

THE BOUNDARY OF YOUNG GRAPH WITH JACK EDGE MULTIPLICITIES

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ABSTRACT. Consider the lattice of all Young diagrams ordered by inclusion, and denote by \mathbb{Y} its Hasse graph. Using the Pieri formula for Jack symmetric polynomials, we endow the edges of the graph \mathbb{Y} with formal multiplicities depending on a real parameter θ . The multiplicities determine a potential theory on the graph \mathbb{Y} . Our main result identifies the corresponding Martin boundary with an infinite-dimensional simplex, the “geometric boundary” of the Young graph \mathbb{Y} , and provides a canonical integral representation for non-negative harmonic functions.

For three particular values of the parameter, the theorem specializes to known results: the Thoma theorem describing characters of the infinite symmetric group, the Kingman’s classification of partition structures, and the description of spherical functions of the infinite hyperoctahedral Gelfand pair.

1. The main result. We use [13] as a basic reference for the standard notations and terminology related to integer partitions and symmetric functions (see also Appendix for basic background definitions).

Let Λ denote the \mathbb{R} -algebra of symmetric functions in infinitely many variables $x = (x_1, x_2, \dots)$. We denote by $P_\mu(x; \theta)$ the Jack symmetric polynomial with the parameter θ , indexed by an integer partition μ (see [13], VI.10, or Section A.4 for the definition; note that our parameter θ is inverse to Macdonald’s $\alpha = 1/\theta$). The family $\{P_\mu\}$ forms a linear basis in the symmetric function algebra Λ . We aim to describe the set $X(\theta)$ of algebra homomorphisms $\varphi : \Lambda \rightarrow \mathbb{R}$, such that

$$\varphi(P_\mu(\cdot; \theta)) \geq 0 \tag{1.1}$$

for a fixed value of θ and all integer partitions μ .

Recall that the power sum symmetric polynomials $p_m(x) = \sum_{j \geq 1} x_j^m$ are algebraically independent and generate the algebra Λ . As a result, every homomorphism $\varphi : \Lambda \rightarrow \mathbb{R}$ is entirely determined by an arbitrary real sequence $\varphi_1, \varphi_2, \dots$ of its values $\varphi_m = \varphi(p_m)$ at the polynomials p_m . The main result of the present paper can be stated as follows.

The authors were partially supported by the grants 96-01-00676 (S.K.) and 95-01-00814 (A.O. and G.O.) of Russian Foundation for Basic Research.

Theorem A. Assume that $\theta > 0$, and let $\varphi : \Lambda \rightarrow \mathbb{R}$ be an algebra homomorphism, such that $\varphi(p_1) = 1$. Then the following two conditions are equivalent:

- (1) $\varphi(P_\mu(\cdot; \theta)) \geq 0$ for all integer partitions μ ;
- (2) there exists a pair of non-negative weakly decreasing sequences

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0),$$

subject to the condition $\sum \alpha_j + \sum \beta_j \leq 1$, and

$$\varphi(p_m) = \sum_{j=1}^{\infty} \alpha_j^m + (-\theta)^{m-1} \sum_{j=1}^{\infty} \beta_j^m$$

for all $m = 2, 3, \dots$

The sequences α, β are uniquely determined by the homomorphism φ .

This statement confirms a particular case of the Conjecture of Section 7.3 in [6]. In the special case $\theta = 1$ the Theorem is equivalent to Schoenberg's conjecture on totally positive sequences, proved in [1], [3], and to a theorem by E. Thoma [23] describing the characters of the infinite symmetric group.

The limiting case $\theta = 0$, in a different form, was studied by Kingman [11], see also [5]. Yet another particular case where the Theorem was already known to be true is that of $\theta = 1/2$. In this case it admits a representation theoretical interpretation, see [18], [21].

Theorem A can be restated in terms of discrete potential theory. In this form it provides a Poisson-type integral representation of non-negative harmonic functions on the Young graph, with respect to an edge multiplicity function depending on a real parameter θ , see Theorem B in Section 5.

In a more general setup one starts with a Pieri-type formula

$$p_1 \cdot P_\lambda = \sum_{\nu} \varkappa(\lambda, \nu) P_\nu$$

where $\{P_\lambda\}$ is a linear homogeneous basis in the algebra Λ . The coefficients $\varkappa(\lambda, \nu)$ (assumed to be non-negative) determine formal multiplicities of edges of the Young graph \mathbb{Y} . One looks for an integral representation of non-negative functions φ on the set \mathbb{Y} , harmonic in the sense that

$$\varphi(\lambda) = \sum_{\nu} \varkappa(\lambda, \nu) \varphi(\nu).$$

Theorem B solves the problem for the basis of Jack symmetric polynomials $P_\lambda(x; \theta)$. The proof relies substantially on a formula for θ -dimension of skew Young diagrams obtained in [20]. We also use many ideas and constructions of papers [5-9], [24] though the present paper can be read independently.

A similar integral representation is known for Schur Q -functions (a special case of Hall – Littlewood symmetric polynomials, see [13], III.8). The corresponding graph

is that of shifted Young diagrams (see Nazarov [15] and Ivanov [4]), it describes the branching of projective representations of symmetric groups.

A part of the conjecture in [6] related to general Macdonald symmetric polynomials still remains open.

The plan of the paper is as follows. In Section 2 we recall the Martin boundary construction for graded graphs with multiplicities. In Sections 3, 4 we present the well-known integral representations of non-negative harmonic functions for two special cases of Jack graphs: the Young lattice and the Kingman's graph. The general case of arbitrary Jack graphs is considered in Section 5. In Sections 6, 7 we derive two basic ingredients in the proof of our main Theorems A and B. The proof itself is given in Section 8. There is also an appendix where we recall necessary background definitions related to integer partitions and symmetric polynomials.

2. The Martin boundary of a branching graph. In this paper we consider a particular case of the following general problem: find the Martin boundary of an infinite graph Γ with respect to a given multiplicity function on the set of its edges.

More precisely, the graph Γ is graded

$$\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n,$$

the vertices in Γ_n representing the admissible states of a process at the moment n . The kinematics of the process is described by the edges of the graph pointing out from a vertex $\lambda \in \Gamma_n$ to some vertex ν at the next level Γ_{n+1} . We write $\lambda \nearrow \nu$ to show that the pair (λ, ν) is an oriented edge of the graph Γ . The process always starts with a distinguished vertex $\emptyset \in \Gamma_0$ at the zero level of the graph Γ .

Another piece of information for our general problem is provided by a *multiplicity function* \varkappa on the set of edges of Γ , with real positive values. A pair (Γ, \varkappa) consisting of a graded graph Γ and a multiplicity function \varkappa will be referred to as a *branching graph* (the origin of the term will be motivated by an example in Section 3).

A function $\varphi : \Gamma \rightarrow \mathbb{R}$ defined on the set of vertices in Γ is called *harmonic*, if the following variant of the “mean value theorem” holds for all vertices $\lambda \in \Gamma$:

$$\varphi(\lambda) = \sum_{\nu: \lambda \nearrow \nu} \varkappa(\lambda, \nu) \varphi(\nu). \tag{2.1}$$

We are interested in the space \mathcal{H} of all non-negative harmonic functions normalized at the vertex \emptyset by the condition $\varphi(\emptyset) = 1$. The topology in the space \mathcal{H} is that of pointwise convergence. It is clear that \mathcal{H} is convex and compact.

A general solution for such a problem is based on the Martin boundary construction (see, for instance, [2]). One starts with the *dimension function* $\dim(\mu, \nu)$ defined recurrently by the formulae $\dim(\mu, \mu) \equiv 1$,

$$\dim(\mu, \nu) = \sum_{\lambda: \lambda \nearrow \nu} \dim(\mu, \lambda) \varkappa(\lambda, \nu), \tag{2.2}$$

and $\dim(\mu, \nu) = 0$, if there is no oriented path t from μ to ν . Associating a weight $w(t) = \prod \varkappa(\lambda_{i-1}, \lambda_i)$ with every such path $t = (\lambda_0 = \mu, \lambda_1, \dots, \lambda_m = \nu)$, one can write the dimension function as a sum of weights over all oriented paths between the vertices μ and ν , that is, $\dim(\mu, \nu) = \sum w(t)$.

From the point of view of potential theory, $G(\mu, \nu) = \dim(\mu, \nu)$ is the Green function with respect to ‘‘Laplace operator’’

$$(\Delta\varphi)(\lambda) = -\varphi(\lambda) + \sum_{\nu: \lambda \nearrow \nu} \varkappa(\lambda, \nu) \varphi(\nu).$$

This means that if $\varphi_\nu(\mu) = G(\mu, \nu)$ for a fixed vertex ν , then $-(\Delta\varphi_\nu)(\mu) = \delta_{\mu\nu}$ for all $\mu \in \Gamma$. The ratio

$$K(\mu, \nu) = \frac{\dim(\mu, \nu)}{\dim \nu} \tag{2.3}$$

is usually called the *Martin kernel*.

