated singular points, say at a_1, \dots, a_m , then our formula above gives

$$\text{deg}(\text{grad}(h)) = (d - 1)^n - \sum_{i=1}^m \mu(V(h), a_i) ,$$

where $\mu(V, a)$ is the Milnor number of the isolated hypersurface singularity (V, a) ; see [D1, p. 161]. We conjecture that when $n > 2$ and $d > 2$ one has $\text{deg}(\text{grad}(h)) > 1$ in this situation, unless $V(h)$ is a cone and then of course $\text{deg}(\text{grad}(h)) = 0$. For more details on this conjecture and its relation to the work by A. duPlessis and C. T. C. Wall in [dPW], see [D3].

4. Complements of hyperplane arrangements

Proof of Lemma 5. We are going to derive this easy result from [H, Th. 5], by using the key homological features of arrangement complements. By Hamm’s theorem, the equality between $(-1)^n \chi(D(Q) \setminus H) = (-1)^n \chi(D(Q), D(Q) \cap H)$ and $b_n(D(Q))$ is equivalent to

$$(*) \quad b_{n-1}(D(Q) \cap H) = b_{n-1}(D(Q)).$$

All we can say in general is that $b_{n-1}(D(Q) \cap H) \geq b_{n-1}(D(Q))$. In the arrangement case, the other inequality follows from two standard facts (see

[OT1, Cor. 5.88 and Th. 5.89]): $H^{n-1}(D(Q) \cap H)$ is generated by products of cohomology classes of degrees $< n-1$ (if $n > 2$); $H^1(D(Q)) \rightarrow H^1(D(Q) \cap H)$ is surjective (which in particular settles the case $n = 2$). See also Proposition 2.1 in [Da3].

Proof of Corollary 4. (1) This claim follows directly from Theorem 1 and Lemma 5.

(2)(a) To complete this proof we only have to explain why the claims (ii) and (iii) are equivalent. This in turn is an immediate consequence of the well-known equality: $\deg P_{\mathcal{A}}(t) = \text{codim}(\cap_{i=1}^d H_i) - 1$, where $P_{\mathcal{A}}(t)$ is the Poincaré polynomial of $D(Q)$; see [OT1, Cor. 3.58, Ths. 3.68 and 2.47].

(2)(b) The inequality can be proved by induction on d by the method of deletion and restriction; see [OT1, p. 17].

Proof of Corollary 6. Using the affine Lefschetz theorem of Hamm (see Theorem 5 in [H]), we know that for a generic projective hyperplane H , the space M has the homotopy type of a space obtained from $M \cap H$ by attaching n -cells. The number of these cells is given by

$$(-1)^n \chi(M, M \cap H) = (-1)^n \chi(M \setminus H) = b_n(M);$$

see Corollary 4 above.

To finish the proof of the minimality of M we proceed by induction. Start with a minimal cell structure, K , for $M \cap H$, to get a cell structure, L , for M , by attaching $b_n(M)$ top cells to K . By minimality, we know that K has trivial cellular incidences. The fact that the number of top cells of L equals $b_n(L)$ means that these cells are attached with trivial incidences, too, whence the minimality of M .

Finally, M' has the homotopy type of $M \times S^1$, being therefore minimal, too.

Remark 12. (i) Let μ_e be the cyclic group of the e -roots of unity. Then there are natural algebraic actions of μ_e on the spaces $F \setminus N$ and $F \cap N$ occurring in Theorem 3. The corresponding weight equivariant Euler polynomials (see [DL] for a definition) give information on the relation between the induced μ_e -actions on the cohomology $H^*(F \setminus N, \mathbb{Q})$ and $H^{n-1}(F \cap N, \mathbb{Q})$ and the functorial Deligne mixed Hodge structure present on cohomology.

When N is a hyperplane arrangement \mathcal{A}' and f is an \mathcal{A}' -generic function, these weight equivariant Euler polynomials can be combinatorially computed from the lattice associated to the arrangement (see Corollary (2.3) and Remark (2.7) in [DL]) by the fact that the weight equivariant Euler polynomial of the μ_e -variety F is known; see [MO] and [St]. This gives in particular the characteristic polynomial of the monodromy associated to the function $f : N \rightarrow \mathbb{C}$.

(ii) The minimality property turns out to be useful in the context of *homology with twisted coefficients*. Here is a simple example. (More results along this line will be published elsewhere.) Let X be a connected CW-complex of finite type. Set $\pi := \pi_1(X)$. For a left $\mathbb{Z}\pi$ -module N , denote by $H_*(X, N)$ the homology of X with local coefficients corresponding to N . One knows that $H_*(X, N)$ may be computed as the homology of the chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} N$, where $C_*(\tilde{X})$ denotes the π -equivariant chain complex of the universal cover of X ; see [W, Ch. VI].

Assume now that X is minimal. If N is a finite-dimensional \mathbb{K} -representation of π over a field \mathbb{K} , we obtain, from the above description of twisted homology, that

$$\dim_{\mathbb{K}} H_q(X, N) \leq (\dim_{\mathbb{K}} N) \cdot b_q(X) , \text{ for all } q .$$

When X is, up to homotopy, an arrangement complement, and $\mathbb{K} = \mathbb{C}$, we thus recover the main result of [C]. (Twisted cohomology may be treated similarly.)

Example 13. In this example we explain why special care is needed when doing Morse theory on noncompact manifolds as in [OT2] and [R2]. Let's start with a very simple case, where computations are easy. Consider the Milnor fiber, $X \subset \mathbb{C}^2$, given by $\{xy = 1\}$. The hyperplane $\{x + y = 0\}$ is generic with respect to the arrangement $\{xy = 0\}$, in the sense of [R2, Prop. 2]. Set $\sigma := |x + y|^2$. When trying to do proper Morse theory with boundary, as in [R2, Th. 3], one faces a delicate problem on the boundary. Denoting by B_R (S_R) the closed ball (sphere) of radius R in \mathbb{C}^2 , we see easily that $X \cap B_R = \emptyset$, if $R^2 < 2$, and that the intersection $X \cap S_R$ is not transverse, if $R^2 = 2$. In the remaining case ($R^2 > 2$), it is equally easy to check that the restriction of σ to the boundary $X \cap S_R$ always has eight critical points (with $\sigma \neq 0$). In our very simple example, all these critical points are 'à gradient sortant' (in the terminology of [HL, Def. 3.1.2]). The proof of this fact does not seem obvious, in general. At the same time, this property seems to be needed, in order to get the conclusion of [R2, Th. 3] (see [HL, Th. 3.1.7]).

One can avoid this problem as follows. (See also Theorem 3 in [R3].) Let X denote the affine Milnor fiber and $\ell : X \rightarrow \mathbb{C}$ the linear function induced by the equation of any hyperplane. Then for a real number $r > 0$, let D_r be the open disc $|z| < r$ in \mathbb{C} . The function ℓ having a finite number of critical points, it follows that X has the same homotopy type (even diffeo type) as the cylinder $X_r = \ell^{-1}(D_r)$ for $r \gg 0$. Fix such an r . For $R \gg r$, we have that $Y = X_r \cap B_R$ has the same homotopy type as X_r .

Moreover, if the hyperplane $\ell = 0$ is generic in the sense of Lê/Némethi, it follows that the real function $|\ell|$ has no critical points on the boundary of Y except those corresponding to the minimal value 0 which do not matter. In the example treated before, one can check that the critical values

corresponding to the eight critical points tend to infinity when $R \rightarrow \infty$; i.e., the eight singularities are no longer in Y for a good choice of r and R !

Note that Y is a noncompact manifold with a noncompact boundary, but the sets $Y^{r_0} = \{y \in Y \mid |\ell(y)| \leq r_0\}$ are compact for all $0 \leq r_0 < r$. If R is chosen large enough, then Y^0 has the same homotopy type as $X \cap \{\ell = 0\}$, and hence we can use proper Morse theory on manifolds with boundary to get the result.

