

EIGENVALUES, PSEUDOSPECTRUM AND STRUCTURED PERTURBATIONS

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Abstract. We investigate the behavior of eigenvalues under structured perturbations. We show that for many common structures such as (complex) symmetric, Toeplitz, symmetric Toeplitz, circulant and others the structured condition number is equal to the unstructured condition number for normwise perturbations, and prove similar results for real perturbations. An exception are complex skewsymmetric matrices. We also investigate componentwise complex and real perturbations. Here Hermitian and skew-Hermitian matrices are exceptional for real perturbations. Furthermore we characterize the structured (complex and real) pseudospectrum for a number of structures and show that often there is little or no significant difference to the usual, unstructured pseudospectrum.

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1. Introduction. Let A be a complex $n \times n$ matrix. We will investigate the behavior of the eigenvalues of A with respect to structured perturbations. We first look at the condition number, i.e. infinitely small perturbations, and then at the pseudospectrum, i.e. finite perturbations. For the condition number, assume λ to be a simple eigenvalue of A with (normalized) right and left eigenvectors x and y , respectively, i.e.

$$(1.1) \quad Ax = \lambda x, \quad y^H A = \lambda y^H \quad \text{with } \|x\| = \|y\| = 1.$$

Throughout this paper $\|\cdot\|$ denotes the spectral norm $\|\cdot\|_2$, for vectors and for matrices. For a perturbation ΔA of A with $\|\Delta A\| \leq \varepsilon$ and sufficiently small ε , the eigenvalue λ is uniquely perturbed into some $\lambda + \Delta\lambda$. Hence a commonly used definition [16] of the condition number of λ is

$$(1.2) \quad \kappa(A, \lambda) := \limsup_{\varepsilon \rightarrow 0} \left\{ \frac{|\Delta\lambda|}{\varepsilon} : \Delta A \in \mathbb{C}^{n \times n}, \|\Delta A\| \leq \varepsilon, \lambda + \Delta\lambda \in \Lambda(A + \Delta A) \right\},$$

where $\Lambda(A)$ denotes the spectrum of A . It is well known [11] that $\kappa(A, \lambda) = 1/|y^H x|$. To underline that perturbations are complex, we also use $\kappa_{\mathbb{C}}(A, \lambda)$. The condition number for perturbations restricted to real ones is denoted by $\kappa_{\mathbb{R}}(A, \lambda)$ and can decrease $\kappa_{\mathbb{C}}(A, \lambda)$ by at most a factor $1/\sqrt{2}$ [7]. Our definition of the condition number reflects the *absolute* change of λ ; for a *relative* condition number of $\lambda \neq 0$ divide by $|\lambda|$. This is not important for this paper because we are interested in the difference between the condition numbers for general and for structured perturbations.

It seems reasonable for a structured matrix, for example symmetric or Toeplitz or circulant, to ask for the sensitivity of λ with respect to structure-preserving perturbations. This leads to the structured condition number. For linear structures this has been investigated in [16, 13], for other structures see [19, 21]. In the present paper we will treat several linear structures. Some of these structures but also others like Hamiltonians have been investigated in [29, 22]. Let

$$(1.3) \quad \text{struct} \in \{\text{sym, Herm, skewsym, skewHerm, persym, Toep, symToep, Hankel, persymHankel, circ}\}$$

denote structures such that $A \in M_{\mathbb{C}}^{\text{struct}}$ implies $A \in \mathbb{C}^{n \times n}$ to be symmetric, Hermitian, skewsymmetric, skew-Hermitian, persymmetric, (general) Toeplitz, symmetric Toeplitz, Hankel, persymmetric Hankel or circulant, respectively. Moreover, $A \in M_{\mathbb{R}}^{\text{struct}}$ shall imply A to be structured and real. Then the structured condition number of λ restricts perturbations ΔA in the definition (1.2) to (real or complex) structured ones [16]:

$$(1.4) \quad \kappa_{\mathbb{K}}^{\text{struct}}(A, \lambda) := \limsup_{\varepsilon \rightarrow 0} \left\{ \frac{|\Delta\lambda|}{\varepsilon} : \Delta A \in M_{\mathbb{K}}^{\text{struct}}, \|\Delta A\| \leq \varepsilon, \lambda + \Delta\lambda \in \Lambda(A + \Delta A) \right\},$$

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where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A given matrix may belong to more than one structure. For example, for a real symmetric Toeplitz matrix $A \in M_{\mathbb{R}}^{\text{symToep}}$ also $A \in M_{\mathbb{C}}^{\text{symToep}}$, $A \in M_{\mathbb{C}}^{\text{sym}}$ or $A \in M_{\mathbb{R}}^{\text{sym}}$, possibly resulting in different condition numbers. As we will see, this is not the case.

Definitions (1.2) and (1.4) may also be applied to a matrix A not belonging to the same structure, or real perturbations to a complex matrix. With few exceptions we will not treat these cases. Note that, provided A belongs to the structure, for all structures in (1.3) the definition of the real or complex structured condition number does not change when replacing $\Delta A \in M_{\mathbb{K}}^{\text{struct}}$ by $A + \Delta A \in M_{\mathbb{K}}^{\text{struct}}$.

We furthermore investigate the condition number subject to (real or complex) componentwise perturbations, i.e.

$$(1.5) \quad \text{cond}_{E, \mathbb{K}}(A, \lambda) := \limsup_{\varepsilon \rightarrow 0} \left\{ \frac{|\Delta \lambda|}{\varepsilon} : \Delta A \in \mathbb{K}^{n \times n}, |\Delta A| \leq \varepsilon |E|, \lambda + \Delta \lambda \in \Lambda(A + \Delta A) \right\}$$

where E denotes a weight matrix and comparison and absolute value for matrices are to be understood componentwise. Similarly, the structured condition number restricts perturbations to structured ones, i.e.

$$(1.6) \quad \text{cond}_{E, \mathbb{K}}^{\text{struct}}(A, \lambda) := \limsup_{\varepsilon \rightarrow 0} \left\{ \frac{|\Delta \lambda|}{\varepsilon} : \Delta A \in M_{\mathbb{K}}^{\text{struct}}, |\Delta A| \leq \varepsilon |E|, \lambda + \Delta \lambda \in \Lambda(A + \Delta A) \right\}.$$

A common choice for the weight matrix is $E = A$, which implies componentwise relative perturbations of each matrix entry.

The structured condition number for eigenvalues was defined and investigated in [16], see also [13]. In this paper we will prove that for most structures listed in (1.3) the structured and unstructured condition numbers are equal, for complex as well as for real perturbations. In other words, amongst the worst case perturbations there is a structured one, and such a perturbation will be identified by our constructive proofs.

For normwise perturbations, there is one extreme exception to that statement, namely skewsymmetric matrices. In this case the (complex) unstructured and structured condition number can differ by an arbitrarily large factor. However, complex skewsymmetric matrices seem not very common.

For componentwise real perturbations there are two exceptions to the former statement for the structures under investigation, namely Hermitian and skew-Hermitian matrices. For both the general condition number may be equal to one, whereas the condition number for relative and real perturbations of each entry of the matrix is zero. However, this is for real perturbations applied to a complex matrix.

The (structured) condition number investigates the sensitivity of an eigenvalue under infinitely small perturbations. The behavior of eigenvalues under finite perturbations of the matrix is characterized by the pseudospectrum, investigated and popularized by Trefethen [31], [32], [10]. The pseudospectrum of a matrix A is defined by

$$(1.7) \quad \Lambda_{\varepsilon}(A) := \{\lambda \in \mathbb{C} : \exists E \in \mathbb{C}^{n \times n}, \|E\| \leq \varepsilon, \lambda \in \Lambda(A + E)\}$$

with the well known characterization [31]

$$(1.8) \quad \Lambda_{\varepsilon}(A) = \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| \geq \varepsilon^{-1}\}.$$

The latter is clear by interpreting E as a perturbation of $A - \lambda I$ into a singular matrix, and using the famous result by Eckart and Young [17, Theorem 6.5] that the distance to singularity in the 2-norm equals the reciprocal of the norm of the inverse, which is the reciprocal of the condition number for a matrix of norm 1.

In [28] we generalized this theorem to structured distances and structured condition numbers. For most structures out of (1.3) the structured distance in the 2-norm of a matrix to the nearest singular one is equal to the unstructured distance, and equal to the reciprocal of the structured (and the unstructured) condition number for a matrix of norm 1. In other words, restricting perturbations to structured ones changes nothing, amongst the worst case perturbations is a structured one. This implies for example that the (complex) structured and unstructured pseudospectrum coincides for Toeplitz and circulant matrices (see [12]).

Note that the pseudospectrum generalizes the condition number in two ways: i) finite rather than infinitely small perturbations are treated, and ii) there is no restriction to simple eigenvalues. The former complicates the matter because terms of higher order cannot be neglected. A number of our results on the condition number also follow by the corresponding ones on the pseudospectrum so that their independent proofs could be omitted. However, we feel that the separate and constructive proofs have their own value and may provide additional insight into the matter.

The (general) pseudospectrum has many interesting properties and reveals insights into certain properties of the matrix [31], [32]. So for structured matrices it seems also reasonable to look at the structured pseudospectrum by limiting finite perturbations E to some structure. This has been done in different ways in the literature. In control theory perturbations of the form $E = PMQ$ with fixed matrices P and Q are studied, see [18] and [30]. Those ideas are closely related to the μ -number [9], [25]. Results on componentwise distances can be found in [23]. In this paper we use

$$(1.9) \quad \Lambda_\varepsilon^{\text{struct}}(A) := \{\lambda \in \mathbb{C} : \exists E \in M_{\mathbb{C}}^{\text{struct}}, \|E\| \leq \varepsilon, \lambda \in \Lambda(A + E)\}.$$

This borrows from the corresponding definitions in sensitivity analysis and condition numbers for linear systems [15], [28], and is similar to the definition (1.4) for eigenvalues. It is also used in [12] and [22]. A similar definition is used by Böttcher et al. [4], [3], where perturbations are restricted to banded Toeplitz structures, and it is shown that the banded Toeplitz-structured and unstructured pseudospectrum do, in general, not coincide.

We aim to characterize $\Lambda_\varepsilon^{\text{struct}}(A)$ for most structures in (1.3). In fact, for many of those structures we will show $\Lambda_\varepsilon(A) = \Lambda_\varepsilon^{\text{struct}}(A)$, especially for $\text{struct} = \text{Toep}$. As noted by Albrecht Böttcher, here is a beautiful example for the fact that assertions being valid for all finite matrices need not extend to infinite operators [2]: Definitions (1.7) and (1.9) make also sense for bounded linear operators. But [5, Theorem 8.2] implies that for Toeplitz operators A , that is, for operators generated by infinite Toeplitz matrices on $\ell^2(\mathbf{N})$, the equality $\Lambda_\varepsilon(A) = \Lambda_\varepsilon^{\text{Toep}}(A)$ is in general *not true*. This is in remarkable contrast to our Theorem 4.3, which, among other things, says that this equality is always valid for finite Toeplitz matrices!

Note that perturbations in (1.9) are complex structured; we will also characterize the real structured pseudospectrum

$$(1.10) \quad \Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A) := \{\lambda \in \mathbb{C} : \exists E \in M_{\mathbb{R}}^{\text{struct}}, \|E\| \leq \varepsilon, \lambda \in \Lambda(A + E)\}.$$

for most structures in (1.3). In many cases the real structured pseudospectrum is the intersection of the unstructured pseudospectrum with the real line. Although the (complex) structured and unstructured pseudospectrum coincide for persymmetric matrices, substantial differences may occur when restricting perturbations to real ones. This is nicely demonstrated in Figure 4.2. Again, if $A \in M_{\mathbb{K}}^{\text{struct}}$, then $E \in M_{\mathbb{K}}^{\text{struct}}$ may be replaced by $A + E \in M_{\mathbb{K}}^{\text{struct}}$ without changing definitions (1.9) or (1.10).