Consider the space \mathcal{F} of all functions $f : \Gamma \rightarrow \mathbb{R}$ with the topology of pointwise convergence, and let \tilde{E} be the closure of the subset $\tilde{\Gamma} \subset \mathcal{F}$ of functions $\mu \mapsto K(\mu, \nu)$, $\nu \in \Gamma$. Since those functions are uniformly bounded, $0 \leq K(\mu, \nu) \leq 1$, the space \tilde{E} (called the *Martin compactification*) is indeed compact. One can easily check that $\tilde{\Gamma} \subset \tilde{E}$ is a dense open subset of \tilde{E} . Its boundary $E = \tilde{E} \setminus \tilde{\Gamma}$ is called the *Martin boundary* of the branching graph (Γ, \varkappa) .

By definition, the Martin kernel (2.3) may be extended by continuity to the function $K : \Gamma \times \tilde{E} \rightarrow \mathbb{R}$, and we keep denoting it by the same letter K . For each boundary point $\omega \in E$ the function $\varphi_\omega(\mu) = K(\mu, \omega)$ is non-negative, harmonic, and normalized. Moreover, all such functions have an integral representation similar to the classical Poisson integral for non-negative harmonic functions in the disk.

Theorem (cf. [2]). *Every normalized non-negative harmonic function $\varphi \in \mathcal{H}$ admits an integral representation*

$$\varphi(\lambda) = \int_E K(\lambda, \omega) M(d\omega). \tag{2.4}$$

where M is a probability measure. For every probability measure M on E the integral (2.4) provides a non-negative harmonic function $\varphi \in \mathcal{H}$.

All *indecomposable* (i.e., *extreme*) functions in \mathcal{H} can be represented in the form $\varphi_\omega(\mu) = K(\mu, \omega)$, for appropriate boundary point $\omega \in E$, and we denote by E_{min} the corresponding subset of the boundary E . It is known that E_{min} is a non-empty G_δ subset of E . One can always choose the measure M in the integral representation (2.4) to be supported by E_{min} . Under this assumption, the measure M representing a function $\varphi \in \mathcal{H}$ via (2.4) is unique.

In all the examples considered in the present paper the linear span of the functions $\omega \mapsto K(\mu, \omega)$, $\mu \in \Gamma$ is dense in the space $C(E)$ of continuous functions on the boundary E . As a consequence, the minimal boundary E_{min} will always coincide with the entire Martin boundary E .

Given a concrete example of a branching graph, one looks for an appropriate “geometric” description of the abstract Martin boundary. For instance, the Martin boundary of a domain in the complex plane interior to a Jordan curve can be identified with this curve, its geometric boundary. For a general simply connected domain the Martin boundary is more complicated; its geometric version is provided by the Perron – Carathéodory theory of “boundary elements”, see [14], Chapter V, §3.

The purpose of the present paper is to give an explicit integral representation of the form (2.4) for the Young graph \mathbb{Y} endowed with some edge multiplicities $\varkappa_\theta(\lambda, \nu)$ depending on a real parameter θ . The multiplicities arise in the Pieri formula for the Jack symmetric polynomials $P_\mu(x; \theta)$. We refer to the corresponding branching as to the Jack branching graph, and denote it as $\mathbb{J}(\theta)$. Presently, we proceed with a couple of well-studied concrete examples of such graphs. Both examples will appear as particular instances in the family $\mathbb{J}(\theta)$.

3. Example: the Young lattice. By definition, the vertices of the *Young graph* \mathbb{Y} are Young diagrams (representing partitions of natural numbers). The edges join the pairs of diagrams which only differ by a single box (i.e., if the second diagram in a pair (λ, ν) covers the first one with respect to inclusion order).

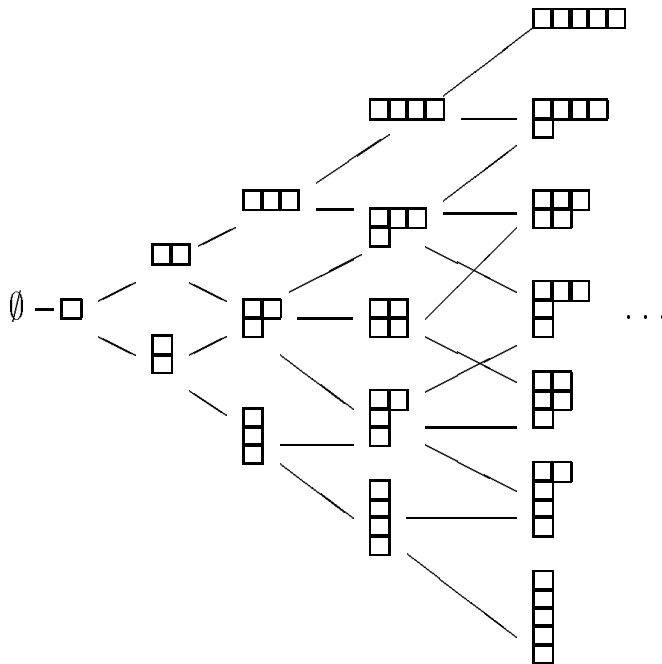


Fig. 1 The Young graph.

The graph \mathbb{Y} plays a fundamental role in representation theory of finite and infinite symmetric groups, see Vershik and Kerov [7], [9], [24]. The reason is that the set \mathbb{Y}_n of Young diagrams with n boxes labels characters of irreducible representations of the symmetric group \mathfrak{S}_n of degree n , and the edges of the Young graph describe the branching of characters of \mathfrak{S}_n when restricted to the subgroup \mathfrak{S}_{n-1} . The multiplicity

function is trivial: $\varkappa \equiv 1$ for all edges. The first five levels of the Young graph are represented in Fig. 1.

The set of all central positive definite functions on the infinite symmetric group $\mathfrak{S}_\infty = \bigcup \mathfrak{S}_n$, normalized at the identity, can be naturally identified with the set \mathcal{H} of positive harmonic functions on the Young graph. The *characters* of the group \mathfrak{S}_∞ , i.e., the indecomposable normalized central positive-definite functions, correspond to the points of the set $E_{min} = E$.

The space of characters of the infinite symmetric group \mathfrak{S}_∞ was found by E. Thoma in the pioneering paper [23]. In equivalent terms of potential theory on the Young graph his result can be stated as follows.

Definition. The space Ω of all pairs $\omega = (\alpha, \beta)$ of weakly decreasing sequences of non-negative real numbers,

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq \dots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq \beta_n \geq \dots \geq 0), \quad (3.1)$$

such that

$$\gamma = 1 - \sum \alpha_n - \sum \beta_n \geq 0,$$

is called the *Thoma simplex*. The topology of Ω is that of pointwise convergence.

Given a Young diagram ν , we denote by $d = \max\{j : \nu_j \geq j\}$ the length of its diagonal, and by

$$a_j = \nu_j - j, \quad b_j = \nu'_j - j + 1; \quad j = 1, \dots, d \quad (3.2)$$

the so called *Frobenius parameters* of the diagram ν . We associate with ν a point

$$\omega_\nu = \left(\frac{a_1}{n}, \dots, \frac{a_d}{n}, 0, \dots; \frac{b_1}{n}, \dots, \frac{b_d}{n}, 0, \dots \right) \quad (3.3)$$

of the Thoma simplex Ω . Remark that a Young diagram ν is entirely determined by the point ω_ν , along with the value of $n = |\nu|$. Thus, we may identify the set of non-empty Young diagrams with the subset

$$\tilde{\mathbb{Y}} = \bigcup_{n=1}^{\infty} \bigcup_{\nu \in \mathbb{Y}_n} \left(\frac{1}{n}, \omega_\nu \right) \quad (3.4)$$

of the product space $[0, 1] \times \Omega$. The set $\tilde{\mathbb{Y}}$ is discrete (i.e., each of its points has a neighborhood in $[0, 1] \times \Omega$ free of other points of $\tilde{\mathbb{Y}}$), and its boundary $\{0\} \times \Omega$ is homeomorphic to Thoma simplex. By this reason we shall call Ω the *geometric boundary of the Young lattice*.

We are now in a position to describe the Martin boundary of the Young graph.

Theorem (cf. [23], [24], [16]). *The Martin boundary E of the Young graph is naturally homeomorphic to its geometric boundary, the Thoma simplex Ω . Every non-negative*

normalized harmonic function $\varphi \in \mathcal{H}$ on the Young graph admits a unique integral representation of the form

$$\varphi(\lambda) = \int_{\Omega} s_{\lambda}(\omega) M(d\omega). \quad (3.5)$$

The Martin kernel $K(\lambda, \omega) = s_{\lambda}(\omega)$ is given by the extended Schur functions (see Section A.3 for the definition of extended symmetric functions).

