GEOMETRY OF WHITNEY-TYPE FORMULAS

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Dedicated to V. I. Arnold on the occasion of his 65th birthday

ABSTRACT. The article contains a generalization of the classical Whitney formula for the number of double points of a plane curve. This formula is split into a series of equalities, and also extended to curves on a torus, to non-pointed curves, and to wave fronts. All the theorems are given geometric proofs employing logarithmic Gauss-type maps from suitable configuration spaces to $\mathbb{C}$.

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1. Whitney Formula and its Generalizations

1.1. Introduction. The classical Whitney formula [4] relates the algebraic number of double points of a generic immersed pointed plane curve to the winding number (Whitney index) of this curve. Higher-dimensional versions of this formula were considered in [3] for immersions $S^n \to \mathbb{R}^{2n}$ and in [1] for immersions $S^n \to \mathbb{R}^{n+1}$.

The original Whitney’s proof of the formula is based on an interpretation of the winding number as a local degree of a Gauss-type map from the configuration space of pairs of points on the curve to $\mathbb{R}^2$. Two sides of the formula represent two different ways to calculate this degree. A similar approach was used in [3].

In [2] the Whitney formula was split into a family of equalities for the algebraic numbers of double points of specific types. Similar formulas with a base point pushed off the curve were also introduced there. All these formulas, though, were proved in a purely combinatorial manner and were lacking clear geometric interpretation since the Gauss map could not be used in this setting.

In this paper we replace the Gauss map with a logarithmic map from the same configuration space to $\mathbb{C} \cong \mathbb{R}^2$. This allows us to interpret the left-hand sides of equalities of [2] as local degrees. An application of the same technique to the case...
of the base point pushed off the curve leads to a new Whitney-type formula, which uses another definition of the sign of a double point.

Using a similar technique, we establish various Whitney-type formulas for curves (both with and without the base point) on a 2-torus $T^2$. These formulas are series of identities indexed by the homology classes in $H_1(T^2, \mathbb{Z})$.

We also provide a generalization of all the results to wave fronts on $\mathbb{R}^2$ and $T^2$.

The paper is organized in the following way. In Section 1 we quote the classical Whitney formula for plane curves (Theorem 1), describe its splitting into a series of equalities (Theorem 2), and formulate the corresponding result for the case when the base point lies outside the curve (Theorem 3). Section 2 contains the formulas for smooth curves on a 2-torus (Theorems 4 and 6). In Section 3 we describe configuration spaces of pairs of points on the circle and Gauss-type maps from these spaces to $\mathbb{R}^2$, using these maps to prove the results of the preceding sections. The last section contains a generalization of the theorems to wave fronts.

1.2. The Whitney formula. Throughout the paper by a plane curve $\gamma$ we will mean an immersion $\gamma: S^1 \to \mathbb{R}^2$ of an oriented circle into the oriented plane, its only singularities being transversal double points.

We will often view $\gamma$ as a pointed curve, i.e. a map $\gamma: [0, 1] \to \mathbb{R}^2$ with the base point $p = \gamma(0) = \gamma(1)$, and assume that $p$ is not a double point. For a double point $d = \gamma(a) = \gamma(b)$ the linear order $0 < a < b < 1$ determines an ordering of two branches of $\gamma$ in $d$. The sign $\varepsilon_d(p) = \pm 1$ of $d$ is determined by the orientation of the corresponding frame of two tangent vectors to $\gamma$ in $d$. Following Whitney [4], we use the sign convention shown in Figure 1a (thus, the sign is positive if the orientation of the basis $(\gamma'(a), \gamma'(b))$ is opposite to the orientation of the plane).

The Gauss map $S^1 \to S^1$ sends a point $x \in S^1$ to the direction of the tangent vector $\gamma'(x)$. The winding number $w(\gamma)$ of $\gamma$ is the degree of this map, i.e. the number of turns made by the tangent vector as we pass along $\gamma$. Up to deformations in the class of immersions, the winding number is the only invariant of $\gamma$, see [4].

For $x \in \mathbb{R}^2 \setminus \gamma$ the direction of the vector connecting $x$ with a point $y$ moving along $\gamma$ defines another map $S^1 \to S^1$. The index $\text{ind}(\gamma, x)$ of $x$ with respect to $\gamma$ is the degree of this map, i.e. the number of turns made by the vector $\overrightarrow{xy}$ as $y$ moves along $\gamma$.