The paper is organized as follows. In the next section we collect some facts we need to prove our main results for the complex and real, normwise and componentwise structured condition number of a simple eigenvalue presented in Section 3, and for the complex and real structured pseudospectrum presented in Section 4. In an appendix we outline a computer-assisted proof of some explicit example for Toeplitz structures.

Our results on structured condition numbers are proved by explicit construction of a structured perturbation. Some of our results and also more have recently been shown by Françoise Tisseur [29] using Lie algebras, see also [22]. This very elegant approach provides unified proofs for a number of our structures plus others like Hamiltonians; however, it does not, for example, apply to Toeplitz-like structures.

Concerning notation we denote by $I_n = I$ the $n \times n$ identity matrix, by $J_n = J$ the $n \times n$ “flip-matrix” (with ones on the anti-diagonal and zero everywhere else). The index is omitted when clear from the context. Furthermore, $\bar{x} \in \mathbb{C}^n$ denotes the conjugate of $x \in \mathbb{C}^n$, and e_i the i th column of I . The spectrum of A is denoted by $\Lambda(A)$, $\sigma_{\min}(A)$ denotes the smallest singular value of A , and $U_\varepsilon(\lambda) := \{z \in \mathbb{C} : |z - \lambda| \leq \varepsilon\}$.

2. Auxiliary results. In this section we collect some facts we need to prove our main results in the next two sections. Throughout this section we suppose (1.1) for a simple eigenvalue λ of A . Multiplying $(A + \Delta A - (\lambda + \Delta\lambda)I)(x + \Delta x) = 0$ from the left by y^H yields

$$(2.1) \quad \Delta\lambda = \frac{y^H \Delta A x}{y^H x} + \mathcal{O}(\varepsilon^2),$$

so that the definition (1.4) implies for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$$(2.2) \quad \kappa_{\mathbb{K}}^{\text{struct}}(A, \lambda) = \frac{\max\{|y^H \Delta A x| : \Delta A \in M_{\mathbb{K}}^{\text{struct}}, \|\Delta A\| \leq 1\}}{|y^H x|} \leq \frac{1}{|y^H x|} = \kappa_{\mathbb{C}}(A, \lambda).$$

Hence the analysis of $\kappa_{\mathbb{K}}^{\text{struct}}$ focusses on

$$(2.3) \quad \varphi_{\mathbb{K}}^{\text{struct}}(x, y) := \max\{|y^H \Delta A x| : \Delta A \in M_{\mathbb{K}}^{\text{struct}}, \|\Delta A\| \leq 1\},$$

where x, y satisfy (1.1). Then (2.2) implies

$$(2.4) \quad \kappa_{\mathbb{K}}^{\text{struct}}(A, \lambda) = \varphi_{\mathbb{K}}^{\text{struct}}(x, y) \cdot \kappa_{\mathbb{C}}(A, \lambda).$$

Although not included in the definition, we mostly assume the matrix to be real when analyzing real perturbations, structured or unstructured.

The value $\varphi_{\mathbb{K}}^{\text{struct}}(x, y)$ does not change when scaling the eigenvectors x and y by a complex scalar of modulus one. To prove $\varphi_{\mathbb{C}}^{\text{struct}}(x, y) = 1$ and therefore $\kappa_{\mathbb{C}}^{\text{struct}}(A, \lambda) = \kappa_{\mathbb{C}}(A, \lambda)$ (or its real counterpart) for a number of structures, we use this freedom to choose appropriate left and right eigenvectors. The results of the following Lemmata 2.1 and 2.2 are well known; the proofs, however, are so short that we choose to include rather than to reference them.

LEMMA 2.1. *Let $Ax = \lambda x$ and $\|x\| = 1$ for $A \in \mathbb{C}^{n \times n}$. Then $y \in \mathbb{C}^n$ with $y^H A = \lambda y^H$ and $\|y\| = 1$ can be chosen such that*

- a) $y = x$ if A is normal ($A^H A = A A^H$),
- b) $y = \bar{x}$ if A is symmetric ($A^T = A$),
- c) $y = J\bar{x}$ if A is persymmetric ($A^T = JAJ$).

PROOF. Part a) follows by $A = Q\Lambda Q^H$, and b) follows from $x^T A = \lambda x^T$. Concerning c), $(J\bar{x})^H A = x^T J A = (A^T J x)^T = (J A x)^T = \lambda (J x)^T = \lambda (J\bar{x})^H$. ■

LEMMA 2.2. *For $A \in M_{\mathbb{C}}^{\text{sym}} \cap M_{\mathbb{C}}^{\text{persym}}$ and $\lambda \in \Lambda(A)$ there exists an eigenvector x to λ with $x = \alpha Jx$ and $\alpha \in \{-1, +1\}$. If A is real, x can be chosen real.*

PROOF. Let v be an eigenvector of A to λ . Then $A = A^T = JAJ$ implies $A(v + \alpha Jv) = \lambda(v + \alpha Jv)$ for every scalar α . The vector $x := v + \alpha Jv$ satisfies $x = \alpha Jx$ for $\alpha \in \{-1, +1\}$, and is nonzero for at least one value of α . ■

The following Lemma 2.3 is the key to certain Toeplitz and Hankel structures. It has been given in [28, Lemma 10.1].

LEMMA 2.3. *Let $x \in \mathbb{C}^n$ be given. Then there exists $H \in M_{\mathbb{C}}^{\text{Hankel}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. If x is real, H can be chosen real so that $Hx = x$.*

The following lemma extends this result to situations, where a symmetric Toeplitz (persymmetric Hankel) matrix is looked for.

LEMMA 2.4. *Let $x \in \mathbb{C}^n$ with $x = \alpha Jx$, $\alpha \in \{-1, +1\}$ be given. Then there exists a symmetric Toeplitz matrix $T \in M_{\mathbb{C}}^{\text{symToep}}$ with $Tx = \bar{x}$ and $\|T\| = 1$. If x is real, T can be chosen real with $Tx = x$.*

PROOF. We extend the proof in [28, Lemma 10.1]. Define $\Psi_x \in \mathbb{C}^{n \times (2n-1)}$ to be the Toeplitz matrix with first column $(x_1, 0, \dots, 0)^T$ and first row $(x_1, \dots, x_n, 0, \dots, 0)$, where there are $n-1$ zeros in each case. For

$n = 3$ we have

$$(2.5) \quad \Psi_x = \begin{pmatrix} x_1 & x_2 & x_3 & & \\ & x_1 & x_2 & x_3 & \\ & & x_1 & x_2 & x_3 \end{pmatrix}$$

with omitted entries equal to zero. Every $p \in \mathbb{C}^{2n-1}$ uniquely defines a Hankel matrix $H \in \mathbb{C}^{n \times n}$ with first column $(p_1, \dots, p_n)^T$ and last row (p_n, \dots, p_{2n-1}) . For $n = 3$ we have

$$(2.6) \quad H = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \\ p_3 & p_4 & p_5 \end{pmatrix}.$$

Then a computation yields $Hx = \Psi_x p$. Following the ideas in [28] we embed Ψ_x into the $(2n-1) \times (2n-1)$ circulant C_x with first row identical to that of Ψ_x . For $n = 3$ we have

$$(2.7) \quad C_x = \begin{pmatrix} x_1 & x_2 & x_3 & & \\ & x_1 & x_2 & x_3 & \\ & & x_1 & x_2 & x_3 \\ x_3 & & & x_1 & x_2 \\ x_2 & x_3 & & & x_1 \end{pmatrix}.$$

Define

$$(2.8) \quad p := C_x^+ C_x^H e_1,$$

where C_x^+ denotes the Moore-Penrose inverse of C_x . For $\tilde{J} := J_n \oplus J_{n-1} \in \mathbb{C}^{(2n-1) \times (2n-1)}$ and using $J_n x = \alpha x$ we have by construction

$$(2.9) \quad \tilde{J} C_x J_{2n-1} = C_{J_n x} = \alpha C_x.$$

Then $\tilde{J}^2 = J_{2n-1}^2 = I_{2n-1}$, (2.9) and $\alpha^2 = 1$ imply $C_x^+ = \alpha J_{2n-1} C_x^+ \tilde{J}$, and $C_x^H e_1 = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}$ in conjunction with (2.8) and $J\bar{x} = \alpha\bar{x}$ yields

$$(2.10) \quad \begin{aligned} J_{2n-1} p &= J_{2n-1} \cdot C_x^+ \cdot C_x^H e_1 = J_{2n-1} \cdot \alpha J_{2n-1} C_x^+ \tilde{J} \cdot \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} \\ &= \alpha C_x^+ \begin{pmatrix} \alpha \bar{x} \\ 0 \end{pmatrix} = C_x^+ C_x^H e_1 = p. \end{aligned}$$

That means, the Hankel matrix defined by p is persymmetric. Denote by $P \in \mathbb{R}^{n \times (2n-1)}$ the first n rows of I_{2n-1} . Then $\Psi_x = PC_x$. Following the arguments in the proof of Lemma 10.1 in [28] we conclude

$$Hx = \Psi_x p = PC_x C_x^+ C_x^H e_1 = P \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix} = \bar{x},$$

so that $T := \alpha JH$ is symmetric Toeplitz with $Tx = \bar{x}$. The proof of $\|T\| = \|H\| \leq 1$ is identical to the one in [28], and $\|\bar{x}\| = \|Tx\| \leq \|T\| \|x\|$ implies $\|T\| = 1$. If x is real, so are by construction H and T . \blacksquare

We will apply Lemma 2.4, for example, to x being an eigenvector of a symmetric Toeplitz matrix. Then by Lemma 2.2 we can choose $x = \alpha Jx$. This assumption is mandatory for Lemma 2.4. To see this let $x = (p, q, r)^T \in \mathbb{C}^3$ and assume there is symmetric Toeplitz T with $Tx = \bar{x}$. Denote the first row of T by (a, b, c) . Then

$$(2.11) \quad \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} p & q & r \\ q & p+r & 0 \\ r & q & p \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix}.$$

Choosing $x = (p, q, r)^T = (1, i, -1)^T$ we can solve (2.11) uniquely for a, b, c and obtain

$$Tx = \bar{x} \quad \text{for} \quad T = \begin{pmatrix} -1 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \quad \text{but} \quad \|T\| = 3.$$

Note that $x \neq \alpha Jx$ for $\alpha \in \{-1, +1\}$.

LEMMA 2.5. *Let $z \in \mathbb{C}^n$ with $\|z\| = 1$ be given. Then there exists a real symmetric matrix C and $\alpha \in \mathbb{C}$ with $\|C\| = 1$, $|\alpha| = 1$ and $Cz = \alpha \bar{z}$.*

PROOF. Let $z = x + iy$ for $x, y \in \mathbb{R}^n$ and denote the singular value decomposition of the matrix $[x \ y] \in \mathbb{R}^{n \times 2}$ with columns x and y by $[x \ y] = U\Sigma V^T$. Then

$$C := U \operatorname{diag}(1, -1, 0, \dots, 0)U^T \quad \text{is real symmetric with } \|C\| = 1.$$

Furthermore,

$$\begin{aligned} C \cdot [x \ y] &= U \operatorname{diag}(1, -1, 0, \dots, 0)U^T \cdot U\Sigma V^T \\ &= U\Sigma V^T \cdot V \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} V^T = [x \ y]Q \end{aligned}$$

with real orthogonal $Q \in \mathbb{R}^{2 \times 2}$. By construction, Q is a reflection, so $Q = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$. Hence, $Cx = px + qy$, $Cy = qx - py$ and

$$Cz = Cx + iCy = (p + iq)x - i(p + iq)y = (p + iq)\bar{z}.$$

Choosing $\alpha := p + iq$ and observing $\|Q\| = 1 = |\alpha|$ finishes the proof. ■

LEMMA 2.6. *Let $A \in M_{\mathbb{R}}^{\text{skewsym}}$ and $Ax = \lambda x$ for $\lambda \neq 0$ and $0 \neq x \in \mathbb{C}^n$. Then there exists $\Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$ with $\Delta Ax = ix$ and $\|\Delta A\| = 1$.*

Let $A \in M_{\mathbb{R}}^{\text{skewsym}}$ and assume $0 \in \Lambda(A)$ is not simple. Then there is $0 \neq x \in \mathbb{C}^n$ and $\Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$ with $Ax = 0$, $\Delta Ax = ix$ and $\|\Delta A\| = 1$.