The boundary Ω of the Young graph \mathbb{Y} has quite a number of interpretations. For instance, the equation (A.2) establishes a bijection between Ω and the set of all totally positive real sequences $\{h_n\}_{n=0}^{\infty}$. Such sequences were studied by I. Schoenberg and his school [1]. Schoenberg's conjecture about the structure of totally positive sequences was proved by A. Edrei [3] long before the paper [23] was published, and independently of the representation theory of the group \mathfrak{S}_{∞} .

Yet another interpretation is based on the observation that the formula

$$\psi(h_n) = h_n(\alpha, \beta); \quad (\alpha; \beta) \in \Omega, \quad n = 1, 2, \dots$$

determines a general homomorphism $\psi : \Lambda \rightarrow \mathbb{R}$ of the ring Λ of symmetric polynomials, which is non-negative on the basis of Schur functions, $\psi(s_{\lambda}) \geq 0$ for all $\lambda \in \mathbb{Y}$, see [23].

Young graph governs the branching of irreducible characters of the symmetric groups \mathfrak{S}_n . There exists a similar branching graph \mathbb{K} responsible for the branching of conjugacy classes in these groups.

The underlying graph of \mathbb{K} coincides with the Young graph \mathbb{Y} , but this time the multiplicities of edges are non-trivial. More precisely,

$$\varkappa(\lambda, \nu) = m_k(\nu), \quad (4.1)$$

where k denotes the length of the row of the diagram ν containing a box to be removed from ν in order to obtain λ , and $m_k(\nu)$ is the total number of rows of length k in the diagram ν (see Fig. 2).

Motivated by a problem of population genetics, J. F. C. Kingman introduced in [11] the notion of *partition structure*. It can be defined as a family $\{M_n\}_{n=1}^{\infty}$ where M_n is a central (i.e., constant on conjugacy classes) probability distribution on the symmetric group \mathfrak{S}_n . The family is assumed to be *coherent* in the sense that $d(M_{n+1}) = M_n$, for all $n = 1, 2, \dots$, where the permutation $d(w) \in \mathfrak{S}_n$ is obtained from $w \in \mathfrak{S}_{n+1}$ by crossing the element $n+1$ out of its cycle in w .

The description of Martin boundary for the branching graph \mathbb{K} is equivalent to the Kingman's classification of partition structures, and can be stated as follows.

Theorem (cf. [11] and [5]). *Every non-negative function φ on the set of Young diagrams, harmonic with respect to the multiplicities (4.1), admits a unique integral representation*

$$\varphi(\lambda) = \int_E m_{\lambda}(\alpha) M(d\alpha), \quad (4.2)$$

where M is a probability measure, and the boundary $E = E_{min}$ is that part of the Thoma simplex (3.2) for which $\beta_n = 0$ for all $n = 1, 2, \dots$. The Martin kernel $m_\lambda(\alpha)$ is given by the extended¹ monomial symmetric functions.

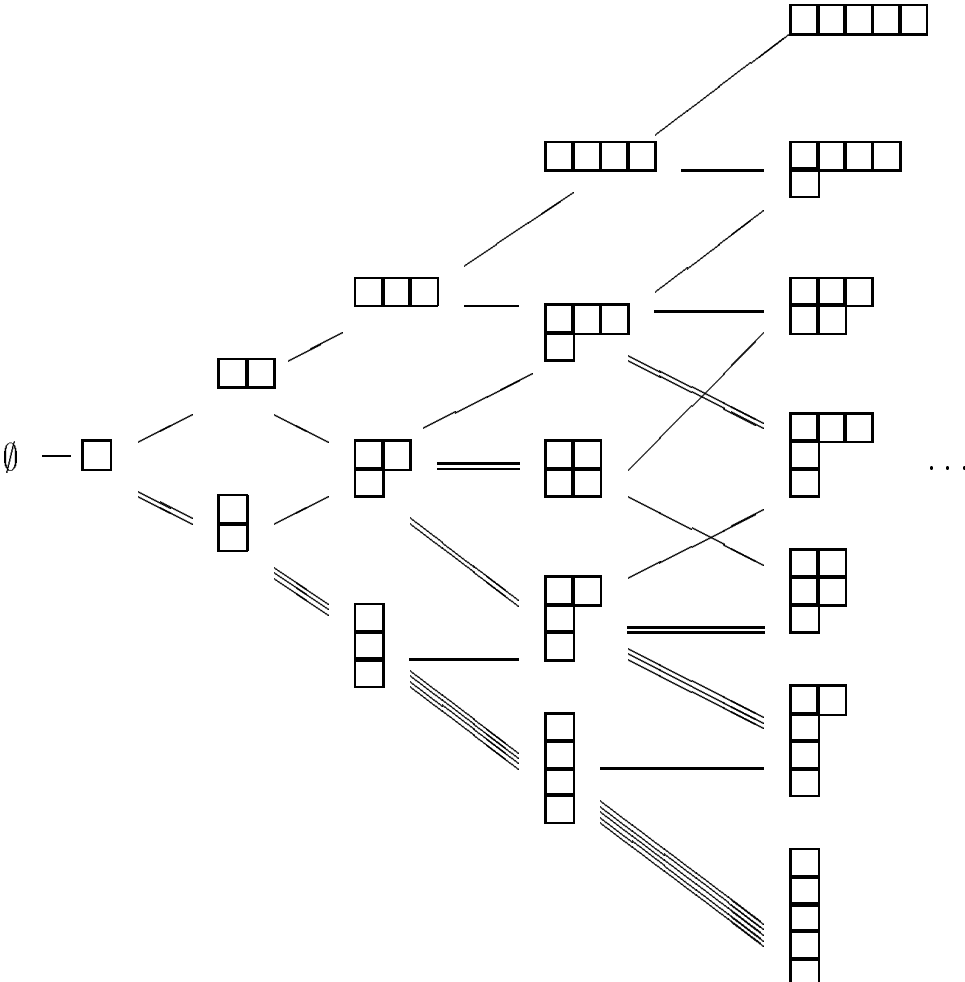


Fig. 2 The Kingman's branching graph.

Kingman's result is much simpler than Thoma theorem. See [5] for another proof of this result.

5. Jack's graphs and their boundaries. In this Section we introduce a one-parameter family of multiplicity functions on the Young graph, interpolating between the trivial multiplicities and those of Kingman. Our main Theorem B identifies the corresponding boundary and the Martin kernel.

¹This tiny extension concerns only the γ parameter; all β -parameters are supposed to vanish, see Section A.3.

We refer to [13] for the theory of Jack symmetric polynomials (see also equations (A.5) – (A.7) for some definitions). We shall use the simplest particular case of Pieri formula for Jack symmetric polynomials which reads

$$p_1(x) P_\lambda(x; \theta) = \sum_{\nu: \lambda \nearrow \nu} \varkappa_\theta(\lambda, \nu) P_\nu(x; \theta), \quad (5.1)$$

where the multiplicities $\varkappa_\theta(\lambda, \nu)$ are given by an explicit formula

$$\varkappa_\theta(\lambda, \nu) = \prod_b \frac{(a(b) + (l(b) + 2)\theta)(a(b) + 1 + l(b)\theta)}{(a(b) + 1 + (l(b) + 1)\theta)(a(b) + (l(b) + 1)\theta)}. \quad (5.2)$$

Here b runs over all boxes in the j -th column of the diagram λ , provided that the new box $b_0 = \nu \setminus \lambda$ belongs to the j -th column of ν . The number $a(i, j) = \lambda_i - j$ is called the *arm length*, and $l(i, j) = \lambda'_j - i$ is the *leg length* of a box $b = (i, j)$ in the diagram λ . (See [13], Ch. VI, (10.10) and (6.24.iv).)

It can be checked that the multiplicities (5.2) tend, as $\theta \rightarrow 0$, to those of Kingman (4.1) (the limit behavior of (5.2) only depends on the boxes b with $a(b) = 0$).

Lemma 5.1. *The coefficients (5.2) are all positive iff $\theta \geq 0$.*

Proof. Clearly, the condition is sufficient. Since $\varkappa_\theta((1^n), (1^{n+1})) = (1+n)/(1+n\theta)$, it is necessary, too. \square

Definition. We denote by $\mathbb{J}(\theta) = (\mathbb{Y}, \varkappa_\theta)$ the Young graph \mathbb{Y} with the edge multiplicities (5.2). By abuse of terminology we call the branching diagram $\mathbb{J}(\theta)$ the *Jack graph* with the parameter θ . We shall always assume that $\theta \geq 0$, so that the multiplicities are positive.

The initial part of the graph $\mathbb{J}(\theta)$ is represented in Fig. 3. Note that $\mathbb{J}(0)$ coincides with the Kingman's branching of Section 4, and that $\mathbb{J}(1)$ is the ordinary Young graph with trivial multiplicities.

Our main concern in this paper is to describe all non-negative harmonic functions on the graph $\mathbb{J}(\theta)$.