The idea behind Proposition 11 is the same: one can do Morse theory on a noncompact manifold with corners if there are no critical points on the boundary, and this is achieved by an argument similar to the above and based on the conditions (cj); see [H] for more details.

5. k -generic sections of aspherical arrangements

The preceding minimality result (Corollary 6) enables us to use the general method of [PS] to get explicit information on higher homotopy groups of arrangement complements, in certain situations. We begin by describing a framework that encompasses all such known computations. (For the basic facts in arrangement theory, we use reference [OT1].)

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a projective hyperplane arrangement in $\mathbb{P}(V)$, with associated central arrangement, $\mathcal{A}' = \{H'_1, \dots, H'_n\}$, in V . Let $M(\mathcal{A}) \subset \mathbb{P}(V)$ and $M'(\mathcal{A}) \subset V$ be the corresponding arrangement complements. Denote by $\mathcal{L}(\mathcal{A})$ the intersection lattice, that is the set of intersections of hyperplanes from \mathcal{A} , X (called *flats*), ordered by reverse inclusion. We will assume that \mathcal{A} is essential. (We may do this, without changing the homotopy types of the complements and the intersection lattice, in a standard way; see [OT1, p. 197].) Then there is a canonical Whitney stratification of $\mathbb{P}(V)$, $\mathcal{S}_{\mathcal{A}}$, whose strata are indexed by the nonzero flats, having $M(\mathcal{A})$ as top stratum; see [GM, III 3.1 and III 4.5]. Thus, the genericity condition from Morse theory takes a particularly simple form, in the arrangement case.

To be more precise, we will need the following definition. Let $U \subset V$ be a complex vector subspace. We say that U is $\mathcal{L}_k(\mathcal{A})$ -generic ($0 \leq k < \text{rank}(\mathcal{A})$) if

- (1) $\text{codim}_V(X) = \text{codim}_U(X \cap U)$, for all $X \in \mathcal{L}(\mathcal{A})$, $\text{codim}_V(X) \leq k + 1$.

It is not difficult to see that (1) forces $k \leq \dim U - 1$ and that $\mathbb{P}(U)$ is transverse to $\mathcal{S}_{\mathcal{A}}$ if and only if U is $\mathcal{L}_k(\mathcal{A})$ -generic, with $k = \dim U - 1$. Consider also the restriction, $\mathcal{A}^U := \{\mathbb{P}(U) \cap H_1, \dots, \mathbb{P}(U) \cap H_n\}$, with complement $M(\mathcal{A}) \cap \mathbb{P}(U)$. A direct application of [GM, Th. II 5.2] gives then the following:

- (2) If $k = \dim U - 1$ and U is $\mathcal{L}_k(\mathcal{A})$ -generic, then the pair $(M(\mathcal{A}), M(\mathcal{A}) \cap \mathbb{P}(U))$ is k -connected.

The next proposition, which will provide our framework for homotopy computations by minimality, upgrades the above implication (2) to the level of cells, for the case of an arbitrary $\mathcal{L}_k(\mathcal{A})$ -generic section, U .

PROPOSITION 14. *Let \mathcal{A} be an arrangement in $\mathbb{P}(V)$, and let $U \subset V$ be a subspace. Assume that both \mathcal{A} and the restriction \mathcal{A}^U are essential. If U is $\mathcal{L}_k(\mathcal{A})$ -generic ($0 \leq k < \text{rank}(\mathcal{A})$), then the inclusion, $M(\mathcal{A}) \cap \mathbb{P}(U) \subset M(\mathcal{A})$, has the homotopy type of a cellular map, $j : X \rightarrow Y$, where:*

- (i) *Both X and Y are minimal CW-complexes.*
- (ii) *At the level of k -skeletons, $X^{(k)} = Y^{(k)}$, and the restriction of j to $X^{(k)}$ is the identity.*

Proof. If $k < \dim U - 1$, we may find a hyperplane $H \subset V$ which is $\mathcal{L}_m(\mathcal{A})$ -generic, $m = \dim H - 1$, and whose trace on U , $H \cap U$, is $\mathcal{L}_{m'}(\mathcal{A}^U)$ -generic, $m' = \dim(H \cap U) - 1$. It follows that \mathcal{A}^H and $H \cap U$ satisfy the assumptions of the proposition, with the same k . The minimal CW-structures for $M(\mathcal{A}) \cap \mathbb{P}(H \cap U)$ and $M(\mathcal{A}) \cap \mathbb{P}(H)$ may be extended to minimal structures for $M(\mathcal{A}) \cap \mathbb{P}(U)$ and $M(\mathcal{A})$, exactly as in the proof of Corollary 6. The second claim of the proposition follows again by induction, together with routine homotopy-theoretic arguments, involving cofibrations and cellular approximations.

If $k = \dim U - 1$, it is not difficult to see that \mathcal{A}^U may be obtained from \mathcal{A} by taking $\dim V - \dim U$ successive generic hyperplane sections. Therefore, the method of proof of Corollary 6 shows that, up to homotopy, $M(\mathcal{A}) \cap \mathbb{P}(U)$ is the k -skeleton of a minimal CW-structure for $M(\mathcal{A})$.

Let \mathcal{A} be an arrangement in $\mathbb{P}(V)$, and let $U \subset V$ be a proper subspace which is $\mathcal{L}_0(\mathcal{A})$ -generic. Assume that both \mathcal{A} and \mathcal{A}^U are essential. The preceding proposition leads to the following combinatorial definition:

$$(3) \quad k(\mathcal{A}, U) := \sup \{0 \leq \ell < \text{rank}(\mathcal{A}) \mid U \text{ is } \mathcal{L}_\ell(\mathcal{A})\text{-generic}\},$$

and to the next topological counterpart:

$$(4) \quad p(\mathcal{A}, U) := \sup\{q \geq 0 \mid b_r(M(\mathcal{A})) = b_r(M(\mathcal{A}^U))\}, \text{ for all } r \leq q\}.$$

PROPOSITION 15. $k(\mathcal{A}, U) = p(\mathcal{A}, U)$.

Proof. The inequality $k(\mathcal{A}, U) \leq p(\mathcal{A}, U)$ follows from Proposition 14. Denoting by r and r' the ranks of \mathcal{A} and \mathcal{A}^U respectively, we know that $b_s(M(\mathcal{A}^U)) = 0$, for $s \geq r'$, and $b_s(M(\mathcal{A})) \neq 0$, for $s < r$. It follows that $p(\mathcal{A}, U) < r'$, since $r' < r$. To show that $p(\mathcal{A}, U) \leq k(\mathcal{A}, U)$, we have to pick an arbitrary independent subarrangement, $\mathcal{B} \subset \mathcal{A}$, of $q + 1$ hyperplanes, $1 \leq q \leq p(\mathcal{A}, U)$, and verify that the restriction \mathcal{B}^U is independent.

To this end, we will use three well-known facts (see [OT1]). Firstly, the natural map, $H^*M(\mathcal{A}) \rightarrow H^*M(\mathcal{A}^U)$, is onto. Secondly, the natural map, $H^*M(\mathcal{B}) \rightarrow H^*M(\mathcal{A})$, is monic. Together with definition (4) above, these two facts imply that the natural map, $H^*M(\mathcal{B}) \rightarrow H^*M(\mathcal{B}^U)$, is an isomorphism, up to degree q . The combinatorial description of the cohomology algebras of arrangement complements by Orlik-Solomon algebras may now be used to deduce the independence of \mathcal{B}^U from \mathcal{B} .

Description of the k -generic framework

We want to apply Theorem 2.10 from [PS] to an (essential) arrangement \mathcal{A} in $\mathbb{P}(U)$, with complement $M := M(\mathcal{A})$. Set $\pi = \pi_1(M)$. The method of [PS] requires both M and $K(\pi, 1)$ to be minimal spaces, with cohomology algebras generated in degree 1. By Corollary 6 (and standard facts in arrangement cohomology) all these assumptions will be satisfied, as soon as $K(\pi, 1)$ is (up to homotopy) an arrangement complement, too.