PROOF. Suppose $\lambda \neq 0$. The eigenvalues of the real skewsymmetric matrix A come in purely imaginary conjugate pairs $\pm\beta i$ with $\beta \in \mathbb{R}$. Since A is normal, we have $A = QDQ^H$ with unitary Q and diagonal D . Without loss of generality we may assume $d_{11} = \beta i, d_{22} = -\beta i$, and that x is a scalar multiple of Qe_1 . Abbreviating $q_\nu = Qe_\nu$ we see $A[q_1 \ q_2] = [q_1 \ q_2] \begin{pmatrix} \beta i & \\ & -\beta i \end{pmatrix}$, so that $[q_1 \ q_2] \begin{pmatrix} \beta i & \\ & -\beta i \end{pmatrix} [q_1 \ q_2]^H$ is real.

Define $\Delta A := [q_1 \ q_2] \begin{pmatrix} i & \\ & -i \end{pmatrix} [q_1 \ q_2]^H$. Then ΔA is real and skewsymmetric with $\|\Delta A\| = 1$. Furthermore,

$$\Delta A q_1 = [q_1 \ q_2] \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i q_1,$$

and the first part of the Lemma is proved.

For the second part assume $\lambda = 0$ is of multiplicity ≥ 2 . Since A is normal, the kernel of A is of dimension ≥ 2 and we find $u, v \in \mathbb{R}^n$ with $Au = Av = 0$ and $u^T v = 0$. We follow the proof of Lemma 5.1 in [28], which in turn borrows ideas from a proof in [6]. Let $Q \in \mathbb{R}^{n \times n}$ be orthogonal with $Q[u \ v] = [e_1 \ -e_2]$. Define $\Delta A := Q^T D Q$ with $D := \operatorname{diag} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, \dots, 0 \right) \in \mathbb{R}^{n \times n}$. Then $\Delta A = -\Delta A^T$, $\|\Delta A\| = 1$, $D e_1 = e_2$, $D e_2 = -e_1$ and $x := u + iv$ yields

$$\Delta Ax = Q^T D(e_1 - ie_2) = Q^T(e_2 + ie_1) = -v + iu = ix. \quad \blacksquare$$

For later use we collect some basic facts about circulants (see, for example, [8], [28]):

(2.12) Every circulant $C \in M_{\mathbb{C}}^{\text{circ}}$ is diagonalized by the Fourier matrix $F = (\omega^{(i-1)(j-1)}/\sqrt{n})$,
 ω denoting the n -th root of unity, i.e. $C = FDF^H$ for diagonal D .

(2.13) The eigenvalues of a circulant $C = FDF^H$ with first row (c_1, \dots, c_n) and $D = \operatorname{diag}(d_{11}, \dots, d_{nn})$
are $d_{kk} = \sum_{\nu=1}^n c_\nu \omega^{-(k-1)(\nu-1)}$ for $k = 1, \dots, n$.

(2.14) The circulant $C = FDF^H$ is real iff $D = PD^HP$ with P denoting the permutation matrix P mapping $(1, \dots, n)$ into $(1, n, \dots, 2)$.

The proofs follow by direct computation.

3. Structured condition numbers. Let $A \in \mathbb{C}^{n \times n}$ be given. In a recent paper [7] Byers and Kressner show that restricting (general) complex perturbations $\Delta A \in \mathbb{C}^{n \times n}$ to (general) *real* perturbations can improve $\kappa_{\mathbb{C}}(A, \lambda)$ by at most a factor $1/\sqrt{2}$. We show a similar result for certain structured perturbations. Moreover we prove that there is no difference between the real and complex unstructured condition number for any real matrix belonging to one of the structures in (1.3).

LEMMA 3.1. *Let $A \in M_{\mathbb{C}}^{\text{struct}}$ be given and λ, x, y with (1.1), λ simple. Suppose struct is such that $B \in M_{\mathbb{C}}^{\text{struct}}$ implies that the real part and the complex part of B are in $M_{\mathbb{R}}^{\text{struct}}$. Then*

$$\frac{1}{\sqrt{2}} \kappa_{\mathbb{C}}^{\text{struct}}(A, \lambda) \leq \kappa_{\mathbb{R}}^{\text{struct}}(A, \lambda) \leq \kappa_{\mathbb{C}}^{\text{struct}}(A, \lambda).$$

If $A \in M_{\mathbb{R}}^{\text{struct}}$ for any of the structures in (1.3), then

$$(3.1) \quad \kappa_{\mathbb{R}}(A, \lambda) = \kappa_{\mathbb{C}}(A, \lambda).$$

PROOF. Let $\tilde{A} \in M_{\mathbb{C}}^{\text{struct}}$ be such that $\|\tilde{A}\| \leq 1$ and $|y^H \tilde{A}x| = \max\{|y^H \Delta Ax| : \Delta A \in M_{\mathbb{C}}^{\text{struct}}, \|\Delta A\| \leq 1\}$. Splitting $\tilde{A} = \tilde{A}_{\mathcal{R}e} + i\tilde{A}_{\mathcal{I}m}$ into real and imaginary part yields $\max\{|y^H \tilde{A}_{\mathcal{R}e}x|, |y^H \tilde{A}_{\mathcal{I}m}x|\} \geq \frac{1}{\sqrt{2}}|y^H \tilde{A}x|$, and $\tilde{A}_{\mathcal{R}e}, \tilde{A}_{\mathcal{I}m} \in M_{\mathbb{R}}^{\text{struct}}$ with $\max(\|\tilde{A}_{\mathcal{R}e}\|, \|\tilde{A}_{\mathcal{I}m}\|) \leq 1$ proves the result.

Using (2.1) the second part is proved if there is a matrix $\Delta A \in \mathbb{R}^{n \times n}$ with $|y^H \Delta Ax| = 1$. A real matrix $A \in M_{\mathbb{R}}^{\text{struct}}$ with struct being one of the structures in (1.3) is normal and/or persymmetric. Using Lemma 2.1 we can choose $\Delta A = I$ for normal A , and $\Delta A = JC$ with a matrix C as in Lemma 2.5 for persymmetric A . ■

The assumption in the first part of Lemma 3.1 is satisfied for all structures in (1.3) except Herm and skewHerm. As we will see, for Hermitian matrices the unstructured and structured condition number, both real and complex, are all equal. This is also true for (complex) persymmetric matrices and circulants. Next we state and prove our main result for structured condition numbers of simple eigenvalues. Because of (3.1) we can omit the subscript \mathbb{R} or \mathbb{C} for the unstructured condition number.

THEOREM 3.2. *Let A be a matrix with simple eigenvalue λ and corresponding normalized right and left eigenvector x and y , respectively. Then Table 3.3 shows our results on the (normwise) structured condition number:*

<i>structure</i>	<i>condition number</i>	
<i>symmetric</i>	$A \in M_{\mathbb{R}}^{\text{sym}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{sym}}(A, \lambda) = 1$
	$A \in M_{\mathbb{C}}^{\text{sym}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{sym}}(A, \lambda) = 1/ x^T x $ [<i>complex symmetric</i>]
<i>Hermitian</i>	$A \in M_{\mathbb{C}}^{\text{Herm}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{Herm}}(A, \lambda) = \kappa_{\mathbb{R}}^{\text{sym}}(A, \lambda) = 1$
<i>persymmetric</i>	$A \in M_{\mathbb{C}}^{\text{persym}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{persym}}(A, \lambda) = \kappa_{\mathbb{R}}^{\text{persym}}(A, \lambda) = 1/ x^T Jx $
<i>skewsymmetric</i>	$A \in M_{\mathbb{R}}^{\text{skewsym}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{skewsym}}(A, \lambda) = 1$ for $\lambda \neq 0$
		$\kappa(A, \lambda) = 1$ also for $\lambda = 0$ but $\kappa_{\mathbb{R}}^{\text{skewsym}}(A, 0) = 0$
<i>skew-Hermitian</i>	$A \in M_{\mathbb{C}}^{\text{skewHerm}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{skewHerm}}(A, \lambda) = 1$
<i>Toeplitz</i>	$A \in M_{\mathbb{R}}^{\text{Toep}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{Toep}}(A, \lambda) = 1/ x^T Jx $ for $\lambda \in \mathbb{R}$
		$\frac{1}{\sqrt{2}} \kappa(A, \lambda) \leq \kappa_{\mathbb{R}}^{\text{Toep}}(A, \lambda) \leq \kappa(A, \lambda) = 1/ x^T Jx $ for $\lambda \notin \mathbb{R}$
	$A \in M_{\mathbb{C}}^{\text{Toep}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{Toep}}(A, \lambda) = 1/ x^T Jx $
<i>symmetric Toeplitz</i>	$A \in M_{\mathbb{R}}^{\text{symToep}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{symToep}}(A, \lambda) = 1$
	$A \in M_{\mathbb{C}}^{\text{symToep}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{symToep}}(A, \lambda) = 1/ x^T x $
<i>Hankel</i>	$A \in M_{\mathbb{R}}^{\text{Hankel}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{Hankel}}(A, \lambda) = 1$
	$A \in M_{\mathbb{C}}^{\text{Hankel}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{Hankel}}(A, \lambda) = 1/ x^T x $
<i>persymmetric Hankel</i>	$A \in M_{\mathbb{R}}^{\text{persymHankel}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{R}}^{\text{persymHankel}}(A, \lambda) = 1$
	$A \in M_{\mathbb{C}}^{\text{persymHankel}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{persymHankel}}(A, \lambda) = 1/ x^T x $
<i>circulant</i>	$A \in M_{\mathbb{C}}^{\text{circ}}$	$\Rightarrow \kappa(A, \lambda) = \kappa_{\mathbb{C}}^{\text{circ}}(A, \lambda) = \kappa_{\mathbb{R}}^{\text{circ}}(A, \lambda) = 1$

TABLE 3.3. *The (normwise) structured condition number of an eigenvalue*

The ratio $\kappa_{\mathbb{C}}(A, \lambda)/\kappa_{\mathbb{C}}^{\text{skewsym}}(A, \lambda)$ can be arbitrarily large, also for $\lambda \neq 0$. There exist matrices $A \in M_{\mathbb{R}}^{\text{Toep}}$ with $\kappa_{\mathbb{R}}^{\text{Toep}}(A, \lambda) < 0.95 \cdot \kappa(A, \lambda)$.

REMARK. For the last statement $\kappa_{\mathbb{R}}^{\text{Toep}}(A, \lambda) < 0.95 \cdot \kappa(A, \lambda)$ we will sketch a so-called computer-assisted proof in the appendix. The proof uses our Matlab interval toolbox INTLAB [27].

PROOF. Following (2.3) and (2.4) we construct $\Delta A \in M_{\mathbb{R}}^{\text{struct}}$ with $\|\Delta A\| = 1$ and $|y^H \Delta A x| = 1$. Since λ is simple, x and y with $\|x\| = \|y\| = 1$ are unique up to scalar multiples of modulus 1.