Assuming that $\theta > 0$, we shall prove that the Martin boundary of the graph $\mathbb{J}(\theta)$ does not depend on the parameter θ and may be naturally identified with the Thoma simplex Ω – the geometric boundary of the Young graph. On the contrary, the Martin kernel does depend on θ . In order to describe this kernel, we associate to each point $(\alpha; \beta) \in \Omega$ an algebra homomorphism $\varphi_{\alpha, \beta} : \Lambda \rightarrow \mathbb{R}$ defined by the formulae $\varphi_{\alpha, \beta}(p_1) \equiv 1$ and

$$\varphi_{\alpha, \beta}(p_m) = \sum_{j=1}^{\infty} \alpha_j^m + (-\theta)^{m-1} \sum_{j=1}^{\infty} \beta_j^m; \quad m = 2, 3, \dots \quad (5.3)$$

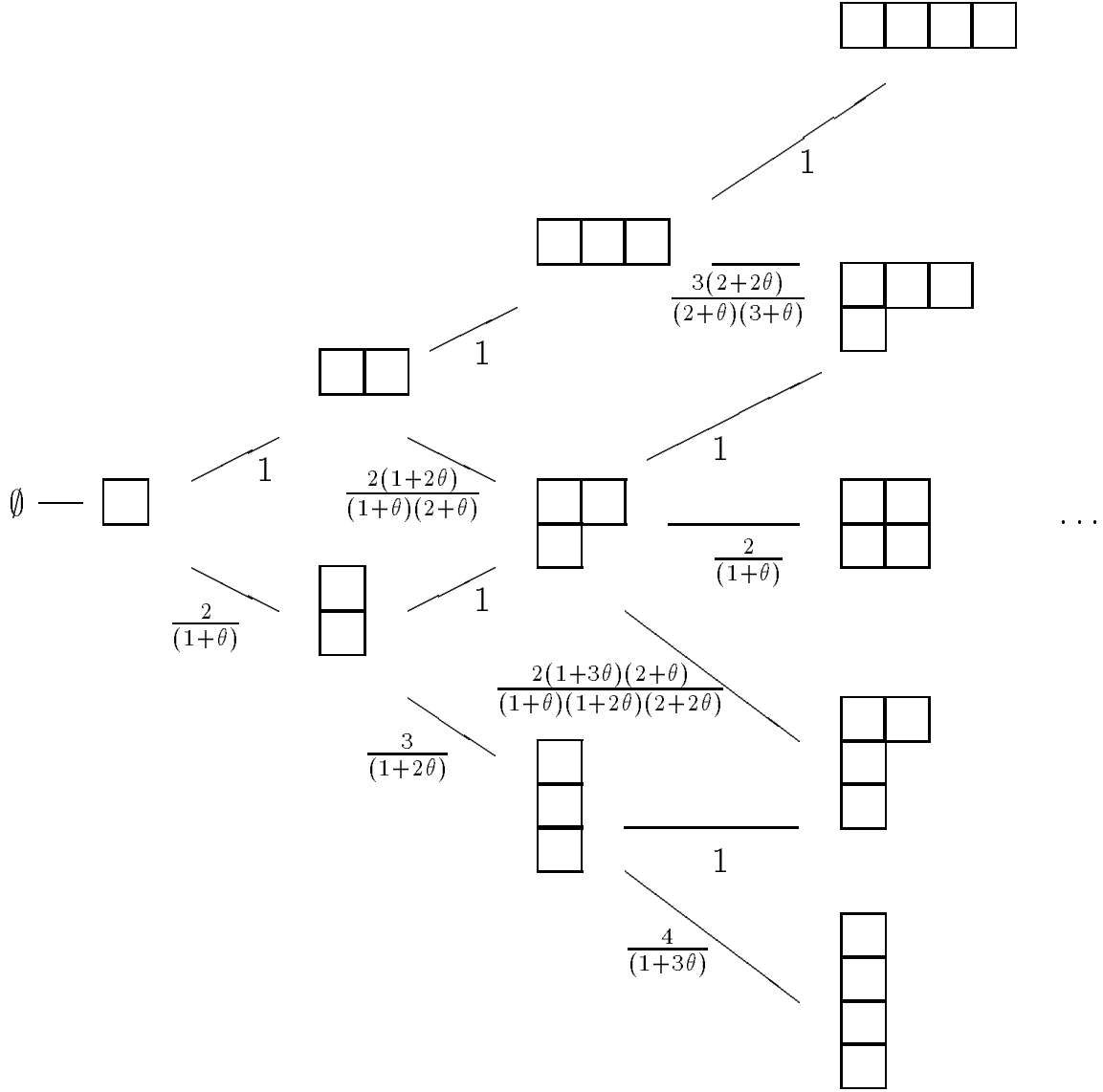


Fig. 3 The Jack's branching graph.

Definition. Given a symmetric polynomial $f \in \Lambda$, we consider a function

$$f(\alpha; \beta | \theta) = \varphi_{\alpha, \beta}(f), \quad (\alpha; \beta) \in \Omega, \quad (5.4)$$

defined on the Thoma simplex Ω , and we call it θ -extension of the polynomial f .

Note that if $\sum \alpha_i = 1$ then all β -variables vanish, $\beta = \mathbf{0}$, and the function $f(\alpha) = f(\alpha; \mathbf{0})$ is identical with the naive evaluation of the polynomial f at the sequence $\alpha = (\alpha_1, \alpha_2, \dots)$. In case of $\theta = 1$ the θ -extended functions coincide with the extended functions of Section A.3.

Theorem B. Assume that $\theta > 0$, and let $\omega = (\alpha; \beta)$ denote a point of Thoma simplex Ω . Then the θ -extension

$$K(\mu; \omega) = \varphi_\omega(P_\mu(\cdot; \theta)), \quad (5.5)$$

of the Jack symmetric polynomial $P_\mu(x; \theta)$ provides the Martin kernel of the Jack graph $\mathbb{J}(\theta)$. The integral representation

$$\varphi(\mu) = \int_{\Omega} K(\mu; \omega) M(d\omega) \quad (5.6)$$

establishes a one - to - one correspondence between the space \mathcal{H} of normalized, non - negative functions $\varphi : \mathbb{Y} \rightarrow \mathbb{R}$, harmonic with respect to Jack multiplicity function \varkappa_θ , and the space \mathcal{M} of probability measures M on the Thoma simplex Ω .

This Theorem will be proved in Section 8 as a corollary of some preliminary work in Sections 6, 7.

Besides the particular cases $\theta = 0, 1$ considered in Sections 3, 4 there is an interesting special case $\theta = 1/2$. The elements of $X(1/2)$ correspond to spherical unitary representations of the Gelfand pair consisting of the “even” infinite symmetric group $\mathfrak{S}(2\infty) = \varinjlim \mathfrak{S}_{2n}$ and its hyperoctahedral subgroup $\varinjlim \mathfrak{S}(n) \times \mathbb{Z}_2^n$. The description of the Martin boundary $X(1/2)$ was obtained (in representation-theoretical terms) in [18], see also [21].

We complete the present Section with a couple of properties of θ -extended symmetric functions to be used later on.

Lemma 5.2. Let $f \in \Lambda$ be a symmetric polynomial. Then its θ -extension $\omega \mapsto f(\omega|\theta)$ is continuous on the Thoma simplex Ω .

Proof. Trivially, the function $p_1(\alpha; \beta|\theta) \equiv 1$ is continuous. Since a sequence α is decreasing, we obtain inequalities $k\alpha_k \leq \alpha_1 + \dots + \alpha_k \leq 1$ and hence $\alpha_k \leq 1/k$. A similar inequality $\beta_k \leq 1/k$ holds for a sequence β . Given a positive $\varepsilon > 0$ and $m = 2, 3, \dots$, there exists such an N that

$$\left| \sum_{k=N}^{\infty} \alpha_k^m \right| + |\theta|^{m-1} \left| \sum_{k=N}^{\infty} \beta_k^m \right| \leq \varepsilon,$$

i.e., the N -th tail of the series for the functions $p_m(\alpha; \beta|\theta)$ is uniformly small on Ω . It follows that the functions $p_m(\alpha; \beta|\theta)$ are continuous. All other θ -extended symmetric functions are polynomials in the generators p_1, p_2, \dots , and the Lemma follows. \square

Lemma 5.3. Assume that $\theta \neq 0$. Then the algebra of θ -extended functions $\omega \mapsto f(\omega|\theta)$, $f \in \Lambda$, is uniformly dense in the space $C(\Omega)$ of continuous functions on the Thoma simplex.