For $x \in \gamma$, define $\text{ind}(\gamma, x) \in \frac{1}{2}\mathbb{Z}$ by averaging the indices of the points lying in the adjacent components of $\mathbb{R}^2 \setminus \gamma$ (two regions if $x$ is a generic point of $\gamma$, and four regions if $x$ is a double point). Denote by $D$ the set of all double points of $\gamma$.

The Whitney formula states that

**Theorem 1** [4]. Let $\gamma$ be a plane curve with a base point $p$ and $\text{ind}(\gamma, p) = j \in \mathbb{Z} + \frac{1}{2}$. Then

$$\sum_{d \in D} \varepsilon_d(p) = w(\gamma) - 2j.$$

1.3. Splitting of the Whitney formula. Consider a plane curve $\gamma: S^1 \to \mathbb{R}^2$ and let $d$ be its double point. Smoothing $\gamma$ at $d$, respecting the orientation, we split $\gamma$ into two parts, see Figure 1b. Denote by $h_d(p) \in \mathbb{Z}$ the index of $p$ with respect to
the part not containing \( p \). Denote by \( D_k(p) \) the set of all double points of \( \gamma \) such that \( h_d(p) = k \).

For every \( j \in \mathbb{Z} + \frac{1}{2} \) and \( k \in \mathbb{Z} \) define the number \( s(j, k) \) as follows. If \( j > 0 \), then

\[
s(j, k) = \begin{cases}
+1, & \text{if } 0 < k < j, \\
 j + \frac{1}{2}, & \text{if } k = 0, \\
0 & \text{otherwise,}
\end{cases}
\]

and if \( j < 0 \), then

\[
s(j, k) = \begin{cases}
-1, & \text{if } j < k < 0, \\
 j - \frac{1}{2}, & \text{if } k = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

The following theorem was proved in [2] by purely combinatorial methods:

**Theorem 2** [2]. Let \( \gamma \) be a plane curve with a base point \( p \) and \( \operatorname{ind}(\gamma, p) = j \in \mathbb{Z} + \frac{1}{2} \). Then for any \( k \in \mathbb{Z} \)

\[
\sum_{d \in D_k(p)} \varepsilon_d(p) = \begin{cases}
\operatorname{w}(\gamma) - s(j, 0), & \text{if } k = 0, \\
-s(j, k) & \text{otherwise.}
\end{cases}
\]

Theorem 2 provides a splitting of the formula of Theorem 1 into an infinite family of equalities; summing them over \( k \) we recover Theorem 1. We give a new geometric proof of Theorem 2 in Section 3.

**1.4. Pushing the base point off the curve.** We would like to modify Theorem 2 for \( p \in \mathbb{R}^2 \setminus \gamma \) (see also [2]). The right-hand side of the equation does not present any problem: the definition of \( \operatorname{w}(\gamma) \) does not involve the base point, and \( \operatorname{ind}(\gamma, p) \) is well-defined also for \( p \in \mathbb{R}^2 \setminus \gamma \) (being in this case an integer). The left-hand side is more problematic, since the definition of the sign \( \varepsilon_d(p) \) of a double point \( d \) uses the linear order of the two branches of \( \gamma \) in \( d \), and hence requires \( \gamma \) to be a pointed curve.

Split \( \gamma \) at \( d \) as we did for pointed curves. Denote by \( \gamma_d^l \) (resp., \( \gamma_d^r \)) the part whose tangent vector turns clockwise (resp., counter-clockwise) in a neighborhood of \( d \), see Figure 1b. Denote \( l_d(p) = \operatorname{ind}(\gamma_d^l, p), r_d(p) = \operatorname{ind}(\gamma_d^r, p). \)
For every \( k \in \mathbb{Z} \) define the sign \( \varepsilon_{k,d}(p) \) as follows:

\[
\varepsilon_{k,d}(p) = \begin{cases} 
-1, & \text{if } l_d(p) = k \neq r_d(p), \\
+1, & \text{if } r_d(p) = k \neq l_d(p), \\
0 & \text{otherwise,}
\end{cases}
\]

(1)

Notice that \( \varepsilon_{k,d}(p) = -\varepsilon_{j-k,d}(p) \).

