Lecture 18. The Gauss and Codazzi equations

In this lecture we will prove the fundamental identities which hold for the extrinsic curvature, including the Gauss identity which relates the extrinsic curvature defined via the second fundamental form to the intrinsic curvature defined using the Riemann tensor.

18.1 The fundamental identities

The definitions (from the last lecture) of the connection on the normal and tangent bundles, and the second fundamental form h and the associated operator $\mathcal{W}$, can be combined into the following two useful identities: First, for any pair of vector fields U and V on M,

$$D_U D_V X = -h(U,V) + DX(\nabla_U V). \tag{18.1}$$

This tells us how to differentiate an arbitrary tangential vector field $D_V X$, considered as a vector field in $\mathbb{R}^N$ (i.e. an N-tuple of smooth functions). Then we have a corresponding identity which tells us how to differentiate sections of the normal bundle, again thinking of them as N-tuples of smooth functions: For any vector field U and section ϕ of NM,

$$D_U \phi = DX(\mathcal{W}(U,\phi)) + \nabla_U \phi. \tag{18.2}$$

Since we can think of vector fields in this way as N-tuples of smooth functions, we can deduce useful identities in the following way: Take a pair of vector fields U and V. Applying the combination $UV - VU - [U,V]$ to any function gives zero, by definition of the Lie bracket. In particular, we can applying this to the position vector X:

$$\begin{aligned}0 &= (UV - VU - [U,V])X\\ &= -h(U,V) + DX(\nabla_U V) + h(V,U) - DX(\nabla_V U) - DX([U,V]).\end{aligned}$$

Since the right-hand side vanishes, both the normal and tangential components must vanish. The normal component is $h(V,U) - h(U,V)$, so this establishes the fact we already knew that the second fundamental form is symmetric. The tangential component is $DX(\nabla_U V - \nabla_V U - [U,V])$, so the vanishing of this tells us that the connection is symmetric (as we proved before).

18.2 The Gauss and Codazzi equations

We will use the same method as above to deduce further important identities, by applying $UV - VU - [U,V]$ to an arbitrary tangential vector field.

Let W be a smooth vector field on M. Then we have

$$\begin{aligned}
0 &= (UV - VU - [U,V])WX \\
&= U(VWX) - V(UWX) - [U,V]Wx \\
&\overset{(18.1)}{=} U\left(-h(V,W) + (\nabla_V W)X\right) - V\left(-h(U,W) + (\nabla_U W)X\right) \\
&\quad - \left(-h([U,V],W) + (\nabla_{[U,V]}W)X\right) \\
&\overset{(18.2)}{=} -DX(\mathcal{W}(U,h(V,W))) - (\nabla h(U,V,W) + h(\nabla_U V,W) + h(V,\nabla_U W)) \\
&\quad + DX(\mathcal{W}(V,h(U,W))) - (\nabla h(V,U,W) + h(\nabla_V U,W) + h(U,\nabla_V W)) \\
&\quad + U(\nabla_V W)X - V(\nabla_U W)X + h([U,V],W) - (\nabla_{[U,V]}W)X \\
&\overset{(18.1)}{=} -DX(\mathcal{W}(U,h(V,W))) - (\nabla h(U,V,W) + h(\nabla_U V,W) + h(V,\nabla_U W)) \\
&\quad + DX(\mathcal{W}(V,h(U,W))) - (\nabla h(V,U,W) + h(\nabla_V U,W) + h(U,\nabla_V W)) \\
&\quad - h(U,\nabla_V W) + (\nabla_U\nabla_V W)X + h(V,\nabla_U W) - (\nabla_V\nabla_U W)X \\
&\quad + h([U,V],W) - (\nabla_{[U,V]}W)X \\
&= DX\left(R(V,U)W - \mathcal{W}(U,h(V,W)) + \mathcal{W}(V,h(U,W))\right) \\
&\quad + \nabla h(U,V,W) - \nabla h(V,U,W).
\end{aligned}$$