Proof. By the Stone – Weierstrass theorem, it suffices to show that the functions $\omega \mapsto p_m(\omega|\theta)$ separate the points of the Thoma simplex Ω . One easily derives from the definition (5.3) the generating series

$$\sum_{m=1}^{\infty} \frac{p_m(\alpha; \beta|\theta)}{z^m} = \sum_{k=1}^{\infty} \frac{\alpha_k}{z - \alpha_k} + \sum_{k=1}^{\infty} \frac{\beta_k}{z + \theta\beta_k} + \frac{\gamma}{z}$$

for θ -extended power sum polynomials. Hence, the values of $\alpha_k, -\theta\beta_k$ can be restored as non-zero poles of this series. \square

Remark. Recall that the polynomial

$$J_\mu(x; \theta) = H'(\mu)P_\mu(x; \theta)$$

where $H'(\mu) = \prod_{s \in \mu} (a(s) + \theta l(s) + \theta)$ is called the *integral form* of the Jack polynomial [13]. Define the coefficients $\kappa_\theta(\mu, \lambda)$ as those in the Pieri formula for polynomials J_μ ,

$$p_1 \cdot J_\mu = \sum_{\lambda} \kappa_\theta(\mu, \lambda) J_\lambda.$$

Since

$$\kappa_\theta(\mu, \lambda) = \frac{H'(\mu)}{H'(\lambda)} \varkappa_\theta(\mu, \lambda), \quad (5.7)$$

the coefficients lead to an equivalent Martin boundary problem for the Young graph. On the other hand, the coefficients $\kappa_\theta(\mu, \lambda)$ have the following especially symmetric expression.

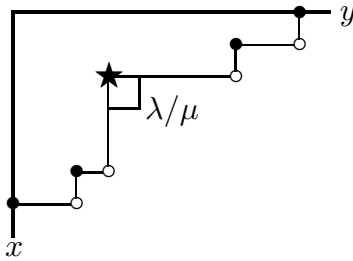


Fig. 4 Inner and outer corners of a diagram.

Fig. 4 represents inner and outer corners of the diagram μ . The corner of μ where the box λ/μ is attached to μ is marked by \star . One easily checks that the products in (5.2), (5.7) telescope to

$$\kappa_\theta(\mu, \lambda) = \frac{1}{\theta} \frac{\prod_{\text{outer corners } \circ \text{ of } \mu} r_\theta(\circ, \star)}{\prod_{\text{inner corners } \bullet \text{ of } \mu} r_\theta(\bullet, \star)}, \quad (5.8)$$

where the θ -axial distance r_θ between two points on the (x, y) plane is defined by

$$r_\theta((x_1, y_1) - (x_2, y_2)) := (y_1 - y_2) - \theta(x_1 - x_2),$$

and the product in the denominator of (5.8) ranges over all inner corners distinct from the corner \star .

For the coefficients $\varkappa_\theta(\mu, \lambda)$ there is a similar but less symmetric expression.

6. Shifted Jack polynomials and the dimension formula. In this Section we establish the first major ingredient in our proof of Theorem B – the explicit formula for the Martin kernel of the Jack graph $\mathbb{J}(\theta)$, found in the paper [20]. To this end we survey some necessary facts from [20] concerning θ -shifted symmetric polynomials in general, and θ -shifted Jack symmetric polynomials in particular.

Denote by $\Lambda^\theta(n)$ the subalgebra of $\mathbb{R}[x_1, \dots, x_n]$ formed by the polynomials symmetric in the ‘shifted’ variables $x'_j = x_j - \theta j$, $j = 1, \dots, n$. We define the projection map $\Lambda^\theta(n) \rightarrow \Lambda^\theta(n-1)$ as the specialization $x_n = 0$, and note that this projection preserves the filtration defined by ordinary degree of polynomials. The projective limit $\Lambda^\theta = \varprojlim \Lambda^\theta(n)$ in the category of filtered algebras is referred to as the *algebra of θ -shifted symmetric polynomials*. The degree of an element $F \in \Lambda^\theta$ is denoted as $\deg F$.

Each element $F \in \Lambda^\theta$ can be evaluated at any sequence $x = (x_1, x_2, \dots)$ with finitely many non-zero terms. In particular, one can evaluate F at any integer partition $\nu \in \mathbb{Y}$, which will be important in what follows.

For each polynomial $F \in \Lambda^\theta$, we denote by $[F] \in \Lambda$ its *leading symmetric term* which is a homogeneous symmetric polynomial of degree $\deg F$. The map $F \mapsto [F]$ provides a canonical isomorphism of the graded algebra $\text{gr } \Lambda^\theta$ associated to the filtered algebra Λ^θ , onto the symmetric function algebra Λ . Assuming that the leading terms $[F_1], [F_2], \dots$ of a sequence $F_1, F_2, \dots \in \Lambda^\theta$ generate the algebra Λ , one readily derives that the latter polynomials generate the algebra Λ^θ .

For an elementary example of a θ -shifted symmetric function, consider a polynomial

$$p_m^*(x; \theta) = \sum_{j=1}^{\infty} \left((x_j - \theta j)^m - (-\theta j)^m \right), \quad (6.1)$$

a shifted analog of the power sum symmetric function $p_m = \sum x_j^m$. Since $[p_m^*] = p_m$, the polynomials p_1^*, p_2^*, \dots generate the algebra Λ^θ .

We are interested in θ -shifted counterparts $P_\mu^*(x; \theta)$ of the Jack symmetric polynomials $P_\mu(x; \theta)$. These polynomials were studied in [12], [17] and [20]; we refer to these papers for a detailed exposition (see also [20] which is dedicated to the special case $\theta = 1$).

The polynomial $P_\mu^*(x; \theta)$ can be characterized as the unique element of the algebra Λ^θ , such that $\deg P_\mu^* = |\mu|$ and

$$P_\mu^*(\lambda; \theta) = \begin{cases} H(\mu), & \lambda = \mu, \\ 0, & \mu \not\subseteq \lambda. \end{cases}$$

Here $H(\mu) = \prod (\mu_i - j + \theta(\mu'_j - i) + 1)$, the product runs over all boxes of the Young diagram μ , the length of the i -th row is denoted by μ_i , and μ'_j is the length of the j -th column of μ . There is also an explicit combinatorial formula for θ -shifted Jack polynomials,

$$P_\mu^*(x; \theta) = \sum_T \psi_T(\theta) \prod_{b \in \mu} (x_{T(b)} - c_\theta(b)), \quad (6.2)$$

similar to the formula (A.7) for ordinary Jack polynomials. In this formula $c_\theta(b) = (j-1) - \theta(i-1)$ stands for the θ -content of the box $b = (i, j)$ on the crossing of the i -th row and j -th column of μ . Note that T ranges in (6.2) over the set of *reverse* tableaux of shape μ . See [17] for details and for the proof of the formula.

It follows directly from the equations (A.7) and (6.2) that the leading symmetric term of the θ -shifted Jack polynomial P_μ^* equals the ordinary Jack polynomial, $[P_\mu^*] = P_\mu$.

We denote by $\dim_\theta(\mu, \nu)$ the dimension function (2.2) with respect to the Jack multiplicity function (5.2), and we use the abbreviation $\dim_\theta \nu \equiv \dim_\theta(\emptyset, \nu)$. A nice *hook formula* is available for θ -dimension of true Young diagrams:

$$\dim_\theta \nu = \frac{|\nu|!}{H_\theta(\nu)} = |\nu|! \prod_{(i,j) \in \nu} ((\nu_i - j) + (\nu'_j - i)\theta + 1)^{-1}.$$

This formula can be deduced from the results of Stanley ([22], Theorem 5.4) and Macdonald ([13], VI.10). To this end we remark that, by definition of dimension function,

$$p_1^n(\cdot) = \sum_{\lambda \in \mathbb{Y}_n} \dim_\theta \lambda P_\lambda(\cdot; \theta),$$

hence

$$\dim_\theta \lambda = \frac{(p_1^n, P_\lambda)}{(P_\lambda, P_\lambda)}.$$

Consider the representation $P_\lambda = \sum_\mu c_\mu p_\mu$ in the basis of power sum functions. Then the coefficient $c = c_{(1^n)}$ of p_1^n equals the leading term in X in the formula (10.20) of [13], VI.10. It follows that the numerator in the last formula is $c(p_1^n, p_1^n) = c n! \alpha^n$. One should also use the identity (10.16) in [13] for $b_\lambda = (P_\lambda, P_\lambda)^{-1}$. For another proof see [20], Section 5.

The importance of θ -shifted Jack polynomials for the evaluation of Martin kernel of the Jack graph $\mathbb{J}(\theta)$ roots in the following basic dimension formula.

Theorem 6.1 ([20], (5.2)). *Let μ and ν be arbitrary Young diagrams and $m = |\mu|$, $n = |\nu|$. Then the Martin kernel (2.3) of the Young graph with Jack edge multiplicities (5.2) can be written in the form*

$$\frac{\dim_\theta(\mu, \nu)}{\dim_\theta \nu} = \frac{P_\mu^*(\nu; \theta)}{n(n-1)\dots(n-m+1)}. \quad (6.3)$$

Remarks. In the important particular case of $\theta = 1$ the formula (6.3) was first established in [20], Section 8. The polynomials $s_\mu^*(x) = P_\mu^*(x; 1)$ are called *shifted Schur polynomials*, they were defined in [20] by a simple determinantal formula

$$s_\mu^*(x_1, \dots, x_n) = \frac{\det[(x_i + n - i \mid \mu_j + n - j)]}{\det[(x_i + n - i \mid n - j)]}, \quad (6.4)$$

similar to the well-known Weyl formula for the ordinary Schur functions. Here and below we use the symbol

$$(x \mid m) = x(x-1)\dots(x-m+1)$$

for the falling factorial powers of x .