This in turn happens whenever \mathcal{A} is a k -generic section, $k \geq 2$, of an (essential) aspherical arrangement, $\widehat{\mathcal{A}}$. That is, if there exists $\widehat{\mathcal{A}}$ in $\mathbb{P}(V)$, $U \subset V$, with $M(\widehat{\mathcal{A}})$ aspherical, such that $\mathcal{A} = \widehat{\mathcal{A}}^U$, and with the property that U is $\mathcal{L}_k(\widehat{\mathcal{A}})$ -generic, as in (1) above.

Indeed, Proposition 14 guarantees that in this case we may replace, up to homotopy, the inclusion $M(\mathcal{A}) \hookrightarrow M(\widehat{\mathcal{A}})$ by a cellular map between minimal CW-complexes, $j : X \rightarrow Y$, which restricts to the identity on k -skeletons. In particular, Y is a $K(\pi, 1)$ and j is a classifying map.

Let us recall now from [PS, Def. 2.7] the *order of π_1 -connectivity*,

$$(5) \quad p(M) := \sup \{q \mid b_r(M) = b_r(Y), \text{ for all } r \leq q\} .$$

Set $p = p(M)$. It follows from Proposition 15 that $p = \infty$ if and only if $U = V$. Moreover, Propositions 14 and 15 imply that $k \leq p$ and that we may actually construct a classifying map j with the property that $j|_{X^{(p)}} = \text{id}$ (with the convention $X^{(\infty)} = X$).

We will also need the π -equivariant chain complexes of the universal covers, \widetilde{X} and \widetilde{Y} , associated to the above Morse-theoretic *minimal* cell structures:

$$(6) \quad C_*(\widetilde{X}) := \{d_q : H_q X \otimes \mathbb{Z}\pi \longrightarrow H_{q-1} X \otimes \mathbb{Z}\pi\}_q ,$$

and

$$(7) \quad C_*(\widetilde{Y}) := \{\partial_q : H_q Y \otimes \mathbb{Z}\pi \longrightarrow H_{q-1} Y \otimes \mathbb{Z}\pi\}_q .$$

They have the property that $C_{\leq p}(\widetilde{X}) = C_{\leq p}(\widetilde{Y})$. (Here we adopt the convention of turning left $\mathbb{Z}\pi$ -modules into right $\mathbb{Z}\pi$ -modules, replacing the action of $x \in \pi$ by that of x^{-1} .)

Denote by $\tilde{j} : C_*(\tilde{X}) \rightarrow C_*(\tilde{Y})$ the π -equivariant chain map induced by the lift of j to universal covers, $\tilde{j} := \{\tilde{j}_q : H_q X \otimes \mathbb{Z}\pi \rightarrow H_q Y \otimes \mathbb{Z}\pi\}_q$, and by $j_* : H_* X \rightarrow H_* Y$ the map induced by j on homology,

$$j_* := \{j_{*q} : H_q X \rightarrow H_q Y\}_q.$$

Define $\mathbb{Z}\pi$ -linear maps, $\{D_q\}_q$, by:

$$(8) \quad D_q := \partial_{q+2} + \tilde{j}_{q+1} : (H_{q+2} Y \otimes \mathbb{Z}\pi) \oplus (H_{q+1} X \otimes \mathbb{Z}\pi) \longrightarrow H_{q+1} Y \otimes \mathbb{Z}\pi .$$

THEOREM 16. *Let \mathcal{A} be an essential k -generic section of an essential aspherical arrangement $\hat{\mathcal{A}}$, with $k \geq 2$. Set $M = M(\mathcal{A})$, $\pi = \pi_1(M)$ and $p = p(M)$, and denote by \tilde{M} the universal cover of M . Then:*

- (i) M is aspherical if and only if $p = \infty$.
- (ii) If $p < \infty$, then the first higher nonzero homotopy group of M is $\pi_p(M)$, with the following finite presentation as a $\mathbb{Z}\pi$ -module:

$$(9) \quad \pi_p(M) = \text{coker} \{D_p : (H_{p+2} M(\hat{\mathcal{A}}) \oplus H_{p+1} M(\mathcal{A})) \otimes \mathbb{Z}\pi \longrightarrow H_{p+1} M(\hat{\mathcal{A}}) \otimes \mathbb{Z}\pi\} .$$

- (iii) If $2k \geq \text{rank}(\mathcal{A})$, then \tilde{M} has the rational homotopy type of a wedge of spheres.

Proof. Proposition 15 may be used to infer that both properties from Part (i) of the theorem are equivalent to $U = V$.

The equality $j|_{X^{(p)}} = \text{id}$ implies that \tilde{M} is $(p - 1)$ -connected; see [PS, Th. 2.10(1)] for details. To deduce the presentation (9) of $\pi_p(M)$, we may use the proof of Theorem 2.10(2) from [PS]. There is an oversight in [PS], namely the fact that, in general, \tilde{j}_{p+1} is different from $j_{*(p+1)} \otimes \text{id}$. (Obviously, $\tilde{j}_q = j_{*q} \otimes \text{id}$, whenever $X^{(q)}$ is a subcomplex of $Y^{(q)}$, and the restriction of j to $X^{(q)}$ is the inclusion, $X^{(q)} \subset Y^{(q)}$.) This is the reason why Δ_p from [PS, (2.3)] has to be replaced by D_p from (9) above. Nevertheless, this change does not affect the results from [PS, Cor. 2.11], since ∂_{p+2} is ε -minimal. In particular, $\pi_p(M) \neq 0$, and the proof of Part (ii) of our Theorem 16 is completed.

For the last part, let us begin by noting that \tilde{M} is a $(k - 1)$ -connected complex, of dimension $r - 1$, $r := \text{rank}(\mathcal{A})$. (The connectivity claim follows from Parts (i)–(ii), due to the already remarked fact that $k \leq p$.) Now the assumption $2k \geq r$ implies firstly that the cohomology algebra $H^*(\tilde{M}; \mathbb{Q})$ has trivial product structure, and secondly that the rational homotopy type of \tilde{M} is determined by its rational cohomology algebra (see [HS, Cor. 5.16]). It is also well-known that the last property holds for wedges of spheres; see for example [HS, Lemma 1.6], which finishes our proof.

Remark 17. Denote, as usual, by $M' := M'(\mathcal{A})$, the cone of $M(\mathcal{A})$. Set $\pi' = \pi_1(M')$. The triviality of the Hopf fibration readily implies that M' is aspherical if and only if M is aspherical, and that $p(M') = p(M)$. Therefore, Theorem 16 (i) also holds for M' . If $p < \infty$, the Hopf fibration may be invoked once more to see that $\pi_p(M') = \pi_p(M)$ is the first higher nonzero homotopy group of M' , with $\mathbb{Z}\pi'$ -module structure induced by restriction from $\mathbb{Z}\pi$.

The examples. The first explicit computation of nontrivial higher homotopy groups was made by Hattori [Hat], for *general position* arrangements \mathcal{A} . Firstly, he showed that in this case $\pi_1(M(\mathcal{A})) = \mathbb{Z}^n$, where $n = |\mathcal{A}| - 1$. Denote by x_i the 1-cell of the i -th S^1 -factor of the torus $(S^1)^n$. One knows that the (acyclic) chain complex (7) coming from the canonical (minimal) cell structure of the n -torus is of the form

$$(10) \quad \{\partial_q : \wedge^q(x_1, \dots, x_n) \otimes \mathbb{Z}\mathbb{Z}^n \longrightarrow \wedge^{q-1}(x_1, \dots, x_n) \otimes \mathbb{Z}\mathbb{Z}^n\}_q,$$

$$\partial_q(x_{i_1} \cdots x_{i_q}) = \sum_{r=1}^q (-1)^{r-1} x_{i_1} \cdots \widehat{x_{i_r}} \cdots x_{i_q} \otimes (x_{i_r}^{-1} - 1).$$

Hattori found out that a certain truncation of the above complex provides a $\mathbb{Z}\mathbb{Z}^n$ -resolution for the first higher nontrivial homotopy group of $M(\mathcal{A})$.