In deriving this we used the definition of curvature in the last step, and used the symmetry of the connection to note that several of the terms cancel out. The tangential and normal components of the resulting identity are the following:

$$R(U,V)W = \mathcal{W}(U,h(V,W)) - \mathcal{W}(V,h(U,W)) \tag{18.3}$$

and

$$\nabla h(U,V,W) = \nabla h(V,U,W). \tag{18.4}$$

Note that the tensor ∇h appearing here is the covariant derivative of the tensor h, defined by

$$\nabla h(U,V,W) = \nabla_U(h(V,W)) - h(\nabla_U V,W) - h(V,\nabla_U W)$$

for any vector fields U, V and W. Here the ∇ appearing in the first term on the right-hand side is the connection on the normal bundle, and the other two terms involve the connection on TM.

Equation (18.3) is called the Gauss equation, and Equation (18.4) the Codazzi equation. Note that the Gauss equation gives us a formula for the intrinsic curvature of M in terms of the extrinsic curvature h.

It is sometimes convenient to write these identities in local coordinates: Given a local chart with coordinate tangent vectors $\partial_1, \ldots, \partial_n$, choose also a collection of smooth sections e_α of the normal bundle, $\alpha = 1, \ldots, N - n$, which are linearly independent at each point. Let g be the metric on TM and $\tilde{g}$ the metric on NM, and write $g_{ij} = g(\partial_i, \partial_j)$ and $\tilde{g}_{\alpha\beta} = \tilde{g}(e_\alpha, e_\beta)$. Then we can write

$$h(\partial_i, \partial_j) = h_{ij}{}^\alpha e_\alpha$$

and

$$\mathcal{W}(\partial_i, e_\beta) = \mathcal{W}_{i\beta}{}^j \partial_j.$$

The relation between h and $\mathcal{W}$ then tells us that $\mathcal{W}_{i\beta}{}^j = g^{jk} \tilde{g}_{\beta\alpha} h_{ik}{}^\alpha$.

The Gauss identity then becomes

$$R_{ijkl} = \left(h_{jk}{}^\alpha h_{il}{}^\beta - h_{jk}{}^\beta h_{il}{}^\alpha \right) \tilde{g}_{\alpha\beta} \tag{18.5}$$

If we write $\nabla h = \nabla_i h_{jk}{}^\alpha dx^i \otimes dx^j \otimes dx^k \otimes e_\alpha$, then the Codazzi identity becomes

$$\nabla_i h_{jk}{}^\alpha = \nabla_j h_{ik}{}^\alpha. \tag{18.6}$$

Since we already know that h_{ij} is symmetric in j and k, this implies that $\nabla_k h_{ij}$ is totally symmetric in i, j and k.

18.3 The Ricci equations

We will complete our suite of identities by applying $UV - VU - [U, V]$ to an arbitrary section ϕ of the normal bundle:

$$\begin{aligned}
0 &= (UV - VU - [U,V])\phi \\
&\overset{(18.2)}{=} U(\mathcal{W}(V,\phi)X + \nabla_V \phi) - V(\mathcal{W}(U,\phi)X + \nabla_U \phi) \\
&\quad - (\mathcal{W}([U,V],\phi)X + \nabla_{[U,V]}\phi) \\
&\overset{(18.1)}{=} -h(U, \mathcal{W}(V,\phi)) + (\nabla\mathcal{W}(U,V,\phi) + \mathcal{W}(\nabla_U V, \phi) + \mathcal{W}(V, \nabla_U \phi))X \\
&\quad + h(V, \mathcal{W}(U,\phi)) - (\nabla\mathcal{W}(V,U,\phi) + \mathcal{W}(\nabla_V U, \phi) + \mathcal{W}(U, \nabla_V \phi))X \\
&\quad + U\nabla_V \phi - V\nabla_U \phi - \mathcal{W}([U,V],\phi)X - \nabla_{[U,V]}\phi \\
&\overset{(18.2)}{=} -h(U, \mathcal{W}(V,\phi)) + (\nabla\mathcal{W}(U,V,\phi) + \mathcal{W}(V, \nabla_U \phi))X \\
&\quad + h(V, \mathcal{W}(U,\phi)) - (\nabla\mathcal{W}(V,U,\phi) + \mathcal{W}(U, \nabla_V \phi))X \\
&\quad + \mathcal{W}(U, \nabla_V \phi)X + \nabla_U \nabla_V \phi - \mathcal{W}(V, \nabla_U \phi)X - \nabla_V \nabla_U \phi - \nabla_{[U,V]}\phi \\
&= R^\perp(V,U)\phi - h(U, \mathcal{W}(V,\phi)) + h(V, \mathcal{W}(U,\phi)) \\
&\quad + (\nabla\mathcal{W}(U,V,\phi) - \nabla\mathcal{W}(V,U,\phi))\, X.
\end{aligned}$$