Since there is an explicit expression for $\dim_\theta \nu$, formula (6.3) yields an expression for the number $\dim_\theta(\mu, \nu)$. In case of $\theta = 1$, we obtain (thanks to the determinantal formula (6.4)) a simple algebraic expression for the number $\dim_1(\mu, \nu)$ of standard Young tableaux of the *skew* shape $\nu \setminus \mu$. ”

7. The asymptotics of θ -shifted symmetric polynomials. The second major ingredient in our proof of Theorem B is the asymptotic formula (7.4) to be proved in the present Section.

Given a box $b = (i, j)$ the number

$$c_\theta(b) = (j - 1) - \theta(i - 1) \quad (7.1)$$

is referred to as θ -content of the box b . A box b is said to be *positive* or *negative* according to the sign of its θ -content. More precisely, b is positive if $c_\theta(b) > 0$, and negative if $c_\theta(b) \leq 0$.

We may consider a Young diagram $\nu = (\nu_1, \dots, \nu_l)$ as a collection of boxes,

$$\nu \equiv \{b_{ij} : 1 \leq i \leq l, 1 \leq j \leq \nu_i\},$$

and we split ν as a union of disjoint subsets of its positive and negative boxes,

$$\nu^+ = \{b \in \nu \mid c_\theta(b) > 0\}, \quad \nu^- = \{b \in \nu \mid c_\theta(b) \leq 0\}.$$

Denote by r the number of rows in ν^+ , and by s the number of columns in ν^- . Let

$$a_1 \geq a_2 \geq \dots \geq a_r > 0; \quad b_1 \geq b_2 \geq \dots \geq b_s > 0 \quad (7.2)$$

denote the lengths of corresponding rows and columns. Clearly,

$$\sum_{i=1}^r a_i + \sum_{j=1}^s b_j = n$$

is the total number of boxes in the diagram ν .

Given a Young diagram ν , we associate with it a point

$$\omega_\nu(\theta) = \left(\frac{a_1}{n}, \dots, \frac{a_r}{n}, 0, \dots; \frac{b_1}{n}, \dots, \frac{b_s}{n}, 0, \dots \right) \quad (7.3)$$

of the Thoma simplex Ω .

Recall that to any symmetric polynomial f there corresponds a continuous function $\omega \mapsto f(\omega|\theta)$ on the Thoma simplex Ω defined by the equation (5.4). We shall show that the normalized value $F(\nu)/n^m$ of a θ -shifted symmetric polynomial $F \in \Lambda^\theta$ of degree m at a Young diagram ν with n boxes gets close to the value $f(\omega_\nu(\theta)|\theta)$ of its leading symmetric term $f = [F]$ at the point $\omega_\nu(\theta) \in \Omega$, as $n \rightarrow \infty$.

Theorem 7.1. Denote by ν a Young diagram with n boxes, and by $F \in \Lambda^\theta$ a θ -shifted symmetric polynomial of degree $\deg F = m$ with the leading symmetric term $f = [F] \in \Lambda$. Then

$$\left| \frac{F(\nu)}{n^m} - f(\omega_\nu(\theta)|\theta) \right| \leq \frac{C}{\sqrt{n}}, \quad (7.4)$$

where the constant C depends on F and θ , but not on n .

Proof. It suffices to prove (7.4) for the polynomials $F = \tilde{p}_m$, where

$$\tilde{p}_m(x) = \sum_{i=1}^{\infty} \left((x_i - \theta(i-1) \downarrow m) - (-\theta(i-1) \downarrow m) \right)$$

is a variant of θ -shifted power sum polynomial, similar to that of (6.1). Note that the leading symmetric term of \tilde{p}_m equals $[\tilde{p}_m] = p_m$, the conventional power sum symmetric function. The polynomials p_m generate the algebra Λ , hence every element $F \in \Lambda^\theta$ is a polynomial in the variables $\tilde{p}_1, \tilde{p}_2, \dots$. Since both functions $\nu \mapsto F(\nu)/n^m$, $\nu \mapsto f(\omega_\nu(\theta); \theta)$ are bounded on \mathbb{Y} , the Theorem will follow.

Recall (see Section A.1) that $(x \downarrow m)$ denotes the descending factorial power of x . The basic idea of the proof is to evaluate a sum

$$S_m = m \sum_{b \in \nu} (c_\theta(b) \downarrow m - 1)$$

in two different ways (a similar approach was employed in [8]).

First of all we note that S_m coincides with the value of \tilde{p}_m at the diagram ν . Indeed, since $(c+1 \downarrow m) - (c \downarrow m) = m(c \downarrow m - 1)$, we get an elementary summation formula

$$m \sum_{c=c_{\min}}^{c_{\max}-1} (c \downarrow m - 1) = (c_{\max} \downarrow m) - (c_{\min} \downarrow m). \quad (7.5)$$

Splitting the diagram ν into its rows, and applying (7.5) to each row, we see that

$$S_m = \sum_{i \geq 1} m \sum_{b \in \nu_i} (c_\theta(b) \downarrow m - 1) = \sum_{i \geq 1} ((\nu_i - \theta(i-1) \downarrow m) - (-\theta(i-1) \downarrow m)) = \tilde{p}_m(\nu).$$

In our second calculation of the sum S_m we find partial sums along the rows of the positive part ν^+ , and along the columns of the negative part ν^- of the diagram ν .

Lemma 7.1. For every $m = 1, 2, \dots$,

$$\left| \frac{m}{n^m} \sum_{b \in \nu^+} (c_\theta(b) \downarrow m - 1) - \sum_{i=1}^r \left(\frac{a_i}{n} \right)^m \right| \leq \frac{C_1}{\sqrt{n}},$$

where the constant C_1 does not depend on n .

Proof. Denote by c_i the θ -content (7.1) of the leftmost box in the i -th row ν_i^+ of the shape ν^+ . By the summation formula (7.5),

$$S_m^+(\nu) \equiv m \sum_{b \in \nu_i^+} (c_\theta(b) \downarrow m - 1) = (a_i + c_i \downarrow m) - (c_i \downarrow m). \quad (7.6)$$

An important feature of the partition $\nu = \nu^+ \cup \nu^-$ is that the θ -contents of “diagonal” boxes are uniformly bounded, $-1 < c_i \leq 1$. As a result, one can derive from the equation (7.6) an estimate

$$\left| m \sum_{b \in \nu_i^+} (c_\theta(b) \downarrow m - 1) - a_i^m \right| \leq \text{const } a_i^{m-1}, \quad (7.7)$$

where *const* depends on m , but not on n and i .

Remark that the Young diagram ν always contains a rectangle on the crossing of its first r rows and s columns. Since $r \approx \theta s$, it follows that

$$r \leq \text{const } \sqrt{n}, \quad s \leq \text{const } \sqrt{n}, \quad (7.8)$$

where *const* depends on θ only.

Now divide both sides of (7.7) by n^m and take into account the inequalities $a_i \leq n$. We obtain an estimate

$$\left| \frac{m}{n^m} \sum_{b \in \nu_i^+} (c_\theta(b) \downarrow m - 1) - \left(\frac{a_i}{n} \right)^m \right| \leq \frac{\text{const}}{n}$$

uniform in i , and the Lemma follows from the inequalities (7.8). \square

Let us now deal with the column sum

$$S_m^-(\nu) = m \sum_{b \in \nu^-} (c_\theta(b) \downarrow m - 1).$$

We reduce its evaluation to that of the row sum

$$S_m^+(\mu) = m \sum_{b \in \mu^+} (c_\theta(b) \downarrow m - 1)$$

for the transposed Young diagram $\mu = \nu'$. Remark that the positive part μ^+ of μ is now taken with respect to the $1/\theta$ -content. The transposition map $(i, j) \mapsto (j, i)$ provides a bijection $b \mapsto b'$ between the boxes in ν^- and those in μ^+ (up to a minor asymmetry in the definitions of μ^+ and ν^- which is not essential for our purposes). Clearly,

$$c_\theta(b) = (j - 1) - \theta(i - 1) = (-\theta)((i - 1) - \theta^{-1}(j - 1)) = (-\theta) c_{1/\theta}(b').$$

Lemma 7.2. *Let $\mu = \nu'$ denote the transposed diagram of the Young diagram ν . Then*

$$\frac{m}{n^m} \left| \sum_{b \in \nu^-} (c_\theta(b) \downarrow m - 1) - (-\theta)^{m-1} \sum_{b \in \mu^+} (c_{1/\theta}(b') \downarrow m - 1) \right| \leq \frac{C_2}{\sqrt{n}},$$

where the constant C_2 depends on m and θ , but not on n .