**Theorem 3** (cf. [2, Theorem 4’]). Let \( \gamma \) be a plane curve, \( p \in \mathbb{R}^2 \setminus \gamma \), and \( \text{ind}(\gamma, p) = j \in \mathbb{Z} \). Then for any \( k \in \mathbb{Z} \)

\[
\sum_{d \in D} \varepsilon_{k,d}(p) = \begin{cases} 
w(\gamma) - j, & \text{if } k = 0 \neq j, \\
j - w(\gamma), & \text{if } k = j \neq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

We give a geometric proof of Theorem 3 in Section 3.

2. Curves on the Torus

2.1. Pointed curves. Let \( \gamma : [0, 1] \to \mathbb{T}^2 = S^1 \times S^1 \) be a smooth curve on a 2-torus with a base point \( p = \gamma(0) = \gamma(1) \), its only singularities being transversal double points. Every double point \( d \) can be assigned a sign \( \varepsilon_d(p) \) like for the plane curves, see Figure 1a.

Let \( \pi : \mathbb{R}^2 \to \mathbb{T}^2 \) be a universal covering with \( \pi(0) = p \). The covering space \( \mathbb{R}^2 \) contains a lattice \( L = \pi^{-1}(p) \) naturally identified with the homology group \( H_1(\mathbb{T}^2, \mathbb{Z}) \).

Denote by \( \tilde{\gamma} : [0, 1] \to \mathbb{R}^2 \) the lifting of the curve \( \gamma \) to the covering space such that \( \tilde{\gamma}(0) = 0 \), and let \( \tilde{\gamma}(1) = j \). Apparently, \( j \in L \) is the homology class represented by \( \gamma \). If \( j \neq 0 \) then the curve \( \tilde{\gamma} \) is not closed, but nevertheless one has \( \tilde{\gamma}(0) = \tilde{\gamma}(1) \), so that the winding number \( w(\gamma) = w(\tilde{\gamma}) \) still can be defined exactly as in Section 1.2.

Consider now the union \( \tilde{\tau} \) of the curve \( \tilde{\gamma}(t), 0 \leq t \leq 1 \), and the curve \( j - \tilde{\gamma}(t), 0 \leq t \leq 1 \). The curve \( \tilde{\tau} \) is smooth except for cusps at the points 0 and \( j \); tangent lines in these points are parallel. Since \( \gamma \) is generic, the curve \( \tilde{\tau} \) avoids points of the lattice \( L \) except 0 and \( j \). Define a new curve \( \tau \) slightly pushing \( \tilde{\tau} \) off the points 0 and \( j \) in any direction transversal to the tangent line. The curve \( \tau \) does not intersect the lattice \( L \), so we can define an integer \( s(\gamma, k) = \text{ind}(\tau, k) \) for all \( k \in L \). It is easy to see that the values of \( s(\gamma, k) \), including \( s(\gamma, 0) \) and \( s(\gamma, j) \), do not depend on the push-off direction.

Smooth the curve \( \gamma \) at a double point \( d \) and denote by \( h_d(p) \in L \) the homology class represented by the part of \( \gamma \) not containing the base point. For any \( k \in L \) denote by \( D_k(p) \) the set of all double points \( d \) such that \( h_d(p) = k \).

**Theorem 4.** Let \( \gamma \) be a curve on \( \mathbb{T}^2 \) with a base point \( p \). Then for any \( k \in H_1(\mathbb{T}^2, \mathbb{Z}) \)

\[
\sum_{d \in D_k(p)} \varepsilon_d(p) = \begin{cases} 
w(\gamma) - s(\gamma, 0), & \text{if } k = 0, \\
-s(\gamma, k) & \text{otherwise.}
\end{cases}
\]
Corollary 5.  
\[ \sum_{d \in D} \varepsilon_d(p) = w(\gamma) - s(\gamma) \]
where \( s(\gamma) = \sum_{k \in L} s(\gamma, k) \) is the total index of the lattice \( L \).

2.2. Non-pointed curves. Consider now a generic non-pointed curve \( \gamma: S^1 \to \mathbb{T}^2 \) on the torus. Smoothing \( \gamma \) at a double point \( d \) we get two curves \( \gamma_d^+ \) and \( \gamma_d^- \), as in Figure 1b. Denote by \( l_q \) and \( r_d \) their homology classes, and for every \( k \in H_1(\mathbb{T}^2, \mathbb{Z}) \) define the sign \( \varepsilon_{k,d} \) by the same formula (1) as for the plane curves.