As before, this gives us two sets of identities, one from the tangential component and one from the normal component. In fact the tangential component is just the Codazzi identities again, but the normal component gives a new identity, called the *Ricci identity*, which expresses the curvature of the normal bundle in terms of the second fundamental form:

$$R^{\perp}(U,V)\phi = h(V,\mathcal{W}(U,\phi)) - h(U,\mathcal{W}(V,\phi)). \tag{18.7}$$

In local coordinates, with a local basis for the normal bundle as above, this identity can be written as follows: If we write $R^{\perp}_{ij\alpha\beta} = \tilde{g}(R^{\perp}(\partial_i,\partial_j)e_\alpha, e_\beta)$ and $h_{ij\alpha} = \tilde{g}(h(\partial_i,\partial_j),e_\alpha)$, then we have

$$R^{\perp}_{ij\alpha\beta} = g^{kl}(h_{ik\alpha}h_{jl\beta} - h_{jk\alpha}h_{il\beta}). \tag{18.8}$$

18.4 Hypersurfaces

In the case of hypersurfaces the identities we have proved simplify somewhat: First, since the normal bundle is one-dimensional, the normal curvature vanishes and the Ricci equations become vacuous.

Also, the basis $\{e_\alpha\}$ for NM can be taken to consist of the single unit normal vector $\mathbf{n}$, and the Gauss and Codazzi equations become

$$R_{ijkl} = h_{ik}h_{jl} - h_{jk}h_{il}$$

and

$$\nabla_i h_{jk} = \nabla_j h_{ik}.$$

The curvature tensor becomes rather simple in this setting: At any point $x \in M$ we can choose local coordinates such that $\partial_1,\ldots,\partial_n$ are orthonormal at x and diagonalize the second fundamental form, so that

$$h_{ij} = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases}$$

Then we have an orthonormal basis for the space of 2-planes $\Lambda^2 T_x M$, given by $\{e_i \wedge e_j :\ i < j\}$. The Gauss equation gives for all $i < j$ and $k < l$

$$\mathrm{Rm}(e_i \wedge e_j, e_k \wedge e_l) = \begin{cases} \lambda_i\lambda_j, & i = k,\ j = l \\ 0, & \text{otherwise.} \end{cases}$$

In particular, this basis diagonalizes the curvature operator, and the eigenvalues of the curvature operator are precisely $\lambda_i\lambda_j$ for $i < j$. Note that all of the eigenvectors of the curvature operator are simple planes in $\Lambda^2 TM$. It follows that if M is any Riemannian manifold which has a non-simples 2-plane as a eigenvector of the curvature operator at any point, then M cannot be immersed (even locally) as a hypersurface in $\mathbb{R}^{n+1}$.