Proof. Note that $(-\theta c \downarrow m - 1)$ and $(-\theta)^{m-1}(c \downarrow m - 1)$ are both polynomials of degree $m - 1$ in the variable c , with one and the same leading term $(-\theta c)^{m-1}$. It follows that

$$\left| \sum_{b \in \nu_j^-} \left((c_\theta(b) \downarrow m - 1) - (-\theta)^{m-1}(c_{1/\theta}(b') \downarrow m - 1) \right) \right| \leq \text{const } b_j^{m-1},$$

where *const* does not depend on n . Using inequalities $b_j \leq n$, we get

$$\frac{m}{n^m} \left| \sum_{b \in \nu_j^-} \left((c_\theta(b) \downarrow m - 1) - (-\theta)^{m-1}(c_{1/\theta}(b') \downarrow m - 1) \right) \right| \leq \frac{\text{const}}{n},$$

and the Lemma follows from the estimates (7.8). \square

By Lemma 7.1, we obtain formulae

$$\frac{m}{n^m} \sum_{b \in \nu^+} (c_\theta(b) \downarrow m - 1) = \sum_{i=1}^r \left(\frac{a_i}{n} \right)^m + O\left(\frac{1}{\sqrt{n}} \right)$$

and

$$\frac{m}{n^m} \sum_{b \in \mu^+} (c_{1/\theta}(b) \downarrow m - 1) = \sum_{j=1}^s \left(\frac{b_j}{n} \right)^m + O\left(\frac{1}{\sqrt{n}} \right),$$

so that the Lemma 7.2 and the equation $S_m = S_m^+(\nu) + S_m^-(\nu) = \tilde{p}_m(\nu)$ imply

$$\frac{\tilde{p}_m(\nu)}{n^m} = \sum_{i=1}^r \left(\frac{a_i}{n} \right)^m + (-\theta)^{m-1} \sum_{j=1}^s \left(\frac{b_j}{n} \right)^m + O\left(\frac{1}{\sqrt{n}} \right).$$

which completes the proof of the Theorem. \square

In the course of the proof of Theorem 7.1 we have used the notation $\omega_\nu(\theta)$ for a point of the Thoma simplex introduced by equation (7.3). It depends on the parameter θ and differs from the point $\omega_\nu = \omega_\nu(1)$ introduced in a similar way by equation (3.3). Let us now remark that if $\theta \neq 0$ and $n = |\nu|$ goes to infinity, the elements $\omega_\nu(\theta)$ and ω_ν are asymptotically equivalent. In fact, for each fixed $k = 1, 2, \dots$ the corresponding coordinates only differ by a constant (depending on k).

Corollary 7.1. *Given a sequence $\nu^{(n)} \in \mathbb{Y}_n$ of Young diagrams, $n = 1, 2, \dots$, assume that their images (3.3) in the Thoma simplex converge to a point $\omega \in \Omega$. Then, in the notations of Theorem 7.1,*

$$\lim_{n \rightarrow \infty} \frac{F(\nu^{(n)})}{n^m} = f(\omega|\theta). \quad (7.9)$$

8. Regular sequences of Young diagrams. In this Section we identify the Martin compactification of the Jack graph $\mathbb{J}(\theta)$, $\theta > 0$, with the space

$$\tilde{\Omega} = (\{0\} \times \Omega) \cup \tilde{\mathbb{Y}} \subset [0, 1] \times \Omega \quad (8.1)$$

(considered as a sort of geometric compactification). Recall that the discrete subset $\tilde{\mathbb{Y}} \subset [0, 1] \times \Omega$ was introduced in Section 3 by the equations (3.3) and (3.4). Both spaces $\tilde{\mathbb{Y}}$ and $\tilde{\Omega}$ were defined with no reference to θ .

Theorem 8.1. *Let $\nu^{(n)}$, $n = 1, 2, \dots$ be a sequence of Young diagrams. Then the following two conditions are equivalent:*

- (i) *the sequence $(|\nu^{(n)}|^{-1}, \nu^{(n)})$ converges in $\tilde{\Omega}$, as $n \rightarrow \infty$, to a point $\tilde{\omega} \in \tilde{\Omega}$;*
- (ii) *for each Young diagram μ there exists the limit $K(\mu, \tilde{\omega}) = \lim_{n \rightarrow \infty} K(\mu, \nu^{(n)})$.*

The Martin kernel of the Jack graph $\mathbb{J}(\theta)$ is provided by the θ -extended versions of Jack symmetric polynomials,

$$K(\mu, \omega) = P_\mu(\omega|\theta). \quad (8.2)$$

Here $P_\mu(\omega|\theta) = \varphi_{\alpha, \beta}(P_\mu(\cdot; \theta))$ denotes the value of θ -extended Jack polynomial indexed by μ at the point $\omega = (\alpha; \beta)$ of Thoma simplex, see Definition (5.3).

Proof. Let us write $N(n) = |\nu^{(n)}|$ for the number of boxes in a diagram $\nu^{(n)}$. Since the set $\tilde{\mathbb{Y}}$ is discrete, there will be no loss of generality in assuming that $\lim N(n) = \infty$. Assume also that the condition (i) holds, so that the sequence $\omega_{\nu^{(n)}}$ converges in the Thoma simplex Ω to a point $\omega = (\alpha; \beta)$. By definition of convergence in Ω this means that

$$\lim_{n \rightarrow \infty} \frac{(\nu^{(n)})_k}{N(n)} = \alpha_k; \quad \lim_{n \rightarrow \infty} \frac{(\nu^{(n)})'_k}{N(n)} = \beta_k; \quad k = 1, 2, \dots \quad (8.3)$$

By the equation (6.3) we know that

$$K(\mu, \nu^{(n)}) = \frac{P_\mu^*(\nu^{(n)}; \theta)}{N(n)^m} + O\left(\frac{1}{N(n)}\right),$$

for all $\mu \in \mathbb{Y}_m$, $m = 1, 2, \dots$. Applying Corollary 7.1 with $F(\nu) = P_\mu^*(\nu, \theta)$, we derive that

$$\lim_{n \rightarrow \infty} K(\mu, \nu^{(n)}) = P_\mu(\omega|\theta)$$

for every $\mu \in \mathbb{Y}$.

In the opposite direction, assume that the condition (ii) holds. Since the space Ω is compact, one can choose a subsequence of the sequence $\nu^{(n)}$, converging to a point $\omega \in \Omega$. But the Jack polynomials form a linear basis in the algebra Λ , and the θ -extended

symmetric functions corresponding to polynomials in Λ are uniformly dense in the space $C(\Omega)$ of continuous functions on the Thoma simplex (by Lemma 5.3). Hence, all partial limits of the sequence $\nu^{(n)}$ are equal, and the condition (i) holds, too. \square

The sequences $\nu^{(n)} \in \mathbb{Y}_n$ of Young diagrams subject to the condition (ii) of Theorem 8.1 are called *regular sequences*.

Corollary 8.1. *Assume that a regular sequence $\nu^{(n)} \in \mathbb{Y}_n$ converges to a point $\omega \in \Omega$. Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\dim_{\theta}(\mu, \nu^{(n)})}{\dim_{\theta} \nu^{(n)}} = P_{\mu}(\omega|\theta) \quad (8.4)$$

exists for all $\mu \in \mathbb{Y}$.

Corollary 8.2. *For every point $\omega \in \Omega$ the function $\mu \mapsto P_{\mu}(\omega|\theta)$ is non-negative, normalized, and harmonic with respect to the Jack multiplicities (5.2).*

Proof. The harmonicity condition (2.1) coincides with the Pieri formula (5.1) for Jack symmetric polynomials. The positivity follows from Corollary 8.1, since the dimension functions are non-negative. \square

The remaining part of the proof of Theorems A, B is quite standard and general; it does not depend on the specific features of Jack graphs.

Lemma 8.1. *Let $\phi \in \mathcal{H}$ be a normalized non-negative harmonic function. Then*

$$M_n(\lambda) = \dim_{\theta} \lambda \phi(\lambda), \quad \lambda \in \mathbb{Y}_n \quad (8.5)$$

is a probability distribution.

Proof. By the harmonicity condition (2.1),

$$M_{n-1}(\lambda) = \sum_{\nu: \lambda \nearrow \nu} \frac{\dim_{\theta} \lambda \kappa_{\theta}(\lambda, \nu)}{\dim_{\theta} \nu} M_n(\nu),$$

for every diagram $\lambda \in \mathbb{Y}_{n-1}$. By the definition (2.2) of the dimension function,

$$\sum_{\lambda: \lambda \nearrow \nu} \frac{\dim_{\theta} \lambda \kappa_{\theta}(\lambda, \nu)}{\dim_{\theta} \nu} = 1,$$

which implies

$$\sum_{\nu \vdash n} M_n(\nu) = \sum_{\lambda \vdash (n-1)} M_{n-1}(\lambda),$$

and hence $\sum_{\nu \vdash n} M_n(\nu) = 1$ for all $n = 1, 2, \dots$ \square

Proof of Theorem B. By the Corollary 8.2, a function

$$\varphi(\mu) = \int_{\Omega} K(\mu, \omega) M(d\omega); \quad \mu \in \mathbb{Y}, \quad (8.6)$$

is an element of \mathcal{H} , for every probability distribution M on the Thoma simplex Ω . By Lemma 5.3 such an integral representation is unique. It only remains to find a measure M representing a given harmonic function φ .