The second class of examples, studied by Randell in [R1], consists of *iterated generic hyperplane sections*, $\mathcal{A} = \widehat{\mathcal{A}}^U$, of essential *aspherical* arrangements $\widehat{\mathcal{A}}$. It is immediate to see that these are particular cases of (essential) k -generic sections, with $k = \dim U - 1$. Randell gave a formula for the π_1 -coinvariants of the first higher nontrivial homotopy group of $M'(\mathcal{A})$ (without determining the full $\mathbb{Z}\pi_1$ -module structure).

The last known examples are the *hypersolvable* arrangements, introduced in [JP1]. This class contains Hattori's general position class, and also the (aspherical) *fiber-type* arrangements of Falk and Randell [FR]; see [JP1]. It follows from [JP1, Def. 1.8 and Prop. 1.10] and [JP2, Cor. 3.1] that the (essential) hypersolvable arrangements coincide with 2-generic sections of (essential) fiber-type arrangements. Within the hypersolvable class, a detailed analysis of the structure of π_2 as a $\mathbb{Z}\pi_1$ -module was carried out in [PS, §5]. See also [PS, Th. 4.12(3)] for a combinatorial formula describing the π_1 -coinvariants of the first higher nontrivial homotopy group of the complement of hypersolvable arrangements.

Beyond the first higher nontrivial homotopy group, it seems appropriate to point out here that Theorem 16 (iii) indicates the presence of a very rich higher homotopy structure. Indeed, there are many rank 3 hypersolvable arrangements \mathcal{A} for which $\pi_2 M(\mathcal{A})$ is an infinitely generated free abelian group; see [PS, Rem. 5.5 and Th. 5.7]. It follows that $\widetilde{M}(\mathcal{A})$ is rationally an infinite

wedge of 2-spheres. Following [Q2], the rational homotopy Lie algebra (under Whitehead product) $\pi_{>1}M(\mathcal{A}) \otimes \mathbb{Q}$ is a free graded Lie algebra on infinitely many generators [Hil].

6. Some higher homotopy groups of arrangements

In this section, we are going to apply Theorem 16 to iterated generic hyperplane sections of aspherical arrangements, in Theorem 18 below. When specialized to the hypersolvable class, this result will enable us to prove that the associated graded module of the first higher nonvanishing homotopy group of the complement is combinatorially determined.

THEOREM 18. *Let $\widehat{\mathcal{A}}$ be an essential aspherical arrangement in \mathbb{P}^{m-1} , and let $\mathcal{A} = \widehat{\mathcal{A}}^U$ be an iterated generic hyperplane section. (Here, a hyperplane H is generic if it is \mathcal{L}_k -generic, with $k = \dim H - 1$; see §5(1).) Assume $\dim U \geq 3$. Set $M = M(\mathcal{A})$, $M' = M'(\mathcal{A})$, $\pi = \pi_1(M)$, $\pi' = \pi_1(M')$, and $p = p(M) = p(M')$. Then,*

- (i) $p = \dim U - 1$.
- (ii) *The first higher nontrivial homotopy group of M is $\pi_p(M)$, with finite, free $\mathbb{Z}\pi$ -resolution*

$$(11) \quad 0 \rightarrow H_{m-1}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \rightarrow \dots \\ \dots \rightarrow H_{p+2}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \rightarrow H_{p+1}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \rightarrow \pi_p(M) \rightarrow 0 ,$$

where $\{\partial_q : H_qM(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \rightarrow H_{q-1}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi\}_q$ is the acyclic π -equivariant chain complex of the universal cover, associated to a Morse-theoretic minimal cell decomposition of $M(\widehat{\mathcal{A}})$, as in Section 5(7). In addition, the resolution (11) is minimal; more precisely, $\partial_q \otimes_{\mathbb{Z}\pi} \mathbb{Z} = 0$, for all q , where $\varepsilon : \mathbb{Z}\pi \rightarrow \mathbb{Z}$ is the augmentation of the group ring.

- (iii) $\pi_p(M)$ is a projective $\mathbb{Z}\pi$ -module \iff it is $\mathbb{Z}\pi$ -free $\iff \dim U = m - 1$.
- (iv) *The first higher nontrivial homotopy group of M' is $\pi_p(M') = \pi_p(M)$, with $\mathbb{Z}\pi'$ -module structure induced from $\mathbb{Z}\pi$, by restriction of scalars, $\mathbb{Z}\pi' \rightarrow \mathbb{Z}\pi$, via the projection map of the Hopf fibration.*
- (v) *The universal cover of M has the homotopy type of a wedge of p -spheres. The rational homotopy Lie algebra, $L_* = \bigoplus_{q \geq 1} L_q$, where*

$$L_q := \pi_{q+1}(M) \otimes \mathbb{Q} = \pi_{q+1}(M') \otimes \mathbb{Q}$$

and the Lie bracket is induced by the Whitehead product, is isomorphic to the free graded Lie algebra generated by $L_{p-1} = \pi_p(M) \otimes \mathbb{Q}$. The $\mathbb{Q}\pi$ -module structure of L_* is given by the graded Lie algebra extension of the π -action on the Lie generators, described by (11) $\otimes \mathbb{Q}$; the $\mathbb{Q}\pi'$ -module structure is obtained by restriction of scalars.

Proof. We have already remarked, at the end of the preceding section, that \mathcal{A} is in particular an (essential) k -generic section of $\widehat{\mathcal{A}}$, $k = \dim U - 1 \geq 2$, as in Theorem 16. We also know from [PS, §4.14] that $p = \dim U - 1$. Since $H_{p+1}M(\mathcal{A}) = 0$, $\widetilde{j}_{p+1} = 0$. Therefore, the presentation (9) may be continued to the free resolution (11); see (8). The algebraic minimality claim from Theorem 18(ii) follows at once from the topological minimality property of the cell structure of $M(\widehat{\mathcal{A}})$. Part (iii) is then an immediate consequence of the minimality of (11), by a standard argument in homological algebra (exactly as in [PS, Th. 5.7(1)]). Part (iv) follows from the triviality of the Hopf fibration.

(v) The claim on the homotopy type of the universal cover, \widetilde{M} , may be obtained from the remark that $\overline{H}_*(\widetilde{M}; \mathbb{Z})$ is a free abelian group, concentrated in degree $* = p$. This in turn is a consequence of the following facts: \widetilde{M} is $(p - 1)$ -connected; M has the homotopy type of a p -dimensional complex. The structure of L_* is then provided by Hilton's theorem [Hil]. The $\pi_1(M)$ -action on L_* is easily seen to be as stated, since one knows that it respects the Whitehead product [S, p. 419].

Remark 19. Hattori's general position framework corresponds to the case when $\widehat{\mathcal{A}}$ is boolean (that is, when $\widehat{\mathcal{A}}'$ is given by the m coordinate hyperplanes in \mathbb{C}^m). See [Hat, §2]. In this case, one recovers, from (i)–(iv) above, Hattori's Theorem 3 [Hat].

Randell's formula for the π' -coinvariants of $\pi_p(M')$ ([R1, Th.2 and Prop.9]) readily follows from (i), (ii) and (iv). He also raised the question of $\mathbb{Z}\pi'$ -freeness for $\pi_p(M')$, at the end of Section 2 [R1]. With the obvious remark that no nonzero $\mathbb{Z}\pi'$ -module, which is induced by restriction from $\mathbb{Z}\pi$, can be free (since it has nontrivial annihilator), (iii) and (iv) above completely clarify this question.