Example 18.4.1 (Curvature of the unit sphere). Consider the unit sphere S^n, which is a hypersurface of Euclidean space $\mathbb{R}^{n+1}$ (so we take the immersion X to be the inclusion). With a suitable choice of orientation, we find that the Gauss map is the identity map on S^n — that is, we have $\mathbf{n}(z) = X(z)$ for all $z \in S^n$. Differentiating this gives

$$DX(\mathcal{W}(u)) = D_u\mathbf{n} = DX(u)$$

so that $\mathcal{W}$ is the identity map on TS^n, and $h_{ij} = g_{ij}$. It follows that all of the principal curvatures are equal to 1 at every point, and that all of the sectional curvatures are equal to 1. Therefore the unit sphere has constant sectional curvatures equal to 1.

Example 18.4.2 (Totally umbillic hypersurfaces). A hypersurface M^n in Euclidean space is called *totally umbillic* if for every $x \in M$ the principal curvatures $\lambda_1(x), \ldots, \lambda_n(x)$ are equal — that is, the second fundamental form has the form $h_{ij} = \lambda(x)g_{ij}$. The Codazzi identity implies that a connected totally umbillic hypersurface in fact has constant principal curvatures (hence also constant sectional curvatures by the Gauss identity):

$$(\nabla_k\lambda)g_{ij} = \nabla_k h_{ij} = \nabla_i h_{kj} = (\nabla_i\lambda)g_{kj}$$

for any i, j and k. Fix k, and choose $j = i \neq k$ (we assume $n \geq 2$, since otherwise the totally umbillic condition is vacuous). This gives $\nabla_k\lambda = 0$. Since k is arbitrary, this implies $\nabla\lambda = 0$, hence λ is constant. There are two possibilities: $\lambda = 0$ (in which case M is a subset of a plane), or $\lambda \neq 0$ (in which case M is a subset of a sphere).

Example 18.4.3 Spacelike hypersurfaces in Minkowski space The definitions we have made for the second fundamental form were given for submanifolds of Euclidean space. However the same definitions work with very minor modifications for certain hypersurfaces in the Minkowski space $\mathbb{R}^{n,1}$: A hypersurface M^n in $\mathbb{R}^{n,1}$ is called *spacelike* if the metric induced on M from the Minkowski metric is Riemannian — equivalently, if every non-zero tangent vector u of M has $\langle u, u\rangle_{\mathbb{R}^{n,1}} > 0$. This is equivalent to the statement that M is given as the graph of a smooth function with slope less than 1 over the plane $\mathbb{R}^n \times \{0\}$.

Given a spacelike hypersurface, we can choose at each $x \in M$ a unit normal by taking the unique future-pointing vector $\mathbf{n}$ which is orthogonal to T_xM with respect to the Minkowski metric, normalized so that $\langle \mathbf{n}, \mathbf{n}\rangle = -1$.

The definitions are now identical to those for hypersurfaces in Euclidean space, except that the metric on the normal bundle is now negative definite. The Codazzi identity is unchanged, and the Gauss identity is almost unchanged: The one difference arises from the presence of the $\tilde{g}$ term in Equation (18.5), which is now negative instead of positive. This gives

$$R_{ijkl} = -\left(h_{ik}h_{jl} - h_{jk}h_{il}\right). \tag{18.9}$$

We can now compute the second fundamental form for a very special example, namely Hyperbolic space $\mathbb{H}^n$, which is the set of unit future timelike vectors in Minkowski space. In this case we have $\mathbf{n}(z) = z$ for every $z \in \mathbb{H}^n$, since $0 = D_u\langle z, z\rangle = 2\langle D_u z, z\rangle = 2\langle u, z\rangle$. Differentiating, we find (exactly as in the calculation for the sphere in Example 18.4.1 above) that $h_{ij} = g_{ij}$, so that the principal curvatures are all equal to 1. The Gauss equation therefore gives that the sectional curvatures are identically equal to -1.

More generally, a hypersurface with all principal curvatures of the same sign (i.e. a convex hypersurface) in Minkowski space has negative curvature operator, while a hypersurface with all principal curvatures of the same sign in Euclidean space has positive curvature operator.