We show that for every function $\varphi \in \mathcal{H}$ there exists a weak limit $M = \lim \widetilde{M}_n$, where \widetilde{M}_n denotes the image of the discrete probability distribution (8.5) with respect to the embedding (3.3) of the set \mathbb{Y}_n into the Thoma simplex Ω . The measure M will provide the representation (8.6) of the function φ .

In fact, it follows from (2.1) and (8.5) that

$$\varphi(\mu) = \sum_{\nu \in \mathbb{Y}_n} \frac{\dim_{\theta}(\mu, \nu)}{\dim_{\theta} \nu} M(\nu),$$

for every fixed $\mu \in \mathbb{Y}_m$ and $n > m$. Using equations (6.3) and (7.4) we derive that

$$\varphi(\mu) = \int_{\Omega} P_{\mu}(\omega|\theta) \widetilde{M}_n(d\omega) + O(1/\sqrt{n}). \quad (8.7)$$

Let us choose such a subsequence that the measures \widetilde{M}_n converge to a limiting measure M . Then according to the formula (8.7)

$$\varphi(\mu) = \int_{\Omega} P_{\mu}(\omega|\theta) M(d\omega),$$

and the measure M represents the harmonic function φ . Note that the integral representation is unique, hence the limit $\lim \widetilde{M}_n = M$ does exist. \square

Proof of Theorem A. It is clear that every point $(\alpha, \beta) \in \Omega$ determines a positive evaluation homomorphism $\varphi_{\alpha, \beta}(f) = f(\alpha; \beta)$, $f \in \Lambda$.

By Theorem B, every positive homomorphism $\varphi : \Lambda \rightarrow \mathbb{R}$ (normalized by the condition $\varphi(p_1) = 1$) can be written in the form $\varphi(f) = \int f dM$, for appropriate probability distribution M on the Thoma simplex Ω . We have to show that M is a δ -measure. Assuming this is not true, there exists a symmetric function $f \in \Lambda$ with a non-trivial distribution with respect to the measure M . But the variance of F vanishes,

$$\text{var}(f) = \int_{\Omega} f^2 dM - \left(\int_{\Omega} f dM \right)^2 = \varphi(f^2) - \varphi^2(f) = 0,$$

so that the assumption leads to a contradiction. (The argument is a part of the proof of Theorem 6 in [10].) This implies Theorem A. \square

APPENDIX

We recall in this Section the basic background notation and terminology related to integer partitions and symmetric functions, see [13] for a detailed exposition.

A.1. Young diagrams. Every decomposition $\lambda : n = \lambda_1 + \lambda_2 + \dots + \lambda_l$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l$ are positive integers, is called an *integer partition* of length $l = l(\lambda)$. We identify integer partitions with corresponding Young diagrams, and we denote by $n = |\nu|$ the number of boxes in λ . The set of Young diagrams with n boxes is denoted as \mathbb{Y}_n , and $\mathbb{Y} = \bigcup \mathbb{Y}_n$ is the set of all Young diagrams. We write $m_j = m_j(\lambda)$ for the number of terms in λ equal to j , and we set $z_\lambda = \prod_{j \geq 1} j^{m_j} m_j!$. The transposed (conjugate) diagram of ν is denoted by ν' ; by definition, $\nu'_j = \#\{i : \nu_i \geq j\}$.

The Young diagrams in \mathbb{Y} are naturally ordered by inclusion, and we shall write $\lambda \nearrow \nu$ to indicate that the diagram ν covers λ with respect to this order. Another useful ordering of Young diagrams, denoted $\lambda \geq \nu$, is called *dominance order*. By definition, $\lambda \geq \nu$ iff $|\lambda| = |\nu|$ and $\lambda_1 + \dots + \lambda_k \geq \nu_1 + \dots + \nu_k$ for all $k \geq 1$.

We use the symbol $(x \downarrow m) = x(x-1)\dots(x-m+1)$ for the descending factorial powers of a variable x .

A.2. Symmetric functions. The \mathbb{R} -algebra of all symmetric polynomials in the variables $x = (x_1, x_2, \dots)$ is denoted by Λ . To each monomial $x_1^{r_1} x_2^{r_2} \dots$ of degree n there corresponds a partition $n = (\lambda_1, \lambda_2, \dots)$, the rearrangement of the exponents r_1, r_2, \dots in the decreasing order. The monomial symmetric function $m_\lambda(x)$ is defined as the sum of all monomials with a fixed partition λ . In particular, the functions

$$p_m(x) = m_{(n)}(x) = \sum_{j=1}^{\infty} x_j^m$$

are called the *power sum symmetric polynomials*, and we set $p_\lambda(x) = \prod_{j \geq 1} p_j^{m_j(\lambda)}$, for every integer partition λ . Each one of the families $\{m_\lambda\}$, $\{p_\lambda\}$ forms a linear basis in the algebra Λ .

A.3. Extended symmetric functions. The *extended Schur functions* $s_\lambda(\alpha, \beta)$ can be formally defined by the Jacoby – Trudi determinant

$$s_{(\lambda_1, \dots, \lambda_m)} = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+m-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+m-2} \\ h_{\lambda_3-2} & h_{\lambda_3-1} & h_{\lambda_3} & \dots & h_{\lambda_3+m-3} \\ \dots & \dots & \dots & \dots & \dots \\ h_{\lambda_m-m+1} & h_{\lambda_m-m+2} & h_{\lambda_m-m+3} & \dots & h_{\lambda_m} \end{vmatrix}, \quad (\text{A.1})$$

where the *extended complete homogeneous symmetric functions* $h_n = h_n(\alpha, \beta)$ arise as the coefficients of the generating series

$$e^{z\gamma} \prod_{j=1}^{\infty} \frac{1+z\beta_j}{1-z\alpha_j} = 1 + \sum_{k=1}^{\infty} h_k(\alpha, \beta) z^k. \quad (\text{A.2})$$

More generally, an *extended version* of a symmetric polynomial may be defined as follows. Recall that every symmetric function $f = f(\alpha)$ is a polynomial in the power sum symmetric functions $p_m(\alpha) = \sum \alpha_j^m$, $m = 1, 2, \dots$. In order to obtain the extended

version $f(\alpha; \beta)$ of the function f , one only has to replace each occurrence of $p_1(\alpha)$ in the corresponding polynomial by 1, and each occurrence of $p_m(\alpha)$, $m \geq 2$, by the *extended power sum polynomials*

$$p_m(\alpha; \beta) = \sum_{j=1}^{\infty} \alpha_j^m + (-1)^{m-1} \sum_{j=1}^{\infty} \beta_j^m. \quad (\text{A.3})$$

A.4. Jack polynomials. Let us recall a few basic facts related to Jack symmetric polynomials $P_\mu(x; \theta)$. Consider a scalar product on the algebra Λ defined by the equation

$$(p_\mu, p_\lambda)_\theta = \delta_{\mu, \lambda} z_\lambda \theta^{-l(\lambda)} \quad (\text{A.4})$$

(note that we use the parameter θ inverse to that employed by Macdonald in [13]).

The polynomials $P_\mu(x; \theta)$ can be characterized by the following properties:

$$(P_\lambda(\cdot; \theta), P_\nu(\cdot; \theta))_\theta = 0 \quad \text{unless } \lambda = \nu; \quad (\text{A.5})$$

$$P_\lambda(x; \theta) = m_\lambda(x) + \sum_{\mu: \mu < \lambda} c_{\lambda\mu}(\theta) m_\mu(x), \quad (\text{A.6})$$

where $c_{\lambda\mu}(\theta)$ are rational functions of θ . In particular, the polynomials $P_{(1^m)}(x; \theta) = m_{(1^m)}(x)$ do not depend on θ .

There is a number of equivalent definitions of Jack polynomials; for instance,

$$P_\mu(x; \theta) = \sum_T \psi_T(\theta) \prod_{b \in \mu} x_{T(b)}, \quad (\text{A.7})$$

where $\psi_T(\theta)$ is an explicit rational expression in θ , see [13], VI.10.12.

In the limit $\theta \rightarrow 0$ the polynomials $P_\mu(x; 0)$ degenerate to monomial symmetric functions $m_\mu(x)$, see [13], p.380. The polynomials $s_\mu(x) = P_\mu(x; 1)$ coincide with the Schur symmetric functions.

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