The associated graded chain complex of the universal cover

Let X be a connected CW-complex of finite type, with fundamental group $\pi := \pi_1(X)$, and universal cover \widetilde{X} . The explicit determination of the boundary maps, d_q , of the equivariant chain complex of \widetilde{X} , is a difficult task, in general. Suppose now that X is minimal, with cohomology ring generated in degree one. Under these assumptions, we are going to show that, at the associated graded level, $\text{gr}_I^*(d_q)$ may be described solely in terms of the cohomology ring, H^*X . When X is (up to homotopy) an arrangement complement (either $M(\mathcal{A})$ or $M'(\mathcal{A})$), one knows [OS] that the cohomology ring, H^*X , is *combinatorially determined*, from the intersection lattice $\mathcal{L}(\mathcal{A})$. This fact will lead to purely combinatorial computations of associated graded objects.

Denote by $I \subset \mathbb{Z}\pi$ the augmentation ideal of the group ring, and by $\text{gr}_I^*\mathbb{Z}\pi := \bigoplus_{k \geq 0} (I^k / I^{k+1})$ the associated graded ring, with respect to the I -adic

filtration of $\mathbb{Z}\pi$. Similarly, we may construct the associated graded module of any (right) $\mathbb{Z}\pi$ -module, P , by $\text{gr}_I^*P := \bigoplus_{k \geq 0} (P \cdot I^k / P \cdot I^{k+1})$; it has an induced (right) $\text{gr}_I^*\mathbb{Z}\pi$ -module structure.

The minimality of X implies that $d_q(H_qX \otimes \mathbb{Z}\pi) \subset H_{q-1}X \otimes I$, for all q . Therefore, by passing to the associated graded in (6), we get a chain complex of free graded $\text{gr}_I^*\mathbb{Z}\pi$ -modules and degree zero $\text{gr}_I^*\mathbb{Z}\pi$ -linear maps,

$$(12) \quad \text{gr}_I^* C_*(\tilde{X}) := \{ \text{gr}_I^*(d_q) : H_qX \otimes \text{gr}_I^*\mathbb{Z}\pi \longrightarrow H_{q-1}X \otimes \text{gr}_I^*\mathbb{Z}\pi \}_q ,$$

where we consider on $H_qX \otimes \text{gr}_I^*\mathbb{Z}\pi$ the tensor product grading, with H_qX concentrated in degree q .

Consider now the cup-product map,

$$(13) \quad H^{q-1}X \otimes H^1X \longrightarrow H^qX .$$

Dualizing, we get a map, $\delta_q^R : H_qX \rightarrow H_{q-1}X \otimes H_1X$. Use the standard identifications, $H_1X \equiv \pi_{ab} \equiv I/I^2 = \text{gr}_I^*\mathbb{Z}\pi$ (see [HiS, Lemma VI.4.1]), and $H_qX \equiv H_qX \otimes 1$, to extend it to a degree zero $\text{gr}_I^*\mathbb{Z}\pi$ -linear map,

$$(13') \quad \delta_q^R : H_qX \otimes \text{gr}_I^*\mathbb{Z}\pi \longrightarrow H_{q-1}X \otimes \text{gr}_I^*\mathbb{Z}\pi .$$

THEOREM 20. *If X is minimal and H^*X is generated by H^1X , then*

$$\text{gr}_I^*(d_q) = (-1)^q \delta_q^R , \text{ for all } q .$$

Proof. Set $n := \text{rank}(\pi_{ab})$. Denote by Y the n -torus, endowed with the canonical minimal CW-structure, giving rise to the π_{ab} -equivariant boundary maps, ∂_q , described in (10). Pick a cellular map, $j : X \rightarrow Y$, that induces the abelianization morphism on fundamental groups, $j_\# : \pi \rightarrow \pi_{ab}$. Lift j to a (cellular, $j_\#$ -equivariant) map, $\tilde{j} : \tilde{X} \rightarrow \tilde{Y}$, inducing a $\mathbb{Z}j_\#$ -linear chain map, $\tilde{j} : C_*(\tilde{X}) \rightarrow C_*(\tilde{Y})$.

Passing to the associated graded, we obtain a morphism of graded rings, $\text{gr}_I^*(j_\#) : \text{gr}_I^*\mathbb{Z}\pi \rightarrow \text{gr}_I^*\mathbb{Z}\pi_{ab}$, and a degree zero $\text{gr}_I^*(j_\#)$ -linear chain map,

$$\{ \text{gr}_I^*(\tilde{j}_q) : H_qX \otimes \text{gr}_I^*\mathbb{Z}\pi \rightarrow H_qY \otimes \text{gr}_I^*\mathbb{Z}\pi_{ab} \}_q .$$

Since $\text{gr}_I^0(\tilde{j}_q) \equiv j_{*q}$, by minimality, we infer that:

$$(14) \quad \text{gr}_I^*(\tilde{j}_q) = j_{*q} \otimes \text{gr}_I^*(j_\#) , \text{ for all } q .$$

To identify the action of $\text{gr}_I^*(d_q)$ on the free $\text{gr}_I^*\mathbb{Z}\pi$ -module generators, $\text{gr}_I^*(d_q) : H_qX \rightarrow H_{q-1}X \otimes \pi_{ab}$, consider $\text{gr}_I^*(\partial_q) : H_qY \rightarrow H_{q-1}Y \otimes \pi_{ab}$. Recall that $\text{gr}_I^*(\tilde{j})$ is a chain map, and use (14) to infer that $(j_{*(q-1)} \otimes j_{*1}) \circ \text{gr}_I^*(d_q) = \text{gr}_I^*(\partial_q) \circ j_{*q}$.

At the same time, it is straightforward to use (10) and check that $(-1)^q \text{gr}_I^*(\partial_q)|_{H_qY}$ equals the dual of the cup-product map (13), corresponding

to Y . By naturality, it follows that $(j_{*(q-1)} \otimes j_{*1}) \circ ((-1)^q \delta_q^R)$ equals $\text{gr}_I^*(\partial_q) \circ j_{*q}$, as well.

We also know that H^*X is generated by H^1X . Since $j_{*1} = \text{id}$, this implies that j induces a split injection on homology, and consequently $\text{gr}_I^*(d_q) = (-1)^q \delta_q^R$, as asserted.

Some combinatorially determined situations

In arrangement theory, we may replace $\text{gr}_I^* \mathbb{Q} \pi$ in Theorem 20 by a combinatorially defined object, in two steps. The first one uses a general result of Quillen [Q1]. Let π be an arbitrary group, with descending central series $\{\Gamma_k\}_{k \geq 1}$, and associated graded Lie algebra $\text{gr}_\Gamma^* \pi := \bigoplus_{k \geq 1} (\Gamma_k / \Gamma_{k+1})$. One has a map, $\chi_k : \Gamma_k \rightarrow I^k$, defined by $\chi_k(x) = x - 1$, for each $k \geq 1$. The maps $\{\chi_k\}$ induce a graded algebra surjection,

$$(15) \quad \chi : U \text{gr}_\Gamma^* \pi \longrightarrow \text{gr}_I^* \mathbb{Z} \pi ,$$

which restricts, in degree one, to the standard identification between $\text{gr}_\Gamma^1 \pi = \pi_{ab}$ and $\text{gr}_I^1 \mathbb{Z} \pi$. Moreover, $\chi \otimes \mathbb{Q}$ is an isomorphism.

The second step involves the *holonomy Lie algebra* of X , $\mathcal{H}_*(X)$. It is defined as the quotient of the free Lie algebra on H_1X , $\mathbb{L}^*(H_1X)$, graded by bracket length, by the Lie ideal generated by the image of the reduced diagonal,

$$(16) \quad \nabla : H_2X \longrightarrow H_1X \wedge H_1X \equiv \mathbb{L}^2(H_1X) .$$

If $X \simeq M(\mathcal{A})$, one knows [K] that the graded Lie algebras $(\text{gr}_\Gamma^* \pi) \otimes \mathbb{Q}$ and $\mathcal{H}_*(X) \otimes \mathbb{Q}$ are isomorphic, by an isomorphism which is the identity on degree one pieces.