For $A \in M_{\mathbb{R}}^{\text{sym}}, M_{\mathbb{C}}^{\text{Herm}}, M_{\mathbb{R}}^{\text{symToep}}, M_{\mathbb{R}}^{\text{circ}}$ and $M_{\mathbb{C}}^{\text{circ}}$, the matrix A is normal, so we may choose $y = x$. Furthermore, $\Delta A := I$ belongs to all those structures. Hence $\|\Delta A\| = 1$, $y^H \Delta A x = x^H x = 1$ and (2.4) proves that structured and unstructured condition numbers are equal to $1/|y^H x| = 1$.

Real or complex Hankel matrices are symmetric, so according to Lemma 2.1b we may assume $y = \bar{x}$, and especially $y = x \in \mathbb{R}^n$ for $A \in M_{\mathbb{R}}^{\text{Hankel}}$. According to Lemma 2.3 there exists (real or complex) $\Delta A \in M^{\text{Hankel}}$ with $\|\Delta A\| = 1$ and $\Delta A x = \bar{x}$, so that $y^H \Delta A x = x^T \bar{x} = 1$. Furthermore, $\kappa(A, \lambda) = 1/|y^H x| = 1/|x^T x|$, which is equal to 1 for real A . Since ΔA can be chosen real for $A \in M_{\mathbb{R}}^{\text{Hankel}}$, these cases are finished. For $y = \bar{x}$ we used only the symmetry of A , so $M_{\mathbb{C}}^{\text{Hankel}} \in M_{\mathbb{C}}^{\text{sym}}$ finishes also the complex symmetric case.

For real or complex persymmetric A we may choose $y = J\bar{x}$ by Lemma 2.1c. Let $\Delta A := JC$ with $C \in M_{\mathbb{R}}^{\text{sym}}$,

$\|C\| = 1$ and $Cx = \alpha\bar{x}$, $|\alpha| = 1$ according to Lemma 2.5. Then $\Delta A \in M_{\mathbb{R}}^{\text{persym}}$, $\|\Delta A\| = 1$ and $|y^H \Delta Ax| = |x^T J \cdot J \alpha \bar{x}| = 1$.

Complex skew-Hermitian matrices are normal, so choosing $y = x$ and $\Delta A := \sqrt{-1} \cdot I \in M_{\mathbb{C}}^{\text{skewHerm}}$ implies $|y^H \Delta Ax| = 1$.

Next we treat $M_{\mathbb{C}}^{\text{symToepl}}$, $M_{\mathbb{R}}^{\text{persymHankel}}$ and $M_{\mathbb{C}}^{\text{persymHankel}}$. In those cases A is symmetric and persymmetric, and by Lemma 2.2 we may choose x with $Ax = \lambda x$, $\|x\| = 1$, $x = \alpha Jx$ and $\alpha \in \{-1, +1\}$. Then by Lemma 2.4 there is symmetric Toeplitz $T \in M_{\mathbb{C}}^{\text{symToepl}}$ with $Tx = \bar{x}$ and $\|T\| = 1$, where T can be chosen real if x is real.

For $A \in M_{\mathbb{C}}^{\text{symToepl}}$ we may choose $y = \bar{x}$ according to Lemma 2.1b, and $\Delta A := T \in M_{\mathbb{C}}^{\text{symToepl}}$ implies $y^H \Delta Ax = x^T \bar{x} = 1$ and $\kappa_{\mathbb{C}}^{\text{symToepl}}(A, \lambda) = \kappa(A, \lambda) = 1/|y^H x| = 1/|x^T x|$.

For a real or complex persymmetric Hankel matrix we choose $y = J\bar{x}$ according to Lemma 2.1c and $\Delta A := JT \in M_{\mathbb{C}}^{\text{persymHankel}}$. Then $y^H \Delta Ax = x^T J \cdot J\bar{x} = 1$ and $\kappa_{\mathbb{C}}^{\text{persymHankel}}(A, \lambda) = \kappa(A, \lambda) = 1/|x^T Jx| = 1/|x^T x|$. For real persymmetric Hankel, x can be chosen real since A is symmetric, and Lemma 2.4 closes this case as well.

For complex Toeplitz A we may choose $y = J\bar{x}$ according to Lemma 2.1c. Furthermore, Lemma 2.3 implies the existence of $H \in M_{\mathbb{C}}^{\text{Hankel}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. Then $\Delta A := JH \in M_{\mathbb{C}}^{\text{Toepl}}$, $\|\Delta A\| = \|H\| = 1$ and $y^H \Delta Ax = x^T J \cdot J\bar{x} = 1$, so that $\kappa_{\mathbb{C}}^{\text{Toepl}}(A, \lambda) = \kappa(A, \lambda) = 1/|x^T Jx|$.

For $A \in M_{\mathbb{R}}^{\text{skewsym}}$ assume $\lambda \neq 0$. Then Lemma 2.6 implies the existence of $\Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$ with $\Delta Ax = ix$ and $\|\Delta A\| = 1$. Since A is normal, $y = x$ and $|y^H \Delta Ax| = |ix^H x| = 1$ prove $\kappa_{\mathbb{R}}^{\text{skewsym}}(A, \lambda) = \kappa(A, \lambda) = 1$. A simple eigenvalue $\lambda = 0$ is only possible for odd dimension; but every real skewsymmetric matrix of odd dimension has an eigenvalue 0, so $\kappa_{\mathbb{R}}^{\text{skewsym}}(A, 0) = 0$. Indeed, for $\lambda = 0$ the eigenvector x can be chosen real, so that for all $\Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$

$$y^H \Delta Ax = x^T \Delta Ax = (x^T \Delta Ax)^T = -x^T \Delta Ax = 0.$$

For general perturbations we choose $\Delta A = I \notin M_{\mathbb{R}}^{\text{skewsym}}$ to see $\kappa(A, \lambda) = 1$.

Let $A \in M_{\mathbb{R}}^{\text{Toepl}}$. If $\lambda \in \mathbb{R}$ then x can be chosen real and $y = Jx$ by Lemma 2.1c. By Lemma 2.3 there is $H \in M_{\mathbb{R}}^{\text{Hankel}}$ with $Hx = x$ and $\|H\| = 1$. For $\Delta A := JH \in M_{\mathbb{R}}^{\text{Toepl}}$ we have $y^H \Delta Ax = x^T J \cdot Jx = 1$ and

$$\kappa_{\mathbb{R}}^{\text{Toepl}}(A, \lambda) = \kappa(A, \lambda) = 1/|y^H x| = 1/|x^T Jx| \quad \text{for } \lambda \in \mathbb{R}.$$

The part $A \in M_{\mathbb{R}}^{\text{Toepl}}$ and $\lambda \notin \mathbb{R}$ follows by Lemma 3.1. An explicit example with $\kappa_{\mathbb{R}}^{\text{Toepl}}(A, \lambda) < 0.95\kappa(A, \lambda)$ is given in the appendix.

Finally, consider $A \in M_{\mathbb{C}}^{\text{skewsym}}$ with

$$(3.2) \quad A = A_{\varphi} = \begin{pmatrix} 0 & 1 - \varphi & 0 \\ -1 + \varphi & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad \text{for } \varphi \in \mathbb{R}, 0 < \varphi < 1,$$

$$\text{with } \lambda = \sqrt{2\varphi - \varphi^2} \in \mathbb{R} \text{ and } x = \frac{1}{\sqrt{2}} \begin{pmatrix} i(1 - \varphi) \\ i\lambda \\ 1 \end{pmatrix}.$$

A computation yields $Ax = \lambda x$ and $\|x\| = 1$. Furthermore,

$$y = \begin{pmatrix} -i(1 - \varphi) \\ i\lambda \\ 1 \end{pmatrix} / \sqrt{2} \text{ satisfies } y^H A = \lambda y^H \text{ and } \|y\| = 1.$$

A general (complex) skewsymmetric perturbation has the form

$$\Delta A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

A computation yields

$$y^H \Delta A x = -\lambda(a(1 - \varphi) + ic)$$

and $|y^H \Delta A x| \leq |\lambda|(|a| + |c|) \leq \sqrt{2}|\lambda| \|\Delta A\|$. This implies

$$\kappa_{\mathbb{C}}^{\text{skewsym}}(A, \lambda) \leq \frac{\sup\{|y^H \Delta A x| : \Delta A \in M_{\mathbb{C}}^{\text{skewsym}}, \|\Delta A\| \leq 1\}}{|y^H x|} \leq \frac{\sqrt{2}|\lambda|}{|y^H x|} < 2\sqrt{\varphi} \cdot \kappa(A, \lambda),$$

so that the structured condition number can be better than the unstructured condition number by an arbitrarily large factor. We mention that $A + \Delta A$ is singular for A as in (3.2) and all (complex) skewsymmetric ΔA . So $\kappa_{\mathbb{C}}^{\text{skewsym}}(A, 0) = 0$, but $\kappa(A, 0)$ is nonzero. \blacksquare

In the remark following (1.4) we mentioned that, for example, a matrix $A \in M_{\mathbb{R}}^{\text{symToep}}$ is also in $M_{\mathbb{C}}^{\text{symToep}}$, $M_{\mathbb{C}}^{\text{sym}}$ and $M_{\mathbb{R}}^{\text{sym}}$, and restricting perturbations to those structures may influence the sensitivity of λ . We now conclude that this is not the case since the corresponding eigenvector x is real, $x^T x = 1$ and the real and complex, unstructured and structured condition number is equal to 1 for all mentioned structures.

The exceptional behavior of complex skewsymmetric matrices needs more investigation; we think it is only possible for eigenvalues near 0.

So far we treated normwise perturbations. Next, we consider condition numbers for componentwise perturbations. Componentwise perturbations impose an additional structure on a perturbation ΔA . For example, zero weights can be used to retain bandedness of a matrix. Note that (1.5) and (2.1) imply

$$(3.3) \quad \text{cond}_{E, \mathbb{K}}^{\text{struct}}(A, \lambda) = \frac{\max\{|y^H \Delta A x| : \Delta A \in M_{\mathbb{K}}^{\text{struct}}, |\Delta A| \leq |E|\}}{|y^H x|} \leq \frac{|y^H| |E| |x|}{|y^H x|}$$

including $\text{cond}_{E, \mathbb{K}}$ by setting $M_{\mathbb{K}}^{\text{struct}} := \mathbb{K}^{n \times n}$. So again we have to maximize $|y^H \Delta A x|$, but this time over $|\Delta A| \leq |E|$. For no structure and Hermitian structure this is included in [16].