**Theorem 6.** Let \( \gamma \) be a curve on \( \mathbb{T}^2 \) representing a class \( j \in H_1(\mathbb{T}^2, \mathbb{Z}) \). Then for any \( k \in H_1(\mathbb{T}^2, \mathbb{Z}) \)

\[ \sum_{d \in D} \varepsilon_{k,d} = \begin{cases} w(\gamma), & \text{if } k = 0 \neq j, \\ -w(\gamma), & \text{if } k = j \neq 0, \\ 0 & \text{otherwise}. \end{cases} \]

3. Configuration Spaces

In this section we introduce configuration spaces of pairs of points on \( S^1 \) and their maps to \( \mathbb{R}^2 \). We use these maps to prove the theorems of Sections 1 and 2. To do this, we interpret the two sides of the corresponding equalities as two different ways to compute local degrees of the relevant maps.

3.1. Local degrees of maps. Let \( M \) be a compact oriented manifold of dimension \( n \) (possibly with boundary). Let \( f: M \to \mathbb{R}^n \) be a smooth map and let \( y \in \mathbb{R}^n \) be a regular value of this map.

By definition, the local degree of \( f \) at the point \( y \) (denoted \( \text{deg}(f, y) \)) is the intersection index of an \( n \)-chain \( f_*([M]) \) with a \( 0 \)-chain \( [y] \), so it can be calculated by counting, with appropriate signs, the points \( q \) such that \( f(q) = y \). On the other hand, the intersection index will not change if one replaces \( [y] \) with a \( 0 \)-cycle \( [y] - [\infty] \), composed of \( y \) taken with the positive orientation and a point near infinity (e.g., any point in the non-compact region of \( \mathbb{R}^n \setminus f(M) \) taken with the negative orientation. The intersection index of \( f_*([M]) \) with \( [y] - [\infty] \) is equal to the linking number in \( \mathbb{R}^n \) of an \( (n-1) \)-cycle \( f_*([\partial M]) \) with \( [y] - [\infty] \). This linking number may be computed also as the intersection index of \( f_*([\partial M]) \) with any \( 1 \)-chain \( [Y] \) such that \( \partial[Y] = [y] - [\infty] \). One possible choice for \( Y \) is a ray from \( y \) to infinity. Thus, the local degree of \( f \) at \( y \) can be calculated also by counting (with appropriate signs) the points \( q \in \partial M \) such that \( f(q) \in Y \).

3.2. Configuration spaces for non-pointed curves. We start from a simpler case when the curve is not based, thus proving Theorems 3 and 6 first.

Consider a space \( \mathcal{C} = \{(a, b): a, b \in S^1, a \neq b \} \) of pairs of distinct points on a circle. For two distinct points \( a, b \in S^1 \) we will denote by \( ab \) an oriented arc joining \( a \) with \( b \) and going in the positive direction (i.e. counterclockwise).

The space \( \mathcal{C} \) is an open manifold, so we will consider its “cut-off” compact subset \( \mathcal{C}_\varepsilon \) consisting of all pairs \( (a, b) \), \( a, b \in S^1 \) such that the lengths of the arcs \( ab \) and \( ba \) are both greater or equal to \( \varepsilon \); here \( 0 < \varepsilon \ll 1 \).
Let first $\gamma$ be a plane curve, and $p \in \mathbb{R}^2 \setminus \gamma$. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider a map $f_\gamma : \mathbb{C}_\varepsilon \to \mathbb{C}$ defined by the formula
\[ f_\gamma(a, b) = \frac{1}{2\pi i} \int_a^b \gamma^* \left( \frac{dz}{z-p} \right) \] (2)
where the integral is taken over the arc $ab$.

Proof of Theorem 3. Let us calculate the local degree $\deg(f_\gamma, k)$ of $f_\gamma$ at the point $k \in \mathbb{Z} \subset \mathbb{C}$ as described in Section 3.1 above.