COROLLARY 21. *Let \mathcal{A} be an arbitrary projective hyperplane arrangement, with complement $M(\mathcal{A})$ and fundamental group $\pi := \pi_1(M(\mathcal{A}))$. Let X be an arbitrary minimal complex having the homotopy type of $M(\mathcal{A})$.*

- (i) *The graded algebras $\text{gr}_I^* \mathbb{Q} \pi$ and $U \mathcal{H}_*(M(\mathcal{A})) \otimes \mathbb{Q}$ are isomorphic, by an isomorphism inducing the identity in degree one. Consequently, the chain complex of graded $\text{gr}_I^* \mathbb{Q} \pi$ -modules $\text{gr}_I^* C_*(\tilde{X}) \otimes \mathbb{Q}$ is combinatorially determined.*
- (ii) *If the associated central arrangement, \mathcal{A}' , is hypersolvable, then all the results from Part (i) hold with \mathbb{Z} -coefficients.*

Proof. Part (i) is a direct consequence of Theorem 20, via the results from [Q1] and [K] mentioned above. For Part (ii), it will be plainly enough to establish the asserted isomorphism between $\text{gr}_I^* \mathbb{Z} \pi$ and $U \mathcal{H}_*(M(\mathcal{A}))$.

To this end, we will consider the complement of \mathcal{A}' , denoted, as usual, by $M'(\mathcal{A})$, with fundamental group $\pi' := \pi_1(M'(\mathcal{A}))$. Up to homotopy, $M'(\mathcal{A}) \simeq$

$M(\mathcal{A}) \times S^1$. We are going to exploit this simple remark, to obtain the result we need for $M(\mathcal{A})$ from known facts about $M'(\mathcal{A})$.

Since the group π is a retract of the group π' , the graded Lie algebra $\text{gr}_\Gamma^* \pi$ embeds into $\text{gr}_\Gamma^* \pi'$. One also knows that the latter is a free abelian group; see [JP1, Th. C]. It follows that $U \text{gr}_\Gamma^* \pi$ is a free abelian group as well, by Poincaré-Birkhoff-Witt, and consequently the Quillen map (15) is an isomorphism.

One also knows [MP, Prop. 5.1] that there is a graded Lie algebra surjection, inducing the identity in degree one,

$$(17) \quad \phi_S : \mathcal{H}_*(S) \longrightarrow \text{gr}_\Gamma^* \pi_1(S) ,$$

for any connected space S with finite Betti numbers and without torsion in $H_1 S$. The result from [K] quoted above implies then that $\phi_{M(\mathcal{A})} \otimes \mathbb{Q}$ is an isomorphism. To show that actually $\phi_{M(\mathcal{A})}$ is an isomorphism, and thus finish the proof of our corollary, it will be enough to check that $\mathcal{H}_*(M(\mathcal{A}))$ is a free abelian group. To do this, we may argue as before, starting with the remark that $M(\mathcal{A})$ is a retract of $M'(\mathcal{A})$. By naturality (see definition (16)), this implies that $\mathcal{H}_*(M(\mathcal{A}))$ embeds into $\mathcal{H}_*(M'(\mathcal{A}))$. Finally, $\mathcal{H}_*(M'(\mathcal{A}))$ is a free abelian group, by [JP1, Th. C] (see also [JP1, (7.6)]), which completes our proof.

To obtain our last main result, Theorem 23, we will use the Koszul property as key ingredient. So we start by recalling from [Lof] the definition of Koszulness.

Let U^* be a connected graded algebra of finite type, over a field \mathbb{K} . The algebra U is *Koszul* if $\text{Tor}_{s,t}^U(\mathbb{K}, \mathbb{K}) = 0$ for $s \neq t$, where s is the homological (resolution) degree, and t is the internal degree (coming from the grading of U^*).

The Koszul property from Part (i) of the technical proposition below will be used to obtain an exactness property at the associated graded level, in the (aspherical) cases treated in Part (ii). This result in turn will provide an important step in the proof of Theorem 23.

PROPOSITION 22. *Let \mathcal{A} be a projective arrangement whose cone, \mathcal{A}' , is hypersolvable. Set $M := M(\mathcal{A})$, and $U^* := U\mathcal{H}_*(M)$. Let X be any minimal cell structure of M .*

- (i) *The algebra $U \otimes \mathbb{K}$ is Koszul, for any field \mathbb{K} .*
- (ii) *Assume that \mathcal{A}' is fiber-type. Then the augmentation, $\varepsilon : U \rightarrow \mathbb{Z}$, extends to a free (right) U^* -resolution, $\varepsilon : \text{gr}_\Gamma^* C_*(\tilde{X}) \rightarrow \mathbb{Z}$, where:*

$$(18) \quad \text{gr}_\Gamma^* C_*(\tilde{X}) = \{(-1)^q \delta_q^R : H_q M \otimes U^* \longrightarrow H_{q-1} M \otimes U^*\}_q ,$$

and the restriction, $\delta_q^R : H_q M \otimes 1 \equiv H_q M \rightarrow H_{q-1} M \otimes U^1 \equiv H_{q-1} M \otimes H_1 M$, is dual to the cup-product, $H^{q-1} M \otimes H^1 M \rightarrow H^q M$.

Proof. (i) This will follow from the fact that M is a retract of $M' := M'(\mathcal{A}) \simeq M \times S^1$, as in the proof of Corollary 21 (ii). By naturality, the graded algebra $U \otimes \mathbb{K} = U\mathcal{H}_*(M) \otimes \mathbb{K}$ is a retract of $U' \otimes \mathbb{K} := U\mathcal{H}_*(M') \otimes \mathbb{K}$, and therefore $\text{Tor}_{**}^{U \otimes \mathbb{K}}(\mathbb{K}, \mathbb{K})$ is a bigraded \mathbb{K} -subspace of $\text{Tor}_{**}^{U' \otimes \mathbb{K}}(\mathbb{K}, \mathbb{K})$. We deduce from [JP1, Th. E] that $U' \otimes \mathbb{K}$ is a Koszul algebra, via Koszul duality (see [JP1, 7(2)]). The Koszulness of $U \otimes \mathbb{K}$ follows.

(ii) Everything is a direct consequence of Theorem 20 and Corollary 21 (ii), except the fact that the augmented chain complex (18) is acyclic. It is enough to check the acyclicity property for arbitrary field coefficients \mathbb{K} .

As a preliminary step, we will prove first that the algebra $H^*M \otimes \mathbb{K}$ is Koszul. As before, the topological retraction property implies the existence of a split graded algebra epimorphism, $H^*M' \otimes \mathbb{K} \rightarrow H^*M \otimes \mathbb{K}$, inducing a surjection on $\text{Tor}_{**}^{(\cdot)}(\mathbb{K}, \mathbb{K})$. Since $H^*M' \otimes \mathbb{K}$ is a Koszul algebra, for the fiber-type class (by the main result of [SY]), we infer that $H^*M \otimes \mathbb{K}$ is also Koszul.

At this point, we will need the *Koszul duality*. Let $A^* = \mathbb{T}^*(V)/\text{ideal}(R)$ be a *quadratic algebra*, where the tensor algebra $\mathbb{T}^*(V)$ is graded by tensor degree, and R is a \mathbb{K} -vector subspace of $V^{\otimes 2}$. Denote by $\#(\cdot)$ \mathbb{K} -duals of vector spaces. The *Koszul dual* of A^* is the quadratic algebra $A^{!*} := \mathbb{T}^*(\#V)/\text{ideal}(R^\perp)$, where $R^\perp \subset \#V^{\otimes 2}$ is the annihilator of R .

From the preceding definition and definition (16), and by the fact that $H^*(M, \mathbb{K}) = H^*M \otimes \mathbb{K}$ is a (strictly) graded-commutative algebra, it is not difficult to verify that the graded algebras $(U \otimes \mathbb{K})^{!*}$ and $\wedge_{\mathbb{K}}^*(H^1M)/\text{ideal}(\ker \mu)$ are isomorphic. Here $\wedge_{\mathbb{K}}^*(H^1M)$ denotes the exterior \mathbb{K} -algebra on $H^1M \otimes \mathbb{K}$, graded by exterior degree, and $\mu : \wedge_{\mathbb{K}}^2(H^1M) \rightarrow H^2M \otimes \mathbb{K}$ is the cup-multiplication.