THEOREM 3.4. *Let A be a matrix with simple eigenvalue λ and corresponding normalized right and left eigenvector x and y , respectively. Then*

$$(3.4) \quad \text{cond}_{E, \mathbb{C}}(A, \lambda) = \frac{|y^H| |E| |x|}{|y^H x|}$$

and

$$(3.5) \quad \frac{1}{\sqrt{2}} \text{cond}_{E, \mathbb{C}}(A, \lambda) \leq \text{cond}_{E, \mathbb{R}}(A, \lambda) \leq \text{cond}_{E, \mathbb{C}}(A, \lambda).$$

Let struct be a structure such that $B \in M_{\mathbb{C}}^{\text{struct}}$ implies that the real part and the complex part of B are in $M_{\mathbb{R}}^{\text{struct}}$. Then $E \in M_{\mathbb{C}}^{\text{struct}}$ implies

$$(3.6) \quad \frac{1}{\sqrt{2}} \text{cond}_{E, \mathbb{C}}^{\text{struct}}(A, \lambda) \leq \text{cond}_{E, \mathbb{R}}^{\text{struct}}(A, \lambda) \leq \text{cond}_{E, \mathbb{C}}^{\text{struct}}(A, \lambda).$$

Let $A, E \in M_{\mathbb{C}}^{\text{sym}}$ be given. Then

$$(3.7) \quad \text{cond}_{E, \mathbb{R}}^{\text{sym}}(A, \lambda) = \text{cond}_{E, \mathbb{C}}^{\text{sym}}(A, \lambda) = \text{cond}_{E, \mathbb{C}}(A, \lambda) = \frac{|x^T| |E| |x|}{|x^T x|}.$$

Let $\text{struct} \in \{\text{Herm}, \text{skewHerm}\}$ and $A, E \in M_{\mathbb{C}}^{\text{struct}}$ be given. Then

$$(3.8) \quad \text{cond}_{E, \mathbb{C}}^{\text{struct}}(A, \lambda) = \text{cond}_{E, \mathbb{C}}(A, \lambda) = |x^H| |E| |x|.$$

For $A, E \in M_{\mathbb{C}}^{\text{circ}}$ holds

$$(3.9) \quad \text{cond}_{E, \mathbb{C}}^{\text{circ}}(A, \lambda) = \text{cond}_{E, \mathbb{C}}(A, \lambda) = \sum_{\nu=0}^{n-1} |\varepsilon_{\nu}|$$

for $(\varepsilon_1, \dots, \varepsilon_n)$ denoting the first row of E .

REMARK. Note that for (3.6) the matrix A need not be structured.

PROOF. Let $S_1, S_2 \in \mathbb{C}^{n \times n}$ be signature matrices, i.e. diagonal with diagonal entries of modulus 1, such that $S_1 x = |x|$ and $S_2 y = |y|$. Choosing $\Delta A := S_2^H |E| S_1$ satisfies $|\Delta A| = |E|$, so that the inequality in (3.3) is an equality and proves (3.4).

To show (3.5) and (3.6) we proceed exactly as in the first part of the proof of Lemma 3.1.

For $\text{struct} = \text{sym}$ we may choose $y = \bar{x}$ by Lemma 2.1b, such that $\Delta A := S_1^T |E| S_1 \in M_{\mathbb{R}}^{\text{sym}} \subseteq M_{\mathbb{C}}^{\text{sym}}$ and (3.3) prove (3.7).

A matrix $A \in M_{\mathbb{C}}^{\text{struct}}$ for $\text{struct} \in \{\text{Herm}, \text{skewHerm}\}$ is normal, so we can choose $y = x$. Then $\Delta A := S_1^H |E| S_1 \in M_{\mathbb{C}}^{\text{Herm}}$ and $\Delta A := i S_1^H |E| S_1 \in M_{\mathbb{C}}^{\text{skewHerm}}$, respectively, and in either case $|y^H \Delta A x| = |x^H |E| x|$. This proves (3.8).

Finally let $\text{struct} = \text{circ}$. By (2.12) and (2.13) we know $A = F D F^H$, $\lambda = d_{kk} = \sum_{\nu=1}^n a_{\nu} \omega^{-(k-1)(\nu-1)}$ and $x = F e_k$ for some $k \in \{1, \dots, n\}$, where (a_1, \dots, a_n) denotes the first row of A . Define ΔA to be the circulant with first row $(\delta a_1, \dots, \delta a_n)$ and $\delta a_{\nu} := |\varepsilon_{\nu}| \omega^{(k-1)(\nu-1)}$ for $1 \leq \nu \leq n$. Obviously $|\Delta A| = |E|$. By (2.13), the k th eigenvalue of ΔA is $\mu := \sum_{\nu=1}^n |\varepsilon_{\nu}|$ with eigenvector $F e_k = x$. Therefore $\Delta A x = \mu x$ and, using $y = x$ since A is normal,

$$y^H \Delta A x = x^H \cdot \mu x = \mu.$$

But $|x| = |F e_k| = n^{-1/2} e$, where $e \in \mathbb{R}^n$ denotes the vector of 1's, hence

$$|y^H |E| x| = \frac{1}{n} e^T |E| e = \mu$$

because E is a circulant. So $|y^H x| = |x^H x| = 1$ closes this case and finishes the proof. \blacksquare

Hermitian and skew-Hermitian matrices are the only exceptions in the structures listed in (1.3) for which $B \in M_{\mathbb{C}}^{\text{struct}}$ does not imply that the real part and the complex part of B are in $M_{\mathbb{R}}^{\text{struct}}$, so (3.6) need not be valid. Indeed, if we restrict perturbations to real ones for a (complex) Hermitian matrix, the condition number may drop from a finite value to zero. Consider

$$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

A general real Hermitian matrix ΔA with $|\Delta A| \leq |A|$ is symmetric and has the form $\Delta A = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$. The

eigenvalues of A are $\pm \alpha$ with eigenvectors $x = y = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$, so that $y^H \Delta A x = 0$ for all $\Delta A \in M_{\mathbb{R}}^{\text{Herm}}$

with $|\Delta A| \leq |A|$. But for $\widetilde{\Delta A} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \notin M_{\mathbb{R}}^{\text{Herm}}$ it holds $|\Delta A| \leq |A|$ and $|y^H \widetilde{\Delta A} x| = 1$, hence

$$(3.10) \quad \text{cond}_{A, \mathbb{C}}(A, \lambda) = \text{cond}_{A, \mathbb{R}}(A, \lambda) = 1 \quad \text{but} \quad \text{cond}_{A, \mathbb{R}}^{\text{Herm}}(A, \lambda) = 0.$$

The latter can also be seen from the eigenvalues $\pm \sqrt{1 + \alpha^2} = \pm(1 + \alpha^2/2) + \mathcal{O}(\alpha^4)$ of $A + \Delta A$. For skew-Hermitian structure the condition numbers of the matrix $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ show the same infinite ratio $\text{cond}/\text{cond}^{\text{struct}}$ as in (3.10). Note that $E = A$ reflects the common case of componentwise relative perturbations.

4. The structured pseudospectrum. To characterize the structured pseudospectrum (1.9) of a matrix we first observe $\Lambda_\varepsilon^{\text{struct}}(A) \subseteq \Lambda_\varepsilon(A)$. That means we have to identify those $\lambda \in \Lambda_\varepsilon(A)$ for which a structured perturbation $A + E$ of A exists with $\lambda \in \Lambda(A + E)$. Moreover, we will investigate the real structured pseudospectrum. We will mainly use the following lemma.

LEMMA 4.1. *Let struct be some structure with the property that $M \in M_{\mathbb{C}}^{\text{struct}}$ implies $\alpha M \in M_{\mathbb{C}}^{\text{struct}}$ for all $\alpha \in \mathbb{R}$. Let $A \in \mathbb{C}^{n \times n}$ be given. Suppose for $\lambda \in \Lambda_\varepsilon(A)$ and $s := \sigma_{\min}(A - \lambda I)$ there exists $\Delta A \in M_{\mathbb{C}}^{\text{struct}}$ and $0 \neq x \in \mathbb{C}^n$ with*

$$(4.1) \quad \|\Delta A\| \leq 1 \quad \text{and} \quad (A - \lambda I)x = s\Delta Ax,$$

Then $\lambda \in \Lambda_\varepsilon^{\text{struct}}(A)$. Moreover, if ΔA is real, then $\lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A)$.

PROOF. Let $\lambda \in \Lambda_\varepsilon(A)$ and define $B := A - \lambda I$. If $\lambda \in \Lambda(A)$, then the zero matrix, which is in $M_{\mathbb{C}}^{\text{struct}}$, does the job. Otherwise B is nonsingular and (1.8) implies $s = \|B^{-1}\|^{-1} \leq \varepsilon$. Define $E := -s\Delta A$. Then $E \in M_{\mathbb{C}}^{\text{struct}}$, $\|E\| = s \leq \varepsilon$ and

$$(A - \lambda I + E)x = Bx - s\Delta Ax = 0.$$

By definition (1.9) it follows $\lambda \in \Lambda_\varepsilon^{\text{struct}}(A)$. ■

Note that (4.1) requires the vectors Bx and $s\Delta Ax$ to coincide, not only the absolute value of a number $y^H \Delta Ax$ to be 1 as for condition numbers. All structures in (1.3) satisfy $\alpha A \in M_{\mathbb{C}}^{\text{struct}}$ whenever $A \in M_{\mathbb{C}}^{\text{struct}}$ and $\alpha \in \mathbb{R}$, and of course $\alpha A \in M_{\mathbb{R}}^{\text{struct}}$ if $A \in M_{\mathbb{R}}^{\text{struct}}$. For the construction of suitable ΔA and x we use the following lemma.

LEMMA 4.2. *Let $B \in \mathbb{C}^{n \times n}$ and denote $s := \sigma_{\min}(B)$. Then*

- a) $B \in M_{\mathbb{C}}^{\text{Herm}}$ $\Rightarrow \exists 0 \neq x \in \mathbb{R}^n : Bx = s\alpha x$ and $\alpha \in \{-1, +1\}$.
- b) $B \in M_{\mathbb{C}}^{\text{sym}}$ $\Rightarrow \exists 0 \neq x \in \mathbb{C}^n : Bx = s\bar{x}$.
- c) $B \in M_{\mathbb{C}}^{\text{persym}}$ $\Rightarrow \exists 0 \neq x \in \mathbb{C}^n : Bx = sJ\bar{x}$.
- d) $B \in M_{\mathbb{C}}^{\text{sym}} \cap M_{\mathbb{C}}^{\text{persym}}$ $\Rightarrow \exists 0 \neq x \in \mathbb{C}^n : Bx = s\bar{x}$, $x = \alpha Jx$ and $\alpha \in \{-1, +1\}$.

PROOF. Part a) is obvious.

Part b) follows by Takagi's factorization [20, Corollary 4.4.4] $B = Q\Sigma Q^T$ with unitary Q and diagonal Σ containing the singular values of B .

Part c) follows by $B \in M_{\mathbb{C}}^{\text{persym}} \Rightarrow JB \in M_{\mathbb{C}}^{\text{sym}}$.

Concerning part d) there is $0 \neq x \in \mathbb{C}^n$ with $Bx = s\bar{x}$ by b), and $B = B^T = JBJ$ shows $B \cdot Jx = JBx = sJ\bar{x}$, so $By = s\bar{y}$ for $y = x + \alpha Jx$ and every $\alpha \in \mathbb{R}$. At least one of the vectors $x + Jx$ and $x - Jx$ is nonzero, and the lemma is proved. ■

With these preliminaries we can prove the following theorem.

THEOREM 4.3. *Let $0 \leq \varepsilon \in \mathbb{R}$ be fixed but arbitrary. If $A \in M_{\mathbb{C}}^{\text{Herm}}$, then*

$$(4.2) \quad \Lambda_\varepsilon^{\text{Herm}}(A) = \Lambda_\varepsilon(A) \cap \mathbb{R}.$$

If $A \in M_{\mathbb{C}}^{\text{skewHerm}}$, then

$$(4.3) \quad \Lambda_\varepsilon^{\text{skewHerm}}(A) = \Lambda_\varepsilon(A) \cap i\mathbb{R}.$$

If $\text{struct} \in \{\text{sym}, \text{persym}, \text{Toep}, \text{symToep}, \text{Hankel}, \text{persymHankel}, \text{circ}\}$ and $A \in M_{\mathbb{C}}^{\text{struct}}$, then

$$(4.4) \quad \Lambda_\varepsilon^{\text{struct}}(A) = \Lambda_\varepsilon(A).$$

PROOF OF THEOREM 4.3. Throughout the proof we assume $\lambda \in \Lambda_\varepsilon(A)$ and abbreviate $B := A - \lambda I$ and $s := \sigma_{\min}(A - \lambda I)$. Since $\Lambda_\varepsilon^{\text{struct}}(A) \subseteq \Lambda_\varepsilon(A)$ we have to prove that $\lambda \in \Lambda_\varepsilon(A)$, possibly restricted to real or purely imaginary λ , implies $\lambda \in \Lambda_\varepsilon^{\text{struct}}(A)$.