First, count the preimages of $k$ with signs. A pair $(a, b) \in \mathbb{C}_\varepsilon$ lies in $f^{-1}(\mathbb{Z})$ iff $\gamma(a) = \gamma(b)$, i.e. $d = \gamma(a)$ is a double point of $\gamma$. Each double point $d$ corresponds to two different points, $(a, b)$ and $(b, a)$, of the preimage. Moreover, $f_\gamma(a, b) = k$ if the closed curve obtained by restriction of $\gamma$ to the arc $ab$ makes exactly $k$ turns around the point $p$. For $\varepsilon$ small enough, all the values $k \in \mathbb{Z}$ are regular. An easy check shows that the sign of the point $(a, b)$ equals $+1$ if the arc $ab$ is mapped by $\gamma$ onto the curve $\gamma_d$ and equals $-1$ if it is mapped to $\gamma^*_d$ (see Figure 1b). Comparing these signs with the definition of $\varepsilon_{k,d}(p)$ (see (1)), one obtains
\[ \deg(f_\gamma, k) = \sum_{d \in D} \varepsilon_{k,d}(p). \] (3)

Second, count the intersections of $f_\gamma(\partial \mathbb{C}_\varepsilon)$ with the ray $Y_0 = \{ z \in \mathbb{C} : \text{Re}(z) = k, \text{Im}(z) \geq 0 \}$. The configuration space $\mathbb{C}_\varepsilon$ is an annulus, whose boundary consists of two circles $\partial_+$ and $\partial_-$, corresponding to pairs $(a, b)$ such that the arc $ab$ (resp., $ba$) has the length $\varepsilon$; circles $\partial_+$ and $\partial_-$ are taken with opposite orientations. For $(a, b) \in \partial_+$, the arc $ab$ is very short, and therefore $f_\gamma(a, b)$ is close to zero. For $(a, b) \in \partial_-$, the arc $ab$ is close to the whole of $S^1$, and therefore $f_\gamma(a, b)$ is close to $j = \text{ind}(\gamma, p)$. This implies that $f_\gamma(\partial \mathbb{C}_\varepsilon)$ does not intersect the rays $Y_k$ for $k \neq 0, j$, hence $\deg(f_\gamma, k) = 0$ for such $k$. To calculate $\deg(f_\gamma, 0)$, note that only $f_\gamma(\partial_+)$ intersects $Y_0$. If $(a, b) \in \partial_+$ then $f_\gamma(a, b) = \log(\gamma(b)/\gamma(a))$, so up to $o(\varepsilon)$ one has $f_\gamma(a, b) \sim (\gamma(b) - \gamma(a))/\gamma(a) \sim \gamma'(a)/\gamma(a) \cdot \varepsilon$. As $a$ moves along $S^1$, the complex number $\gamma'(a) \neq 0$ makes $w(\gamma)$ turns around the origin, and the complex number $1/\gamma(a)$ makes $-\text{ind}(\gamma, p) = -j$ turns. Therefore for $\varepsilon$ small enough, the intersection index of $f_\gamma(\partial \mathbb{C}_\varepsilon)$ with $Y_0$ (and thus $\deg(f_\gamma, 0)$) equals $w(\gamma) - j$. The local degree $\deg(f_\gamma, j)$ is calculated in a similar way. Comparison with (3) proves Theorem 3. \hfill \Box

Consider now a curve $\gamma$ on a torus $\mathbb{T}^2$. Identify the covering space $\mathbb{R}^2$ with $\mathbb{C}$; consider on it a complex coordinate $z$, and on $\mathbb{T}^2$, the corresponding complex-valued 1-form $dz$. Define a map $f_\gamma : \mathbb{C}_\varepsilon \to \mathbb{C}$ by
\[ f_\gamma(a, b) = \int_a^b \gamma^* dz \] (4)
where, again, the integral is taken over the arc $ab$.