The vanishing property of the Tor-pieces for $s = 1$ and $s = 2$, guaranteed by the Koszulness of $H^*M \otimes \mathbb{K}$, translates to the fact that the algebra $H^*M \otimes \mathbb{K}$ is generated by $H^1M \otimes \mathbb{K}$, and the defining relations are of degree 2. In other words, $H^*M \otimes \mathbb{K} = (U \otimes \mathbb{K})^{!*}$, as graded algebras.

The Koszul property of $U \otimes \mathbb{K}$ from Part (i) implies, via the above identification, that the augmentation, $\varepsilon : U \otimes \mathbb{K} \rightarrow \mathbb{K}$, extends to a (left) free $U^* \otimes \mathbb{K}$ -resolution, $\varepsilon : C_*^L \rightarrow \mathbb{K}$, where

$$(19) \quad C_*^L := \{-\delta_q^L : (U^* \otimes \mathbb{K}) \otimes H_q(M, \mathbb{K}) \longrightarrow (U^* \otimes \mathbb{K}) \otimes H_{q-1}(M, \mathbb{K})\}_q ,$$

and the restriction, $\delta_q^L : H_q(M, \mathbb{K}) \rightarrow H_1(M, \mathbb{K}) \otimes H_{q-1}(M, \mathbb{K})$, is dual to the cup-product map,

$$(20) \quad H^1(M, \mathbb{K}) \otimes H^{q-1}(M, \mathbb{K}) \longrightarrow H^q(M, \mathbb{K}) .$$

See [Lof, pp. 305–306] for a proof of the fact that the above *resolution* property actually characterizes Koszul algebras, within the quadratic class.

By construction, the graded algebra $U^* \otimes \mathbb{K}$ is the quotient of $\mathbb{T}^*(H_1(M, \mathbb{K}))$ by the ideal generated by $\text{im}(\nabla \otimes \mathbb{K}) \subset \mathbb{T}^2$. The involutive graded algebra anti-automorphism, α , of \mathbb{T}^* , which sends $x_1 \otimes \cdots \otimes x_k \in \mathbb{T}^k$ to $x_k \otimes \cdots \otimes x_1$, induces a graded algebra anti-automorphism, $\alpha : U^* \otimes \mathbb{K} \rightarrow U^* \otimes \mathbb{K}$. Denote by $c : V \otimes W \rightarrow W \otimes V$ the \mathbb{K} -linear isomorphism which interchanges the factors of a tensor product of two \mathbb{K} -vector spaces. Finally, consider the \mathbb{K} -isomorphisms

$$\{(\text{id} \otimes \alpha) \circ c : (U^* \otimes \mathbb{K}) \otimes H_q(M, \mathbb{K}) \longrightarrow H_q(M, \mathbb{K}) \otimes (U^* \otimes \mathbb{K})\}_q .$$

It is easy to check that they define a \mathbb{K} -linear isomorphism between the augmented chain complexes (18) $\otimes \mathbb{K}$ and (19). Therefore, $\varepsilon : \text{gr}_J^* C_*(\tilde{X}) \otimes \mathbb{K} \rightarrow \mathbb{K}$ is acyclic, and this finishes the proof of Proposition 22.

Let \mathcal{A} be an essential projective arrangement of rank $r \geq 3$, with *hypersolvable* cone, \mathcal{A}' . Set $M := M(\mathcal{A})$, and $p := p(M)$. Theorem 18 applies to M (that is, \mathcal{A} is an iterated generic hyperplane section of an essential aspherical arrangement $\hat{\mathcal{A}}$) precisely when $p = r - 1$. In one direction, the above claim follows from Theorem 18 (i). In general, one knows [JP2] that \mathcal{A} is a 2-generic section of an essential fiber-type arrangement, $\hat{\mathcal{A}}$. Assume now that $p = r - 1$. We may then infer from Proposition 15, via our discussion preceding Theorem 16, on the k -generic framework, that actually \mathcal{A} is a p -generic section of $\hat{\mathcal{A}}$, i.e., an iterated generic hyperplane section of $\hat{\mathcal{A}}$.

The set of (essential) hypersolvable arrangements with $p = r - 1$ is combinatorially defined. Indeed, both the definition of the hypersolvable class and the computation of p within this class are purely combinatorial; see [JP1, Def. 1.8] and [PS, Cor. 4.10(1)] respectively. (Note also that the condition $r \geq 3$ is automatically fulfilled, since $p \geq 2$; see [PS, Cor. 4.10(2)].)

The hypersolvable examples with $p = r - 1$ abound. Let us consider for instance the rank 3 hypersolvable case. Here, $p = 2$ if and only if $\ell > 3$ (by [PS, Th. 5.4(1)]), where $\ell = \ell(\mathcal{A}')$ is the length of \mathcal{A}' , introduced in [JP1]. At the same time, rank 3 hypersolvable examples of arbitrarily large length are easily constructed; see [JP1, §1]. One may also obtain hypersolvable examples with arbitrarily large rank r , and $p = r - 1$, by taking generic hyperplane sections of fiber-type arrangements of large rank; see [PS, p. 90].

To formulate our next result, we need to recall one more fact, from [JP1, Prop. 3.2(i)]: one may associate to any hypersolvable arrangement \mathcal{A}' a combinatorially determined collection of positive natural numbers, $\{1 = d_1, d_2, \dots, d_\ell\}$, which are called the *exponents* of \mathcal{A}' .

THEOREM 23. *Let \mathcal{A} be an essential projective arrangement with hypersolvable cone, \mathcal{A}' . Set $M := M(\mathcal{A})$, $\pi := \pi_1(M)$, $p := p(M)$, $r := \text{rank}(\mathcal{A})$. Assume that $p = r - 1$, and realize \mathcal{A} as an iterated generic hyperplane section of an essential fiber-type arrangement, $\hat{\mathcal{A}}$.*

(i) *There is an isomorphism of graded rings, $\text{gr}_I^* \mathbb{Z}\pi \cong U^*$, where $U^* := U\mathcal{H}_*(M(\widehat{\mathcal{A}}))$ and \mathcal{H}_* denotes the holonomy Lie algebra defined by (16), and a compatible isomorphism of graded modules,*

$$(21) \quad \text{gr}_I^* \pi_p(M) \cong \Sigma^{-p-1} \text{coker} \{ \delta_{p+2}^R : H_{p+2}M(\widehat{\mathcal{A}}) \otimes U^* \longrightarrow H_{p+1}M(\widehat{\mathcal{A}}) \otimes U^* \} ,$$

where δ_{p+2}^R is constructed as in Proposition 22(ii), and Σ^{-p-1} denotes the $(p+1)$ -desuspension operation on graded modules. In particular, the graded module $\text{gr}_I^* \pi_p(M)$ is combinatorially determined. More precisely, it depends only on the rank of \mathcal{A} , and on $\mathcal{L}_2(\mathcal{A})$, the elements of rank at most 2 from $\mathcal{L}(\mathcal{A})$.

(ii) *All graded pieces, $\text{gr}_I^i \pi_p(M)$, are finitely-generated free abelian groups. The Hilbert series, $\text{gr}_I^* \pi_p(M)(t) := \sum_{i \geq 0} \text{rank } \text{gr}_I^i \pi_p(M) \cdot t^i$, is equal to*

$$(22) \quad (-1/t)^{p+1} \left\{ 1 - \left[\sum_{j=0}^p (-1)^j \beta_j \cdot t^j \right] / \prod_{i=2}^{\ell} (1 - d_i t) \right\} ,$$

where $\{1 = d_1, d_2, \dots, d_{\ell}\}$ are the exponents of \mathcal{A}' , and the coefficients $\{\beta_j = b_j(M(\widehat{\mathcal{A}}))\}$ are given by

$$(23) \quad (1 + d_2 t) \cdots (1 + d_{\ell} t) = \sum_{j=0}^{\ell-1} \beta_j \cdot t^j .$$

In particular, $\text{gr}_I^* \pi_p(M)(t)$ depends only on the rank and the exponents of \mathcal{A}' .