For $A, E \in M_{\mathbb{C}}^{\text{Herm}}$, also $A + E \in M_{\mathbb{C}}^{\text{Herm}}$, so $\lambda \in \Lambda_{\varepsilon}^{\text{Herm}}(A)$ implies $\lambda \in \mathbb{R}$. Therefore (4.2) is proved if for every real $\lambda \in \Lambda_{\varepsilon}(A)$ we can find $\Delta A \in M_{\mathbb{C}}^{\text{Herm}}$ and $0 \neq x \in \mathbb{C}^n$ with (4.1). For real λ , $B := A - \lambda I$ is Hermitian, so Lemma 4.2a and $\Delta A := \alpha I \in M_{\mathbb{C}}^{\text{Herm}}$ imply $Bx = s\alpha x = s\Delta Ax$.

For $A, E \in M_{\mathbb{C}}^{\text{skewHerm}}$ it follows $A + E \in M_{\mathbb{C}}^{\text{skewHerm}}$, so that $\lambda \in \Lambda_{\varepsilon}^{\text{skewHerm}}(A)$ implies $\lambda \in i\mathbb{R}$. Let $\lambda \in \Lambda_{\varepsilon}(A) \cap i\mathbb{R}$ be given. Then $B = A - \lambda I \in M_{\mathbb{C}}^{\text{skewHerm}}$ is normal, there is $\alpha is \in \Lambda(B) = \Lambda(A) - \lambda$ with $\alpha \in \{-1, +1\}$ and $s = |\alpha is| = \sigma_{\min}(B)$. So there exists $0 \neq x \in \mathbb{C}^n$ with $Bx = \alpha isx$. Then $\Delta A := \alpha iI \in M_{\mathbb{C}}^{\text{skewHerm}}$, $\|\Delta A\| = 1$, $Bx = \alpha isx = s\Delta Ax$ and (4.1) prove (4.3).

Let $A \in M_{\mathbb{C}}^{\text{sym}}$. Then $B = A - \lambda I \in M_{\mathbb{C}}^{\text{sym}}$, and by Lemma 4.2b there exists a nontrivial vector x with $Bx = s\bar{x}$. By Lemma 2.3 there is $\Delta A \in M_{\mathbb{C}}^{\text{Hankel}} \subseteq M_{\mathbb{C}}^{\text{sym}}$ with $\Delta Ax = \bar{x}$ and $\|\Delta A\| = 1$. Hence $Bx = s\bar{x} = s\Delta Ax$, and Lemma 4.1 proves (4.4) for $M_{\mathbb{C}}^{\text{sym}}$. Now $M_{\mathbb{C}}^{\text{Hankel}} \subseteq M_{\mathbb{C}}^{\text{sym}}$, and $\Delta A \in M_{\mathbb{C}}^{\text{Hankel}}$ proves (4.4) for $\text{struct} = \text{Hankel}$.

For $A \in M_{\mathbb{C}}^{\text{persym}}$ it follows $B \in M_{\mathbb{C}}^{\text{persym}}$, and by Lemma 4.2c there is $0 \neq x \in \mathbb{C}^n$ with $Bx = sJ\bar{x}$. By Lemma 2.3 there is $H \in M_{\mathbb{C}}^{\text{Hankel}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. Then $\Delta A := JH \in M_{\mathbb{C}}^{\text{Toep}} \subseteq M_{\mathbb{C}}^{\text{persym}}$, $\|\Delta A\| = 1$ and $Bx = sJ\bar{x} = sJHx = s\Delta Ax$ finish this case. Furthermore, $M_{\mathbb{C}}^{\text{Toep}} \subseteq M_{\mathbb{C}}^{\text{persym}}$, so $\Delta A \in M_{\mathbb{C}}^{\text{Toep}}$ proves (4.4) for $\text{struct} = \text{Toep}$.

For $A \in M_{\mathbb{C}}^{\text{symToep}}$ it follows $B = A - \lambda I \in M_{\mathbb{C}}^{\text{symToep}} \subseteq M_{\mathbb{C}}^{\text{sym}} \cap M_{\mathbb{C}}^{\text{persym}}$, so that by Lemma 4.2d there is $0 \neq x \in \mathbb{C}^n$ with $Bx = s\bar{x}$ and $x = \alpha Jx$, $\alpha^2 = 1$. Now Lemma 2.4 implies existence of $\Delta A \in M_{\mathbb{C}}^{\text{symToep}}$ with $\Delta Ax = \bar{x}$ and $\|\Delta A\| = 1$. Therefore, $Bx = s\bar{x} = s\Delta Ax$, and Lemma 4.1 finishes this part.

For $A \in M_{\mathbb{C}}^{\text{persymHankel}} \subseteq M_{\mathbb{C}}^{\text{sym}} \cap M_{\mathbb{C}}^{\text{persym}}$ we can proceed similarly and obtain $0 \neq x \in \mathbb{C}^n$ with $Bx = s\bar{x}$, $x = \alpha Jx$, $\alpha^2 = 1$ and $H \in M_{\mathbb{C}}^{\text{symToep}}$ with $Hx = \bar{x}$ and $\|H\| = 1$. Then $\Delta A := \alpha HJ \in M_{\mathbb{C}}^{\text{persymHankel}}$ implies $\|\Delta A\| = 1$ and $Bx = s\bar{x} = sH\alpha Jx = s\Delta Ax$, and Lemma 4.1 closes also this part.

Finally, $A \in M_{\mathbb{C}}^{\text{circ}}$ implies that $B = A - \lambda I \in M_{\mathbb{C}}^{\text{circ}}$ is normal. So there is $0 \neq x \in \mathbb{C}^n$ with $Bx = s\beta x$, $|\beta| = 1$. Then $\Delta A := \beta I \in M_{\mathbb{C}}^{\text{circ}}$, $\|\Delta A\| = 1$ and $Bx = s\beta x = s\Delta Ax$ finish this part and the proof. \blacksquare

Theorem 4.3 characterizes the structured pseudospectrum for all structures in (1.3) except skewsymmetry. In this case the (complex) structured pseudospectrum may be significantly smaller than the unstructured one. Consider the matrix given in (3.2) for $\varphi = 10^{-4}$. Then Figure 4.1 displays the unstructured and the skewsymmetric pseudospectrum $\Lambda_{\varepsilon}(A)$ for $\varepsilon = 5 \cdot 10^{-7}$. Pseudoeigenvalues are plotted by circles for 10^6 random perturbations. The pseudoeigenvalue zero remains a single point zero under structured perturbations, and for the other two the radius of the connected component of the structured pseudospectrum is less than $5.01 \cdot 10^{-5}$! For large ε it is not difficult to see that both $\Lambda_{\varepsilon}(A)$ and $\Lambda_{\varepsilon}^{\text{skewsym}}(A)$ approach $U_{\varepsilon}(0)$.

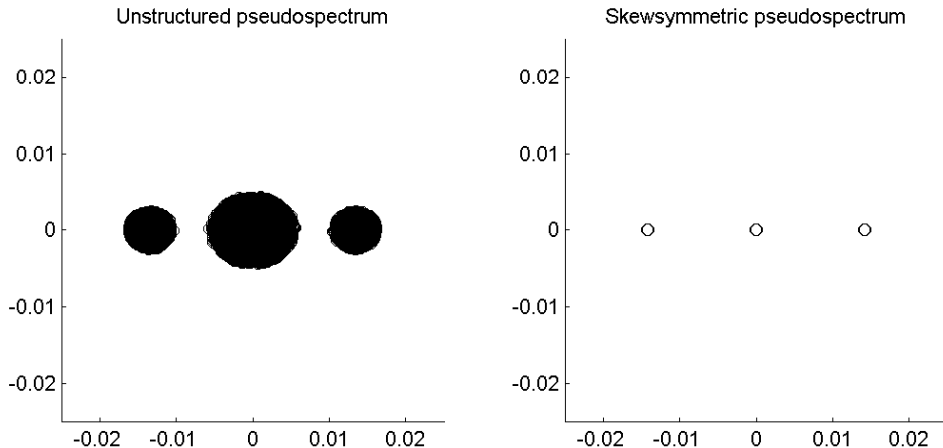


FIG. 4.1. Complex unstructured and skewsymmetric pseudospectrum of the matrix (3.2) for $\varphi = 10^{-4}$ and $\varepsilon = 5 \cdot 10^{-7}$

As for the condition number we may restrict perturbations to real perturbations. This defines the real structured pseudospectrum as in (1.10). Obviously

$$\Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A) \subseteq \Lambda_{\varepsilon}^{\text{struct}}(A) \subseteq \Lambda_{\varepsilon}(A).$$

In the next theorems we characterize the real structured pseudospectrum for a number of structures out of (1.3).

THEOREM 4.4. *If $\text{struct} \in \{\text{sym}, \text{symToep}, \text{Hankel}, \text{persymHankel}\}$ and $A \in M_{\mathbb{R}}^{\text{struct}}$, then*

$$(4.5) \quad \Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A) = \Lambda_{\varepsilon}^{\text{struct}}(A) \cap \mathbb{R} = \Lambda_{\varepsilon}(A) \cap \mathbb{R}.$$

PROOF. First we observe that $A, E \in M_{\mathbb{R}}^{\text{struct}}$ for $\text{struct} \in \{\text{sym}, \text{symToep}, \text{Hankel}, \text{persymHankel}\}$ implies that $A + E$ is symmetric, so $\Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}} \subseteq \mathbb{R}$ and it remains to show $\Lambda_{\varepsilon}(A) \cap \mathbb{R} \subseteq \Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A)$. For given $\lambda \in \Lambda_{\varepsilon}(A) \cap \mathbb{R}$ we abbreviate $B := A - \lambda I$ and $s := \sigma_{\min}(B)$. Following Lemma 4.1 we aim to identify $\Delta A \in M_{\mathbb{R}}^{\text{struct}}$ and $0 \neq x \in \mathbb{C}^n$ with $\|\Delta A\| \leq 1$ and $Bx = s\Delta Ax$ for each individual structure. Then λ belongs to $\Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}$ and proves (4.5). For all structures $B \in M_{\mathbb{R}}^{\text{sym}}$, so there exists $0 \neq x \in \mathbb{R}^n$ with $Bx = s\alpha x$ and $\alpha \in \{-1, +1\}$.

Let $A \in M_{\mathbb{R}}^{\text{sym}}$. Then $\Delta A := \alpha I \in M_{\mathbb{R}}^{\text{sym}}$ gives $\|\Delta A\| = 1$ and $Bx = s\Delta Ax$, and (4.5) follows. The same arguments apply to $A \in M_{\mathbb{R}}^{\text{symToep}}$, and $\Delta A = \alpha I \in M_{\mathbb{R}}^{\text{symToep}}$ finishes this part.

Let $A \in M_{\mathbb{R}}^{\text{Hankel}}$. By Lemma 2.3 there exists $H \in M_{\mathbb{R}}^{\text{Hankel}}$ with $Hx = x$ and $\|H\| = 1$, so $\Delta A = \alpha H \in M_{\mathbb{R}}^{\text{Hankel}}$ implies $Bx = s\Delta Ax$ and (4.5).