Proof of Theorem 6. Calculate the local degree $\deg(f_\gamma, k)$ of $f_\gamma$ at the point $k \in L \subset \mathbb{C}$ as described in Section 3.1 above. The calculation of the left-hand side is
the same as in the proof of Theorem 3; it gives
\[ \deg(f, k) = \sum_{d \in D} \varepsilon_{k,d}. \] (5)

Now count the intersections of \( f, (\partial C) \) with the ray \( Y_k = \{ z \in \mathbb{C} : \text{Re}(z) = \text{Re}(k), \text{Im}(z) \geq \text{Im}(k) \} \). Like in the proof of Theorem 3, note that there are no intersections unless \( k = 0 \) or \( k = j \). For \( k = 0 \), only \( f |_{\partial_k} \) intersects the ray. If \( (a, b) \in \partial_+ \), then up to \( o(\varepsilon) \) one has \( f(a, b) = \gamma'(a)\varepsilon \). Therefore for \( \varepsilon \) small enough the intersection index of \( f, (\partial C) \) with \( Y_0 \) (and thus \( \deg(f, 0) \)) equals \( w(\gamma) \). The reasoning for \( k = j \) is similar. Comparison with (5) proves Theorem 6.

**3.3. Configuration spaces for pointed curves.** Recall that a pointed curve is a map \( \gamma : [0, 1] \to \mathbb{RP}^1 \) (or \( \gamma : [0, 1] \to \mathbb{T}^2 \)), such that \( p = \gamma(0) = \gamma(1) \). Consider a configuration space \( \mathcal{C}^p = \{ (a, b) : 0 < a < b < 1 \} \) of pairs of distinct points on \([0, 1]\). It is an open manifold, so we will consider its “cut-off” compact subset \( \mathcal{C}^p_\varepsilon = \{ (a, b) \in \mathcal{C}^p : a \geq \varepsilon, b \leq 1 - \varepsilon, b - a \geq \varepsilon \} \) where \( 0 < \varepsilon \ll 1 \).

Again, identify \( \mathbb{R}^2 \) with \( \mathbb{C} \). For a pointed plane curve \( \gamma \), define the map \( f_{\gamma} : \mathcal{C}^p_\varepsilon \to \mathbb{C} \) by the formula
\[ f_{\gamma}(a, b) = \frac{1}{2\pi i} \int_a^b \gamma^*(\frac{dz}{z - p}) \] (6)
where the integral is taken over the interval \([a, b] \subset [0, 1]\).

**Proof of Theorem 2.** Calculate the local degree \( \deg(f, k) \) at the point \( k \in \mathbb{Z} \subset \mathbb{C} \) in two ways described in Section 3.1.

First, count the preimages of \( k \) with signs. Apparently \( f_{\gamma}(a, b) = k \) if and only if \( \gamma(a) = \gamma(b) \) and the curve \( \gamma(t), a \leq t \leq b \), makes exactly \( k \) turns around the point \( p \), i.e. \( d = \gamma(a) \) is a double point and belongs to \( D_k(p) \). The sign of the preimage is equal to \( -\varepsilon_d(p) \) by Whitney’s convention, see Figure 1a. Thus,
\[ \deg(f, k) = - \sum_{d \in D_k(p)} \varepsilon_d(p) \] (7)

Suppose for convenience that \( j = \text{ind}(\gamma, p) \geq 0 \) (the opposite case is analogous). Count now intersections of \( f_{\gamma}|_{\partial\mathcal{C}^p_\varepsilon} \) with the ray \( Y_k \) where \( Y_k = \{ z \in \mathbb{C} : \text{Re}(z) = k, \text{Im}(z) \geq 0 \} \) for \( k \neq 0 \), and \( Y_0 = \{ z \in \mathbb{R} \subset \mathbb{C} : z \leq 0 \} \). The configuration space \( \mathcal{C}^p_\varepsilon \) is a triangle with the vertices \( P, Q, R \) corresponding to the points \( (a, b) = (\varepsilon, 2\varepsilon), (\varepsilon, 1 - \varepsilon), \) and \( (1 - 2\varepsilon, 1 - \varepsilon) \), respectively. If \( (a, b) \in [PQ] \), then \( f_{\gamma}(a, b) = \frac{1}{2\pi i} \log(\gamma(b)/\gamma(\varepsilon)) \) (the appropriate branch of the logarithm should be taken as to ensure continuity). Since \( |\gamma(b)| > |\gamma(\varepsilon)| \) for all \( b \in (\varepsilon, 1 - \varepsilon) \), the curve \( f_{\gamma}|_{PQ} \) lies in the upper half-plane, and \( f_{\gamma}(Q) \) is close to the point \( j \). Thus, the intersection index in question is equal to 1 if \( 1 \leq k \leq j \) and is equal to 0 otherwise.