Proof. (i) The $\mathbb{Z}\pi$ -module $\pi_p(M)$ is isomorphic to

$$\text{coker} \{ \partial_{p+2} : H_{p+2}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \longrightarrow H_{p+1}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi \} ,$$

by Theorem 18(ii). The following notation will be useful. Denote by M_i the $\mathbb{Z}\pi$ -modules $H_{p+i}M(\widehat{\mathcal{A}}) \otimes \mathbb{Z}\pi$, for $1 \leq i \leq 3$, and by $\{F_k M_i\}_{k \geq 0}$ the corresponding I -adic filtrations. Set $\partial := \partial_{p+2}$, and $\partial' := \partial_{p+3}$.

Next, we shall isolate three facts which are key to our proof. Firstly, $\partial \partial' = 0$. Secondly, $\partial(F_* M_2) \subset F_{*+1} M_1$, and $\partial'(F_* M_3) \subset F_{*+1} M_2$, by minimality. Finally,

$$(24) \quad \begin{aligned} \text{im} \{ \text{gr}^*(\partial') : \text{gr}^{*-1}(M_3) \longrightarrow \text{gr}^*(M_2) \} \\ = \ker \{ \text{gr}^{*+1}(\partial) : \text{gr}^*(M_2) \longrightarrow \text{gr}^{*+1}(M_1) \} . \end{aligned}$$

This crucial property follows from Proposition 22(ii).

The canonical surjections,

$$F_* M_1 / F_{*+1} M_1 \longrightarrow [F_* M_1 + \text{im}(\partial)] / [F_{*+1} M_1 + \text{im}(\partial)] = \text{gr}_I^* \pi_p(M) ,$$

factor to give a surjective morphism of graded $\text{gr}_I^* \mathbb{Z}\pi$ -modules,

$$(25) \quad \psi : \text{coker} \{ \text{gr}^*(\partial) \} \longrightarrow \text{gr}_I^* \pi_p(M) .$$

We will resort to the above mentioned three basic facts and prove that ψ is injective, in each degree $* = k$.

This means that, given any v^1 such that $v^1 \equiv_k 0$ and $v^1 \equiv_{k+1} \partial u^2$, we have to find w^2 such that $w^2 \equiv_{k-1} 0$ and $v^1 \equiv_{k+1} \partial w^2$. Here the superscripts j indicate that the corresponding elements belong to M_j , and the notation \equiv_q means equality modulo the appropriate filtration term F_q . We will argue by induction and suppose that we have found w_s^2 such that $w_s^2 \equiv_s 0$ and $v^1 \equiv_{k+1} \partial w_s^2$, for $s < k - 1$. (For $s = 0$, we may take $w_0^2 = u^2$.) The equalities $\partial w_s^2 \equiv_{s+2} v^1 \equiv_{s+2} 0$ imply the existence of z^3 such that $z^3 \equiv_{s-1} 0$ and $w_s^2 \equiv_{s+1} \partial' z^3$, due to the exactness property (24). Setting $w_{s+1}^2 = w_s^2 - \partial' z^3$, it is immediate to check that $w_{s+1}^2 \equiv_{s+1} 0$ and $v^1 \equiv_{k+1} \partial w_{s+1}^2$, as needed for the induction step. Finally, we may take $w^2 = w_{k-1}^2$, to complete the proof of the fact that ψ is a $\text{gr}_I^* \mathbb{Z}\pi$ -isomorphism.

We may now finish the proof of Theorem 23(i) as follows. The asserted identification of graded rings, $\text{gr}_I^* \mathbb{Z}\pi \equiv U^*$, is provided by Corollary 21(ii). The compatible isomorphism of graded modules (21) becomes a consequence of Proposition 22(ii). Therefore, the graded module $\text{gr}_I^* \pi_p(M)$ is determined by p and the cohomology ring of $M(\widehat{\mathcal{A}})$.

To see that $H^* M(\widehat{\mathcal{A}})$ is in turn determined by $\mathcal{L}_2(\mathcal{A})$, we may argue as follows. By Proposition 14, the rings $H^* M(\widehat{\mathcal{A}})$ and $H^* M(\mathcal{A})$ are isomorphic, up to degree 2. Since $\mathcal{L}_2(\mathcal{A})$ determines $H^{\leq 2} M(\mathcal{A})$, by [OS], we infer that the \mathbb{Z} -algebra

$$\wedge^*(H^1 M(\widehat{\mathcal{A}})) / \text{ideal} (\ker \{ \mu : \wedge^2(H^1 M(\widehat{\mathcal{A}})) \rightarrow H^2 M(\widehat{\mathcal{A}}) \})$$

depends only on $\mathcal{L}_2(\mathcal{A})$. The above algebra naturally surjects onto $H^* M(\widehat{\mathcal{A}})$. It follows from the proof of Proposition 22(ii) that this surjection induces an isomorphism, with arbitrary field coefficients \mathbb{K} , due to the Koszul property of $H^* M(\widehat{\mathcal{A}}) \otimes \mathbb{K}$. This implies that the isomorphism actually holds over \mathbb{Z} , and we are done.

(ii) By Part (i) and the resolution property from Proposition 22 (ii), we deduce that $\text{gr}_I^* \pi_p(M)$ is isomorphic to

$$\Sigma^{-p-1} \text{im} (\delta_{p+1}^R) \subset \Sigma^{-p-1} (H_p M(\widehat{\mathcal{A}}) \otimes U^*) .$$

The first assertion from Theorem 23(ii) follows then from the fact that the graded pieces of $U^* = U\mathcal{H}_*(M(\widehat{\mathcal{A}}))$ are finitely-generated free abelian groups; see the proof of Corollary 21(ii).

To obtain the I -adic filtration formula (22), we will use \mathbb{Q} -coefficients and the resolution from Proposition 22(ii). Denote by $\{B_q^*\}_{q \in \mathbb{Z}}$ and $\{K_q^*\}_{q \in \mathbb{Z}}$

respectively, the q -boundaries and the q -chains of the acyclic chain complex $\varepsilon : \mathrm{gr}_I^* C_*(\hat{X}) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$. Use the exact sequences $0 \rightarrow B_q^* \rightarrow K_q^* \rightarrow B_{q-1}^* \rightarrow 0$ and straightforward induction to obtain the following formula for the Hilbert series of B_p^* :

$$(26) \quad (-1)^{p+1} B_p(t) = 1 + \sum_{j=0}^p (-1)^{j+1} b_j(M(\hat{\mathcal{A}})) t^j \cdot U \otimes \mathbb{Q}(t) .$$

On the other hand, it follows from (21) that

$$(27) \quad t^{p+1} \mathrm{gr}_I^* \pi_p(M)(t) = B_p(t) .$$

The resolution property also implies that

$$(28) \quad H^*(M(\hat{\mathcal{A}}), \mathbb{Q})(-t) \cdot U \otimes \mathbb{Q}(t) = 1 ,$$

via a standard Euler characteristic argument.

Finally, one also knows that $\hat{\mathcal{A}}'$ and \mathcal{A}' have the same exponents; see [JP2, §2]. We infer from [FR] that $H^*(M'(\hat{\mathcal{A}}), \mathbb{Q})(t) = \prod_{i=1}^{\ell} (1 + d_i t)$, and consequently

$$(29) \quad H^*(M(\hat{\mathcal{A}}), \mathbb{Q})(t) = \prod_{i=2}^{\ell} (1 + d_i t) .$$

Equations (26)–(29) together establish formula (22), thus finishing the proof of Theorem 23.

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