For $A \in M_{\mathbb{R}}^{\text{persymHankel}} \subseteq M_{\mathbb{R}}^{\text{sym}} \cap M_{\mathbb{R}}^{\text{persym}}$ we choose $x = \beta Jx$, $\beta \in \{-1, +1\}$ by Lemma 2.2. Then by Lemma 2.4 there is $T \in M_{\mathbb{R}}^{\text{symToep}}$ with $Tx = x$ and $\|T\| = 1$. Hence $\Delta A = \alpha\beta JT \in M_{\mathbb{R}}^{\text{persymHankel}}$ implies $\|\Delta A\| = 1$ and $Bx = s\alpha x = s\Delta Ax$, and (4.5) follows. \blacksquare

For general Toeplitz and for persymmetric matrices we can at least identify the real part of the real pseudospectrum.

THEOREM 4.5. *If $\text{struct} \in \{\text{Toep}, \text{persym}\}$ and $A \in M_{\mathbb{R}}^{\text{struct}}$, then*

$$(4.6) \quad \Lambda_{\varepsilon, \mathbb{R}}^{\text{struct}}(A) \cap \mathbb{R} = \Lambda_{\varepsilon}(A) \cap \mathbb{R}.$$

PROOF. Let $A \in M_{\mathbb{R}}^{\text{Toep}}$ and $\lambda \in \Lambda_{\varepsilon}(A) \cap \mathbb{R}$. Define $B := A - \lambda I \in M_{\mathbb{R}}^{\text{Toep}}$ and $s := \sigma_{\min}(B)$. Since $JB \in M_{\mathbb{R}}^{\text{sym}}$, there is $0 \neq x \in \mathbb{R}^n$ with $Bx = \alpha s Jx$, $\alpha \in \{-1, +1\}$. By Lemma 2.3 there is $H \in M_{\mathbb{R}}^{\text{Hankel}}$ with $\|H\| = 1$ and $Hx = x$. So $\Delta A := \alpha JH \in M_{\mathbb{R}}^{\text{Toep}}$ satisfies $Bx = \alpha s Jx = s\Delta Ax$, and Lemma 4.1 finishes this part.

For $A \in M_{\mathbb{R}}^{\text{persym}}$ we proceed similarly, where in this case we may simply choose $\Delta A := \alpha J \in M_{\mathbb{R}}^{\text{persym}}$. \blacksquare

If for $A \in M_{\mathbb{R}}^{\text{Toep}}$ perturbations are restricted to real Toeplitz perturbations, then complex $\lambda \in \Lambda_{\varepsilon, \mathbb{R}}(A)$ may be missed by structured real perturbations, so that in general

$$\Lambda_{\varepsilon, \mathbb{R}}^{\text{Toep}}(A) \subsetneq \Lambda_{\varepsilon, \mathbb{R}}(A).$$

This follows by the example (5.1) given in the appendix and sufficiently small ε . The same is true for real persymmetric matrices. Consider

$$(4.7) \quad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 1 \\ -2 & 0 & 1 \end{pmatrix} \in M_{\mathbb{R}}^{\text{persym}} \quad \text{and} \quad E = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \notin M_{\mathbb{R}}^{\text{persym}}.$$

Then $-\frac{1}{2} + \frac{\sqrt{3}}{2}i \in \Lambda(A + E)$ and $\|E\| = \sqrt{2}$. But for every real persymmetric \tilde{E} even with $\|\tilde{E}\| \leq 1.5$ the perturbed persymmetric matrix $A + \tilde{E}$ has only real eigenvalues in the left half plane.

The ‘‘visual proof’’ in Figure 4.2 can be enforced by a computer-assisted proof. Therefore, in general,

$$\Lambda_{\varepsilon, \mathbb{R}}^{\text{persym}}(A) \subsetneq \Lambda_{\varepsilon, \mathbb{R}}(A).$$

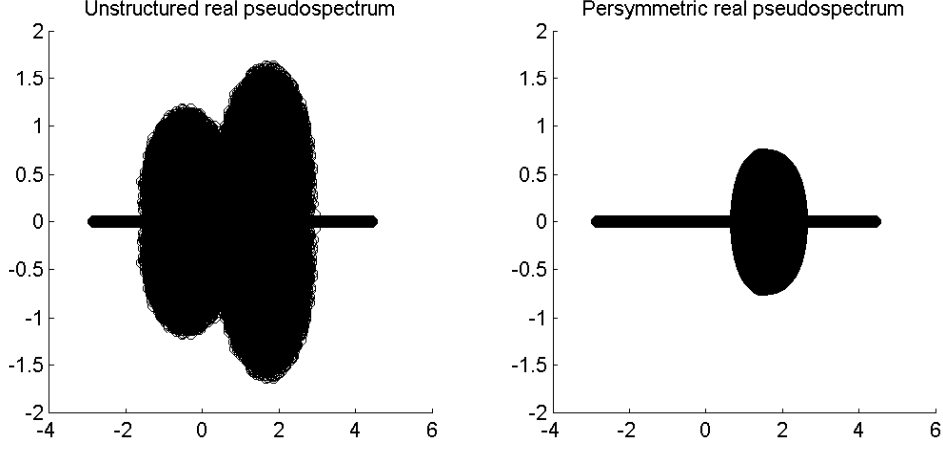


FIG. 4.2. Real unstructured and persymmetric pseudospectrum of A as defined in (4.7) for $\varepsilon = 1.5$

Note that the real parts of $\Lambda_{\varepsilon, \mathbb{R}}^{\text{persym}}$ and Λ_{ε} coincide. For real skewsymmetric matrices of odd dimension, zero is always an eigenvalue, and this must be reflected in the structured pseudospectrum. The following theorem characterizes the real structured pseudospectrum for this structure.

THEOREM 4.6. *Let $A \in M_{\mathbb{R}}^{\text{skewsym}}$ be of even dimension. Then*

$$(4.8) \quad \Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}}(A) = \Lambda_{\varepsilon}(A) \cap i\mathbb{R}.$$

The same is true if A has odd dimension and the eigenvalue zero is not simple. If A is of odd dimension with simple eigenvalue zero, then

$$(4.9) \quad \Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}}(A) = \left(\{0\} \cup \{U_{\varepsilon}(\lambda) : 0 \neq \lambda \in \Lambda(A)\} \right) \cap i\mathbb{R}.$$

So outside $U_{\varepsilon}(0)$ the structured pseudospectrum is always the intersection of the unstructured pseudospectrum with the imaginary axis.

PROOF. With A and E also $A + E$ is real skewsymmetric, so $\Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}} \subseteq i\mathbb{R}$. Since A is normal, $\Lambda_{\varepsilon}(A)$ is the union of all $U_{\varepsilon}(\lambda)$, $\lambda \in \Lambda(A)$.

We first prove that $\tilde{\lambda} \in \Lambda_{\varepsilon}(A) \cap i\mathbb{R}$ with $\tilde{\lambda} \in U_{\varepsilon}(\lambda)$ for $0 \neq \lambda \in \Lambda(A)$ implies $\tilde{\lambda} \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}}(A)$. This proves partly (4.8) and (4.9). For an eigenvector x of A to λ , Lemma 2.6 implies existence of $\Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$ with $\Delta A x = ix$ and $\|\Delta A\| = 1$. But $\lambda, \tilde{\lambda} \in i\mathbb{R}$, so $\tilde{\lambda} - \lambda = i\beta$ for some $\beta \in \mathbb{R}$. Therefore $E := \beta \Delta A \in M_{\mathbb{R}}^{\text{skewsym}}$ gives $\|E\| = |\beta| \leq \varepsilon$ and

$$(A - \tilde{\lambda}I + E)x = (A - \lambda I + E - (\tilde{\lambda} - \lambda)I)x = 0,$$

proving $\tilde{\lambda} \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}}(A)$.

In case $\lambda = 0 \in \Lambda(A)$ is a multiple eigenvalue and $\tilde{\lambda} \in \Lambda_{\varepsilon}(A) \cap i\mathbb{R}$, $|\tilde{\lambda}| \leq \varepsilon$, we can apply Lemma 2.6 the same way as before. This completes the proof of (4.8) because for even dimension a zero eigenvalue is at least double.

Finally, suppose n is odd and zero is a simple eigenvalue. For all $0 \neq \lambda \in \Lambda(A)$ we have to show that $\mu \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{skewsym}}(A) \cap U_{\varepsilon}(0)$ and $\mu \notin U_{\varepsilon}(\lambda)$ imply $\mu = 0$. This proves (4.9) and the theorem. Let $\mu \in \Lambda(A + E)$ for $E \in M_{\mathbb{R}}^{\text{skewsym}}$ with $\|E\| \leq \varepsilon$, and $|\mu| \leq \varepsilon$. Both iA and $i(A + E)$ are Hermitian, so we can order the eigenvalues λ_{ν} and $\tilde{\lambda}_{\nu}$ of A and $A + E$ from $-i\infty$ to $i\infty$, respectively. Applying Weyl's Theorem [20, Theorem 4.3.1] to iA and $i(A + E)$ implies

$$\tilde{\lambda}_k \in U_{\varepsilon}(\lambda_k) \quad \text{for } k = 1, \dots, n.$$

Zero is an eigenvalue of $A + E$, so $\mu = \tilde{\lambda}_p \in U_{\varepsilon}(\lambda_p)$ and $0 = \tilde{\lambda}_q \in U_{\varepsilon}(\lambda_q)$ for some p, q . By assumption, $\mu \in U_{\varepsilon}(\lambda_p)$ implies $\lambda_p = 0 \in \Lambda(A)$. Since A is real skewsymmetric, with λ_q also $-\lambda_q$ is an eigenvalue, and

$|\mu| \leq \varepsilon$ and $|\lambda_q| \leq \varepsilon$ yield $|\mu + \lambda_q| \leq \varepsilon$ or $|\mu - \lambda_q| \leq \varepsilon$. But $\mu \in U_\varepsilon(\pm\lambda_q)$ implies $\lambda_q = 0$. Since zero is a simple eigenvalue of A , the indices p and q must be equal and therefore $\mu = \tilde{\lambda}_p = \tilde{\lambda}_q = 0$. \blacksquare

For real or complex circulants $\Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}$ is a little more involved. However, we can completely characterize the real structured pseudospectrum of real or complex circulants.

THEOREM 4.7. *Let $A \in \mathbb{R}^{n \times n}$ be a circulant matrix with eigenvalues $\lambda_\nu \in \mathbb{C}$, so that $A = F \text{diag}(\lambda_\nu) F^H$. Then the following is true.*

i) For odd n , the eigenvalue λ_1 is real with eigenvector $(1, \dots, 1)^T$ and

$$\Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A) = \{\lambda_1 + [-\varepsilon, \varepsilon]\} \cup \{U_\varepsilon(\lambda_\nu) : \nu \neq 1\}.$$

That means the real structured pseudospectrum of A is identical to the unstructured pseudospectrum outside $U_\varepsilon(\lambda_1)$, and the disc $U_\varepsilon(\lambda_1)$ collapses to its projection on the real axis.

ii) For even n and $m := \frac{n}{2} + 1$, the eigenvalues λ_1 and λ_m are real with eigenvectors $(1, \dots, 1)^T$ and $(1, -1, 1, -1, \dots)^T$, respectively, and

$$\Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A) = \{\lambda_1 + [-\varepsilon, \varepsilon]\} \cup \{\lambda_m + [-\varepsilon, \varepsilon]\} \cup \{U_\varepsilon(\lambda_\nu) : \nu \neq 1, m\}.$$

Therefore, outside $U_\varepsilon(\lambda_1)$ and $U_\varepsilon(\lambda_m)$ the real structured and unstructured pseudospectra are identical, and the discs $U_\varepsilon(\lambda_1)$ and $U_\varepsilon(\lambda_m)$ collapse to their projection on the real axis.