For \( (a, b) \in [QR] \) one has \( f_{\gamma}(a, b) = -\frac{1}{2\pi i} f(a)/f(1 - \varepsilon) \). The curve \( f_{\gamma}|_{QR} \) lies totally in the lower half-plane, and therefore the intersection index in question is 0 for all \( k \).

For \( (a, b) \in [RP] \), one has
\[ f_{\gamma}(a, b) = \frac{1}{2\pi i} \log(\gamma(b + \varepsilon)/\gamma(b)) + o(\varepsilon) = \varepsilon \cdot (\gamma'(b)/\gamma(b)) + o(\varepsilon). \]
So, the curve \( f_\gamma|_{RP} \) lies near the origin, and for \( \varepsilon \) small enough its intersection index with \( Y_k \) is 0 for all \( k \neq 0 \). The intersection index with \( Y_0 \) is equal to the number of turns the curve \( \gamma'(b)/\gamma(b) \) makes around the origin, as \( b \) goes from \( \varepsilon \) to \( 1 - \varepsilon \). Apparently, this number of turns is \( w(\gamma) - j + 1/2 \). Comparing the results obtained with (7), we prove Theorem 2.

For a pointed curve \( \gamma \) on the torus define a map \( f_\gamma: \mathcal{C}_k^P \to \mathbb{C} \) by the formula

\[
f_\gamma(a, b) = \int_a^b \gamma^* dz
\]

where the integral is taken over the interval \([a, b] \subset [0, 1]\).

**Proof of Theorem 4.** Similar to the proof of Theorem 2, calculate the local degree \( \deg(f_\gamma, k) \) at the point \( k \in L \subset \mathbb{C} \) in two ways described in Section 3.1.

First, count the preimages of \( k \) with signs. Apparently \( f_\gamma(a, b) = k \) if and only if \( \gamma(a) = \gamma(b) \) and the curve \( \gamma(t), a \leq t \leq b \), represents the homology class \( k \). Thus,

\[
deag(f_\gamma, k) = - \sum_{d \in D_k(p)} \varepsilon_d(p)
\]

the sign reversal is caused by Whitney convention, cf. (7).

Count now intersections of \( f_\gamma|_{\partial C_k^P} \) with the ray \( Y_k = \{ z \in \mathbb{C} : \text{Re}(z) = \text{Re}(k), \text{Im}(z) \geq \text{Im}(k) \} \). If \( (a, b) \in [PQ] \) then \( f_\gamma(a, b) = \gamma(b) - \gamma(a) \); if \( (a, b) \in [QR] \), then \( f_\gamma(a, b) = \gamma(1 - \varepsilon) - \gamma(a) \), and if \( (a, b) \in [PR] \), then up to \( o(\varepsilon) \) one has \( f_\gamma(a, b) = \cdot \varepsilon \). Thus, as the point \( (a, b) \) moves along the boundary \( \partial C_k^P \) from \( P \) to \( Q \) to \( R \) and back to \( P \), the point \( f_\gamma(a, b) \in \mathbb{C} \) runs along the curve close to the union of \( \tau \) (see Section 2.1 for definition) and the curve \( \gamma'(t) \cdot \varepsilon, t \in [0, 1] \). Comparison with (9) proves Theorem 4.

4. **Wave Fronts**

4.1. **Definitions for fronts.** The formulas of Sections 1 and 2 may be generalized to generic fronts on \( \mathbb{R}^2 \) and \( \mathbb{T}^2 \). By a **front** on a surface \( \Sigma \) we mean a smooth map \( \gamma: S^1 \to \Sigma \), its only singularities being transversal double points and semicubical cusp points. We will always assume that \( \gamma \) is oriented and cooriented (i.e. equipped with a coorienting normal direction). All the constructions above extend to this situation with few minor changes.

We define the Whitney index \( w(\gamma) \) of the front as the degree of the Gauss map given by the coorientation. The definitions of \( \text{ind}(\gamma, p) \) and the signs \( \varepsilon_d(p) \) and \( \varepsilon_{k,d}(p) \) stay the same (note that we are using the orientation, not the coorientation, for them). We should only define signs of cusps and modify the smoothing procedure to take the coorientation into account.

Define the sign \( \varepsilon_c \) of a cusp \( c \) to be +1 if the coorienting vector turns in the positive direction while going through a neighborhood of \( c \) along the orientation, and \(-1\) otherwise (see Figure 2a). To obtain a front “smoothed” in a double point \( d \), we split \( \gamma \) into two parts, respecting both the orientation and the coorientation, see Figure 2b. Other definitions stay the same.
Denote
\[ c(\gamma) = \frac{1}{2} \sum \varepsilon_c. \]

4.2. Theorems for fronts. The only change in theorems of Sections 1 and 2 is the contribution of cusps via \( c(\gamma) \):

**Theorem 2'.** Let \( \gamma \) be a pointed front with the base point \( p \) and \( \text{ind}(\gamma, p) = j \in \mathbb{Z} + \frac{1}{2} \). For any \( k \in \mathbb{Z} \)
\[ \sum_{d \in D_k(p)} \varepsilon_d(p) = \begin{cases} w(\gamma) - c(\gamma) - s(j, 0), & \text{if } k = 0, \\ -s(j, k), & \text{otherwise.} \end{cases} \]

**Corollary** (Whitney’s formula for plane fronts). Let \( \gamma \) be a pointed front with the base point \( p \) and \( \text{ind}(\gamma, p) = j \). Then
\[ \sum_{d \in D} \varepsilon_d(p) = w(\gamma) - c(\gamma) - 2j. \]

A similar statement holds for non-pointed plane fronts:

**Theorem 3'.** Let \( \gamma \) be a plane front, \( p \in \mathbb{R}^2 \setminus \gamma \), and \( \text{ind}(\gamma, p) = j \in \mathbb{Z} \). Then for any \( k \in \mathbb{Z} \)
\[ \sum_{d \in D} \varepsilon_d(p) = \begin{cases} w(\gamma) - j - c(\gamma), & \text{if } k = 0 \neq j, \\ j - w(\gamma) + c(\gamma), & \text{if } k = j \neq 0, \\ 0, & \text{otherwise.} \end{cases} \]

For fronts on the torus we keep the definitions of Section 2.

**Theorem 4'.** Let \( \gamma \) be a front on \( \mathbb{T}^2 \) with a base point \( p \). Then for any \( k \in H_1(\mathbb{T}^2, \mathbb{Z}) \)
\[ \sum_{d \in D_k(p)} \varepsilon_d(p) = \begin{cases} -s(\gamma, 0) + w(\gamma) - c(\gamma), & \text{if } k = 0, \\ -s(\gamma, k), & \text{otherwise.} \end{cases} \]

**Corollary** (Whitney’s formula for fronts on the torus). Let \( \gamma \) be a front on \( \mathbb{T}^2 \) with a base point \( p \).
\[ \sum_{d \in D} \varepsilon_d(p) = w(\gamma) - c(\gamma) - s(\gamma) \]
where \( s(\gamma) = \sum_{k \in L} s(\gamma, k) \) is the total index of the lattice \( L \).
Theorem 6'. Let $\gamma$ be a front on $\mathbb{T}^2$ representing a class $j \in H_1(\mathbb{T}^2, \mathbb{Z})$. Then for any $k \in H_1(\mathbb{T}^2, \mathbb{Z})$

$$\sum_{d \in D} \xi_{k,d} = \begin{cases} w(\gamma) - c(\gamma), & \text{if } k = 0 \neq j, \\ -w(\gamma) + c(\gamma), & \text{if } k = j \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

All these theorems for fronts are corollaries of the corresponding theorems for smooth curves. For a front $\gamma$ consider a curve $\bar{\gamma}$ obtained by smoothing of all the cusps of $\gamma$. The tangent vector of $\bar{\gamma}$ makes a half-turn in the neighbourhood of every cusp $c$ of $\gamma$. The direction of this half-turn is counterclockwise if $\varepsilon_c = -1$ and clockwise if $\varepsilon_c = +1$ (see Figure 2a). Thus,

$$w(\bar{\gamma}) = w(\gamma) - c(\gamma). \quad (10)$$

Away from the cusps, the curve $\bar{\gamma}$ coincides with $\gamma$. In particular, passage from $\gamma$ to $\bar{\gamma}$ preserves the double points and their signs. Applying Theorems 2–6 to curve $\bar{\gamma}$ and combining them with (10), one obtains Theorems 2'–6'.