PROOF. Since A is normal, $\Lambda_\varepsilon(A) = \bigcup \{U_\varepsilon(\lambda_\nu) : \lambda_\nu \in \Lambda(A)\}$. Define the index set \mathfrak{J} by

$$\mathfrak{J} := \begin{cases} \{1\} & \text{for odd } n \\ \{1, m\} & \text{for even } n. \end{cases}$$

Since A is a real circulant, (2.14) implies

$$(4.10) \quad \lambda_k \in \Lambda(A) \text{ real and simple} \iff k \in \mathfrak{J}.$$

Furthermore, $x := Fe_k$ implies $Ax = \lambda_k x$. We prove Theorem 4.7 in three steps:

- a) $k \notin \mathfrak{J}$ and $\lambda \in U_\varepsilon(\lambda_k) \implies \lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$.
- b) $k \in \mathfrak{J}$ and $\lambda \in \lambda_k + [-\varepsilon, +\varepsilon] \implies \lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$.
- c) $\lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$, $k \in \mathfrak{J}$, $\lambda \in U_\varepsilon(\lambda_k)$ and $\lambda \notin U_\varepsilon(\lambda_\nu)$ for $\nu \notin \mathfrak{J} \implies \lambda$ real.

This will prove both parts of Theorem 4.7.

ad a) Define a diagonal matrix D with only nonzero diagonal elements $d_{kk} := \lambda - \lambda_k$ and $d_{pp} := \overline{\lambda - \lambda_k}$, where $p := n + 2 - k$. The index k is mapped into p by the permutation P as in (2.14). Hence $D = PD^H P$, $E := FDF^H \in M_{\mathbb{R}}^{\text{circ}}$ and $\|E\| = \|D\| = |\lambda - \lambda_k| \leq \varepsilon$. Furthermore, $Ex = FDF^H \cdot Fe_k = (\lambda - \lambda_k)x$. Therefore

$$(4.11) \quad (A - \lambda I + E)x = (\lambda_k - \lambda)x + (\lambda - \lambda_k)x = 0$$

implies $\lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$.

ad b) Define diagonal D with only nonzero diagonal element $d_{kk} := \lambda - \lambda_k$. The permutation P in (2.14) maps k into itself, so $\lambda - \lambda_k \in \mathbb{R}$ implies $D = PD^H P$ and $E := FDF^H \in M_{\mathbb{R}}^{\text{circ}}$ with $\|E\| = |\lambda - \lambda_k| \leq \varepsilon$. Furthermore $Ex = (\lambda - \lambda_k)x$ and (4.11) proves this part.

ad c) Since $\lambda \in \Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$, there is $E \in M_{\mathbb{R}}^{\text{circ}}$ with $\|E\| \leq \varepsilon$ and $\lambda \in \Lambda(A + E)$. The Bauer-Fike Theorem [11, Theorem 7.2.2] implies $\lambda \in U_\varepsilon(\lambda_\nu)$ for some eigenvalue λ_ν of A , so that by assumption $\nu \in \mathfrak{J}$. Now (4.10) finishes this part and the theorem is proved. \blacksquare

Note that Theorem 4.7 contains an apparent contradiction. Let, for example, a real circulant $A \in M_{\mathbb{R}}^{\text{circ}}$ of odd order be given, and let λ be a real eigenvalue not equal to $\lambda_1 = \sum a_{1i}$. Then part i) of Theorem 4.7 tells that the real structured pseudospectrum $\Lambda_{\varepsilon, \mathbb{R}}^{\text{circ}}(A)$ contains the complex disc $U_\varepsilon(\lambda)$. But if λ is simple, then a small enough *real* perturbation of the real matrix A produces only real eigenvalues near λ , even for general perturbations: an apparent contradiction. However, (4.10) implies that real $\lambda \neq \lambda_1$ must be multiple! A similar argument applies to even order, and this explains the special role of λ_1 (and λ_m) in Theorem 4.7.

5. Appendix. Recently, computer-assisted proofs have been used successfully in different areas. A convenient way of programming is our Matlab interval toolbox INTLAB [27]. It has been used, for example, to solve five of ten problems of the SIAM 100-digit challenge [1]. We sketch a computer-assisted proof of

$$(5.1) \quad \kappa_{\mathbb{R}}^{\text{Toep}}(A, \lambda) < 0.95 \cdot \kappa(A, \lambda)$$

for some A, λ (cf. Theorem 3.2). After sufficient numerical (Matlab) evidence, we choose

$$(5.2) \quad A = \begin{pmatrix} 399 & -817 & -297 \\ 131 & 399 & -817 \\ 1 & 131 & 399 \end{pmatrix} \text{ with } \lambda \approx 409.3 + 463.3i \text{ and } x \approx \begin{pmatrix} 0.8634 \\ 0.0387 - 0.4839i \\ -0.1365 - 0.0158i \end{pmatrix}.$$

In order to *prove* (5.1), we use Lemma 2.1c and (2.4) and have to check that

$$(5.3) \quad |x^T J \Delta A x| < 0.95 \quad \text{for all } \Delta A \in M_{\mathbb{R}}^{\text{Toep}} \text{ with } \|\Delta A\| \leq 1,$$

where x denotes the (true) normalized eigenvector of A (approximately given in (5.2)). Note that this also implies that for this specific x there is no $\Delta A \in M_{\mathbb{R}}^{\text{Toep}}$ with $\|\Delta A\| \leq 1$ and $\Delta A x = J\bar{x}$.

We use a branch and bound method. To be rigorous, all computations are executed in interval arithmetic (with rounding control) in INTLAB [27], the Matlab interval toolbox; for a nice introduction see [14]. We display the *complete* and *executable* INTLAB code (Algorithm 1) to verify (5.3) and therefore (5.1).

ALGORITHM 1. *Verification of (5.1).*

```

format long, intvalinit('displaymidrad',0)
A = toeplitz([399 131 1],[399 -817 -297])
[V,D] = eig(A); % approximate eigendecomposition of A
[L,X] = verifyeig(A,D(1,1),V(:,1)); % inclusion of eigenpair of A
L % inclusion of eigenvalue of A
X = X/sqrt(sum(X'*X)) % inclusion of normalized eigenvector of A

phi = [ X(1)^2 2*X(1)*X(2) 2*X(1)*X(3)+X(2)^2 2*X(2)*X(3) X(3)^2 ];
Y = infsup(-1,1);
List = { [Y Y infsup(0,1) Y Y] }; % initial box
while ~isempty(List)
    dA = List{end};
    % interval Toeplitz matrix T corresponding to dA, JT:=J*T
    JT = [ dA(1) dA(2) dA(3) ; dA(2) dA(3) dA(4) ; dA(3) dA(4) dA(5) ];
    [V D] = eig(mid(JT)); % approximate eigendecomposition of mid(JT)
    [N,k] = max(abs(diag(D))); % N approximates norm(mid(JT),2)
    v = intval(V(:,k)); % approximate eigenvector to N
    psi = [ v(1)^2 2*v(1)*v(2) 2*v(1)*v(3)+v(2)^2 2*v(2)*v(3) v(3)^2 ];
    if ( 100*abs(sum(phi.*dA)) < 95 ) | ( abs(sum(psi.*dA))>sum(v.*v) )
        List = List(1:end-1); % discard
    else % bisect
        [m,i] = max(rad(dA)); % dA(i) of maximum radius
        dA2 = dA;
        M = mid(dA(i)); % split i-th component
        dA2(i) = infsup(dA(i).inf,M);
        List{end} = dA2; % append first half to List
        dA2(i) = infsup(M,dA(i).sup);
        List{end+1} = dA2; % append second half to List
    end
end
end

```

First we use a self-validating method [26, Chapter 5] to calculate rigorous error bounds for the normalized eigenvector x of A based on the Matlab approximations for λ and x , see rows 3 and 4 in Algorithm 1. The computed inclusion for the eigenvalue and normalized eigenvector is displayed by rows 5 and 6 as

```

intval L =
  1.0e+002 *
< 4.09311491182585 + 4.63324918983671i, 0.000000000000001>
intval X =
< 0.86341362769075 + 0.000000000000000i, 0.000000000000001>
< 0.03870965007446 - 0.48388618626465i, 0.000000000000001>
< -0.13646082868583 - 0.01584523542423i, 0.000000000000001>

```

which is the INTLAB midpoint-radius notation. The maximization over the set of structured matrices is a little delicate. According to (5.3) and assuming ΔA to be the Toeplitz matrix with first column $(\delta a_3, \delta a_2, \delta a_1)^T$ and first row $(\delta a_3, \delta a_4, \delta a_5)^T$, we have to maximize

$$(5.4) \quad |x^T J \Delta A x| = |x_1^2 \delta a_1 + 2x_1 x_2 \delta a_2 + (2x_1 x_3 + x_2^2) \delta a_3 + 2x_2 x_3 \delta a_4 + x_3^2 \delta a_5| \quad \text{subject to} \quad \|\Delta A\| \leq 1.$$

We start with an interval matrix

$$\Delta \mathbf{A} := \begin{pmatrix} \mathbf{dA}_3 & \mathbf{dA}_4 & \mathbf{dA}_5 \\ \mathbf{dA}_2 & \mathbf{dA}_3 & \mathbf{dA}_4 \\ \mathbf{dA}_1 & \mathbf{dA}_2 & \mathbf{dA}_3 \end{pmatrix} \quad \text{with } \mathbf{dA}_3 := [0, 1] \text{ and } \mathbf{dA}_i := [-1, 1] \text{ for } i \neq 3.$$

Obviously every $\Delta A \in M_{\mathbb{R}}^{\text{Toep}}$ with $\delta a_3 \geq 0$ and $\|\Delta A\| \leq 1$ satisfies $\Delta A \in \Delta \mathbf{A}$, so (5.1) is true if (5.3) is satisfied for all $\Delta A \in \Delta \mathbf{A}$. The \mathbf{dA}_i are bisected, each interval vector \mathbf{dA} corresponding to a set $T(\mathbf{dA})$ of Toeplitz matrices. Note that \mathbf{phi} and \mathbf{psi} are calculated such that $x^T J \Delta A x \in \text{sum}(\mathbf{phi}.*\mathbf{dA})$ and $v^T J \Delta A v \in \text{sum}(\mathbf{psi}.*\mathbf{dA})$ for all $\Delta A \in T(\mathbf{dA})$, respectively. This notation has the advantage that the intervals \mathbf{dA}_i occur only once, so there is no overestimation with respect to them (cf. [24]).

A box \mathbf{dA} can be discarded if either (5.3) is satisfied for all $\Delta A \in T(\mathbf{dA})$, or if $\|\Delta A\|_2 > 1$ for all $\Delta A \in T(\mathbf{dA})$ (the if-statement in Algorithm 1). The first condition is verified by straightforward interval evaluation (avoiding the conversion error for 0.95). The second condition is satisfied if $|v^T J \Delta A v| > \|v\|^2$ for some (real) vector v and for all $\Delta A \in T(\mathbf{dA})$. A good choice for v is an approximate eigenvector of J times the midpoint matrix of $T(\mathbf{dA})$ to its norm (note this matrix is Hankel and therefore symmetric).

INTLAB uses interval code if at least one operand of a function, an operator $(+, *, \dots)$ or a comparison is an interval. For example $X > Y$ is true for intervals X, Y iff $x > y$ for all $x \in X$ and all $y \in Y$. Note that $\text{abs}(X) := \{|x| : x \in X\}$ for an interval X . Otherwise we hope the code is self-explaining.

Note that provided the tools, i.e. the software and the hardware in use, are working properly, this is a rigorous mathematical proof and is not based on statistical grounds.

If Algorithm 1 stops, (5.3) and hence (5.1) are proved. This straightforward, not optimized algorithm needs some 10^5 bisections, the maximum depth of `List` is 19, and on a 1.6 GHz Pentium M Laptop the proof takes less than half an hour.

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